Micromorphic Approach to Gradient Plasticity and Damage

Samuel Forest

Contents

S. Forest (✉)
Centre des Matériaux UMR 7633, Mines ParisTech CNRS, Evry Cedex, France
e-mail: samuel.forest@mines-paristech.fr

G. Z. Voyiadjis (ed.), *Handbook of Nonlocal Continuum Mechanics for Materials and Structures*, https://doi.org/10.1007/978-3-319-22977-5_9-1

Abstract
Eringen and Mindlin's original micromorphic continuum model is presented and extended towards finite elastic-plastic deformations. The framework is generalized to any additional kinematic degrees of freedom related to plasticity and/or damage mechanisms. It provides a systematic method to develop size–dependent plasticity and damage models, closely related to phase field approaches, that can be applied to hardening and/or softening material behavior. The regularization power of the method is illustrated in the case of damage in single crystals. Special attention is given to the various possible finite deformation formulations enhancing existing frameworks for finite elastoplasticity and damage.

Keywords
Gradient plasticity · Gradient damage · Micromorphic media · Regularization · Finite deformations · Generalized continua · Microstrain · Microstretch · Strain localization · Elasto-plasticity · Cleavage

Introduction

Micromorphic media are examples of three-dimensional generalized continua including additional degrees of freedom complementing the usual displacement vector. A classification of generalised mechanical continuum theories is proposed in Fig. 1 in order to locate more precisely the class of micromorphic media. The present chapter is limited to continuum media fulfilling the principle of local action, meaning that the mechanical state at a material point $\underline{\boldsymbol{X}}$ depends on variables defined at this point only (Truesdell and Toupin 1960; Truesdell and Noll 1965). The classical Cauchy continuum is called *simple material* because its response at

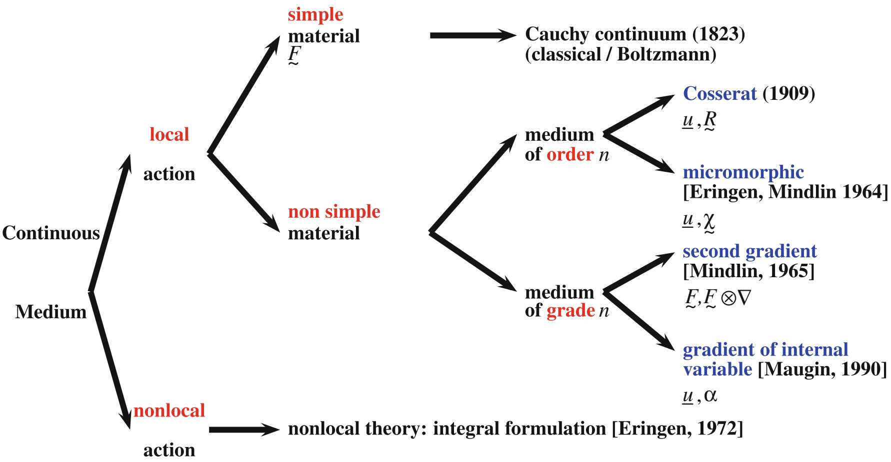

Fig. 1 A classification of the mechanics of generalised continua.

material point $\underline{X}$ to deformations homogeneous in a neighborhood of $\underline{X}$ determines uniquely its response to every deformation at $\underline{X}$. In *higher grade* materials, homogeneous deformations are not sufficient to characterise the material behaviour because they are sensitive to higher gradients of the displacement field. Mindlin formulated for instance the theories that include the second and third gradients of the displacement field (Mindlin 1965). The gradient effect may be limited to the plastic part of deformation which leads to strain gradient plasticity models (Aifantis 1984; Forest and Bertram 2011) or, more generally, theories that include the gradient of some internal variables (Maugin 1990). *Higher order* materials are characterised by additional degrees of freedom of the material points (Eringen 1999). Directors can be attached to each material point that evolve in a different way from the material lines. Cosserat directors can rotate. In the micromorphic continuum designed by Eringen and Suhubi (1964) and Mindlin (1964), the directors can also be distorted, so that a second order tensor is attributed to each material point. Tensors of higher order can even be introduced as proposed in Germain's general micromorphic theory (Germain 1973b; Forest and Sab 2017).

Higher order media are sometimes called continua with *microstructure*. This name is misleading because Cauchy material models can also integrate some aspects of the underlying microstructure as illustrated by classical homogenisation methods used to derive the effective properties of composites. However generalised continua incorporate a feature of the microstructure which is not accounted for by standard homogenisation methods, namely their size–dependent material response. They involve intrinsic lengths directly stemming from the microstructure of the material. The mechanics of generalized continua represents a way of introducing, in the continuum description of materials, some characteristic length scales associated with their microstructure (Mühlhaus 1995). Such intrinsic lengths and generalized constitutive equations can be identified in two ways. Direct identification is possible from experimental curves exhibiting clear size effects in plasticity or fracture or from full–field strain measurements of strongly heterogeneous fields (Geers et al. 1998). The effective properties of such generalized continua can also be derived from scale transition and homogenization techniques by prescribing appropriate boundary conditions on a representative volume of material with microstructure (Cailletaud et al. 2003).

The multiplication of generalized continuum model formulations from Cosserat to strain gradient plasticity in literature may leave an impression of disorder and inconsistency. Recent accounts have shown, on the contrary, that unifying presentations of several classes of generalized continuum theories are possible (Hirschberger and Steinmann 2009; Forest 2009). One of them, called the micromorphic approach, encompasses most theories incorporating additional degrees of freedom from the well–established Cosserat, microstretch and micromorphic continua (Eringen 1999) up to Aifantis and Gurtin strain gradient plasticity theories. Gradient theories are obtained from the micromormphic approach by imposing some internal constraints linking the additional degrees of freedom and other model variables.

The micromorphic theory now arouses strong interest from the materials science and computational mechanics communities because of its regularisation power in the context of softening plasticity and damage and of its rather simple implementation in a finite element program. The number of degrees of freedom is not an obstacle any more with constantly increasing computer power and parallel solvers.

The objective of this chapter is first to present the elastoviscoplasticity theory of micromorphic media at finite deformation. This presentation is based on the fundamental work of Eringen and on recent developments in the context of plasticity. It represents an update of the corresponding chapter in the CISM book (Forest 2012). The second objective of this chapter is to present an extension of the micromorphic theory to other kinds of additional degrees of freedom like plasticity and damage related variables. This micromorphic approach to gradient effects in materials' behavior is a systematic method for incorporating intrinsic lengths in non–linear continuum mechanical models. It is illustrated here in the case of an anisotropic plasticity and damage model. The so–called *microdamage* model takes into account the crystallography of plasticity and fracture in metal single crystals.

The nonlinear theory of micromorphic media is presented in section "The Micromorphic Theory After Eringen and Mindlin and Its Extension to Plasticity". The micromorphic approach is exposed in section "The Micromorphic Approach to Various Gradient Field Theories" together with the closely related phase field approach. Differences and similarities between the micromorphic framework and the phase field approach are pointed out following the general framework provided in Gurtin (1996). A single crystal plasticity and damage model is explored in section "Application to a Continuum Damage Model for Single Crystals and Its Regularization" up to crack propagation simuation. This presentation of the micromorphic approach and the corresponding example are taken from the formulation presented in Aslan and Forest (2011). A first account of the method is given at small strains for the sake of simplicity. It is followed by extensions to finite deformations, following the guidelines given in Forest (2016). The presentation is limited to the static case, the reader being referred to Eringen's original dynamical formulations and to Forest and Sab (2017) for the consideration of inertial terms.

Intrinsic notations are used throughout the chapter. In particular, scalars, vectors, tensors of second, third and fourth ranks are denoted by $a, \underline{\boldsymbol{a}}, \underset{\sim}{\boldsymbol{a}}, \underset{\simeq}{\boldsymbol{a}}, \underset{\approx}{\boldsymbol{a}}$, respectively. Contractions are written as:

$$\underset{\sim}{\boldsymbol{a}} : \underset{\sim}{\boldsymbol{b}} = a_{ij} b_{ij}, \quad \underset{\simeq}{\boldsymbol{a}} \,\vdots\, \underset{\simeq}{\boldsymbol{b}} = a_{ijk} b_{ijk}, \quad \underset{\approx}{\boldsymbol{a}} :: \underset{\approx}{\boldsymbol{b}} = a_{ijkl} b_{ijkl} \tag{1}$$

using the Einstein summation rule for repeated indices. The tensor product is denoted by $\otimes$. For example, the component $\left(\underset{\sim}{\boldsymbol{a}} \otimes \underset{\sim}{\boldsymbol{b}}\right)_{ijkl}$ is $a_{ij} b_{kl}$. A modified tensor product $\boxtimes$ is also used: the component $\left(\underset{\sim}{\boldsymbol{a}} \boxtimes \underset{\sim}{\boldsymbol{b}}\right)_{ijkl}$ is $a_{ik} b_{jl}$.

The gradient operators ∇_x or ∇_X are introduced when the functions depend on microscopic coordinates $\underline{\boldsymbol{x}}$ or macroscopic coordinates $\underline{\boldsymbol{X}}$. The following notation is used:

$$\underline{\boldsymbol{U}} \otimes \nabla_X = U_{i,j} \underline{\boldsymbol{e}}_i \otimes \underline{\boldsymbol{e}}_j, \quad \text{with} \quad U_{i,j} = \frac{\partial U_i}{\partial X_j} \tag{2}$$

$$\underline{\boldsymbol{u}} \otimes \nabla_x = u_{i,j} \underline{\boldsymbol{e}}_i \otimes \underline{\boldsymbol{e}}_j, \quad \text{with} \quad u_{i,j} = \frac{\partial u_i}{\partial x_j} \tag{3}$$

where $(\underline{e}_i)_{i=1,2,3}$ is a Cartesian orthonormal basis.

The Micromorphic Theory After Eringen and Mindlin and Its Extension to Plasticity

Kinematics of Micromorphic Media

The degrees of freedom of the theory are the displacement vector $\underline{\boldsymbol{u}}$ and the microdeformation tensor $\underset{\sim}{\boldsymbol{\chi}}$:

$$DOF := \left\{ \underset{\sim}{\boldsymbol{u}}, \underset{\sim}{\boldsymbol{\chi}} \right\}$$

The current position of the material point is given by the transformation Φ according to $\underline{\boldsymbol{x}} = \Phi(\underline{\boldsymbol{X}}, t) = \underline{\boldsymbol{X}} + \underline{\boldsymbol{u}}(\underline{\boldsymbol{X}}, t)$. The microdeformation describes the deformation of a triad of directors, $\underline{\boldsymbol{\Xi}}^i$ attached to the material point.

$$\underline{\boldsymbol{\xi}}^i(\underline{\boldsymbol{X}}, t) = \underset{\sim}{\boldsymbol{\chi}}(\underline{\boldsymbol{X}}, t) \cdot \underline{\boldsymbol{\Xi}}^i \tag{4}$$

As such, its determinant is taken as strictly positive. The polar decomposition of the generally incompatible microdeformation field $\underset{\sim}{\boldsymbol{\chi}}(\underline{\boldsymbol{X}})$ is introduced

$$\underset{\sim}{\boldsymbol{\chi}} = \underset{\sim}{\boldsymbol{R}}^\sharp \cdot \underset{\sim}{\boldsymbol{U}}^\sharp \tag{5}$$

Internal constraints can be prescribed to the microdeformation. The micromorphic medium reduces to the Cosserat medium when the microdeformation is constrained to be a pure rotation: $\underset{\sim}{\boldsymbol{\chi}} \equiv \underset{\sim}{\boldsymbol{R}}^\sharp$. The microstrain medium is obtained when $\underset{\sim}{\boldsymbol{\chi}} \equiv \underset{\sim}{\boldsymbol{U}}^\sharp$ (Forest and Sievert 2006). Finally, the second gradient theory is retrieved when the microdeformation coincides with the deformation gradient, $\underset{\sim}{\boldsymbol{\chi}} \equiv \underset{\sim}{\boldsymbol{F}}$. A hierarchy of higher order continua can be established by specialising the micromorphic theory and depending on the targeted material class, see Table 1.

The following kinematical quantities are then introduced:

- the velocity field $\underline{\boldsymbol{v}}(\underline{\boldsymbol{x}}, t) := \underline{\dot{\boldsymbol{u}}}\left(\Phi^{-1}(\underline{\boldsymbol{x}}, t)\right)$
- the deformation gradient $\underset{\sim}{\boldsymbol{F}} = \underset{\sim}{\boldsymbol{1}} + \underline{\boldsymbol{u}} \otimes \nabla_X$
- the velocity gradient $\underline{\boldsymbol{v}} \otimes \nabla_x = \underset{\sim}{\dot{\boldsymbol{F}}} \cdot \underset{\sim}{\boldsymbol{F}}^{-1}$

Table 1 A hierarchy of higher order continua

Name	Number of DOF	DOF	References
Cauchy	3	$\underline{u}$	Cauchy (1822)
Microdilatation	4	$\underline{u}, \quad \chi$	Goodman and Cowin (1972) and Steeb and Diebels (2003)
Cosserat	6	$\underline{u}, \quad \underset{\sim}{R}$	Kafadar and Eringen (1971)
Microstretch	7	$\underline{u}, \ \chi, \ \underset{\sim}{R}$	Eringen (1990)
Microstrain	9	$\underline{u}, \quad \underset{\sim}{C}^{\sharp}$	Forest and Sievert (2006)
Micromorphic	12	$\underline{u}, \quad \underset{\sim}{\chi}$	Eringen and Suhubi (1964) and Mindlin (1964)

- the microdeformation rate $\dot{\underset{\sim}{\chi}} \cdot \underset{\sim}{\chi}^{-1}$
- the third rank Lagrangean microdeformation gradient $\underset{\approx}{K} := \underset{\sim}{\chi}^{-1} \cdot \underset{\sim}{\chi} \otimes \nabla_X$
- the gradient of the microdeformation rate tensor

$$\left(\dot{\underset{\sim}{\chi}} \cdot \underset{\sim}{\chi}^{-1}\right) \otimes \nabla_x = \underset{\sim}{\chi} \cdot \dot{\underset{\approx}{K}} : \left(\underset{\sim}{\chi}^{-1} \boxtimes \underset{\sim}{F}^{-1}\right) \tag{6}$$

and the corresponding index notation:

$$\left(\dot{\chi}_{il}\chi^{-1}_{lj}\right)_{,k} = \chi_{ip}\dot{K}_{pqr}\chi^{-1}_{qj}F^{-1}_{rk}$$

Principle of Virtual Power

The method of virtual power is used to introduce the generalised stress tensors and the field and boundary equations they must satisfy (Germain 1973b).

The modelling variables are introduced according to a first gradient theory:

$$MODEL = \left\{\underline{v}, \ \underline{v} \otimes \nabla_x, \dot{\underset{\sim}{\chi}} \cdot \underset{\sim}{\chi}^{-1}, \left(\dot{\underset{\sim}{\chi}} \cdot \underset{\sim}{\chi}^{-1}\right) \otimes \nabla_x\right\}$$

The virtual power of internal forces of a subdomain $\mathcal{D} \subset \mathcal{B}$ of the body is

$$\mathcal{P}^{(i)}\left(\underline{v}^*, \dot{\underset{\sim}{\chi}}^* \cdot \underset{\sim}{\chi}*^{-1}\right) = \int_{\mathcal{D}} p^{(i)}\left(\underline{v}^*, \dot{\underset{\sim}{\chi}}^* \cdot \underset{\sim}{\chi}*^{-1}\right) \, dV$$

The virtual power density of internal forces is a linear form on the fields of virtual modeling variables:

$$
\begin{aligned}
p^{(i)} &= \underset{\sim}{\boldsymbol{\sigma}} : \left(\underset{\sim}{\dot{\boldsymbol{F}}} \cdot \underset{\sim}{\boldsymbol{F}}^{-1}\right) + \underset{\sim}{\boldsymbol{s}} : \left(\underset{\sim}{\dot{\boldsymbol{F}}} \cdot \underset{\sim}{\boldsymbol{F}}^{-1} - \underset{\sim}{\dot{\boldsymbol{\chi}}} \cdot \underset{\sim}{\boldsymbol{\chi}}^{-1}\right) + \underset{\approx}{\boldsymbol{M}} \vdots \left(\left(\underset{\sim}{\dot{\boldsymbol{\chi}}} \cdot \underset{\sim}{\boldsymbol{\chi}}^{-1}\right) \otimes \nabla_x\right) \\
&= \underset{\sim}{\boldsymbol{\sigma}} : \left(\underset{\sim}{\dot{\boldsymbol{F}}} \cdot \underset{\sim}{\boldsymbol{F}}^{-1}\right) + \underset{\sim}{\boldsymbol{s}} : \left(\underset{\sim}{\boldsymbol{\chi}} \cdot \left(\underset{\sim}{\boldsymbol{\chi}}^{-1} \cdot \underset{\sim}{\boldsymbol{F}}\right)^{\cdot} \cdot \underset{\sim}{\boldsymbol{F}}^{-1}\right) \\
&\quad + \underset{\approx}{\boldsymbol{M}} \vdots \left(\underset{\sim}{\boldsymbol{\chi}} \cdot \underset{\approx}{\dot{\boldsymbol{K}}} : \left(\underset{\sim}{\boldsymbol{\chi}}^{-1} \boxtimes \underset{\sim}{\boldsymbol{F}}^{-1}\right)\right)
\end{aligned}
\tag{7}
$$

where the relative deformation rate $\underset{\sim}{\dot{\boldsymbol{F}}} \cdot \underset{\sim}{\boldsymbol{F}}^{-1} - \underset{\sim}{\dot{\boldsymbol{\chi}}} \cdot \underset{\sim}{\boldsymbol{\chi}}^{-1}$ is introduced and expressed in terms of the rate of the relative deformation $\underset{\sim}{\boldsymbol{\chi}}^{-1} \cdot \underset{\sim}{\boldsymbol{F}}$. The virtual power density of internal forces is invariant with respect to virtual rigid body motions so that $\underset{\sim}{\boldsymbol{\sigma}}$ must be symmetric. The generalised stress tensors conjugate to the velocity gradient, the relative deformation rate and the gradient of the microdeformation rate are the simple stress tensor $\underset{\sim}{\boldsymbol{\sigma}}$, the relative stress tensor $\underset{\sim}{\boldsymbol{s}}$ and the double stress tensor $\underset{\approx}{\boldsymbol{M}}$ of third rank.

The Gauss theorem is then applied to the power of internal forces.

$$
\begin{aligned}
\int_{\mathcal{D}} p^{(i)} dV = & \int_{\partial\mathcal{D}} \underline{\boldsymbol{v}}^* \cdot \left(\underset{\sim}{\boldsymbol{\sigma}} + \underset{\sim}{\boldsymbol{s}}\right) \cdot \underline{\boldsymbol{n}}\, dS + \int_{\partial\mathcal{D}} \left(\underset{\sim}{\dot{\boldsymbol{\chi}}}^* \cdot \underset{\sim}{\boldsymbol{\chi}}^{*-1}\right) : \underset{\approx}{\boldsymbol{M}} \cdot \underline{n}\, dS \\
& - \int_{\mathcal{D}} \underline{\boldsymbol{v}}^* \cdot \left(\underset{\sim}{\boldsymbol{\sigma}} + \underset{\sim}{\boldsymbol{s}}\right) \cdot \nabla_x dV \\
& - \int_{\mathcal{D}} \left(\underset{\sim}{\dot{\boldsymbol{\chi}}}^* \cdot \underset{\sim}{\boldsymbol{\chi}}^{*-1}\right) : \left(\underset{\approx}{\boldsymbol{M}} \cdot \nabla_x + \underset{\sim}{\boldsymbol{s}}\right) \cdot dV
\end{aligned}
$$

The form of the previous boundary integral dictates the possible form of the power of contact forces acting on the boundary $\partial\mathcal{D}$ of the subdomain $\mathcal{D} \subset \mathcal{B}$.

$$
\begin{aligned}
\mathcal{P}^{(c)}\left(\underline{\boldsymbol{v}}^*, \underset{\sim}{\dot{\boldsymbol{\chi}}}^* \cdot \underset{\sim}{\boldsymbol{\chi}}^{*-1}\right) &= \textstyle\int_{\partial\mathcal{D}} p^{(c)}\left(\underline{\boldsymbol{v}}^*, \underset{\sim}{\dot{\boldsymbol{\chi}}}^* \cdot \underset{\sim}{\boldsymbol{\chi}}^{*-1}\right) dV \\
p^{(c)}\left(\underline{\boldsymbol{v}}^*, \underset{\sim}{\dot{\boldsymbol{\chi}}}^* \cdot \underset{\sim}{\boldsymbol{\chi}}^{*-1}\right) &= \underline{\boldsymbol{t}} \cdot \underline{\boldsymbol{v}}^* + \underset{\sim}{\boldsymbol{m}} : \left(\underset{\sim}{\dot{\boldsymbol{\chi}}}^* \cdot \underset{\sim}{\boldsymbol{\chi}}^{*-1}\right)
\end{aligned}
$$

where the simple traction $\underline{\boldsymbol{t}}$ and double traction $\underset{\sim}{\boldsymbol{m}}$ are introduced.

The power of forces acting at a distance is defined as

$$
\begin{aligned}
\mathcal{P}^{(e)}\left(\underline{\boldsymbol{v}}^*, \underset{\sim}{\dot{\boldsymbol{\chi}}}^* \cdot \underset{\sim}{\boldsymbol{\chi}}^{*-1}\right) &= \textstyle\int_{\mathcal{D}} p^{(e)}\left(\underline{\boldsymbol{v}}^*, \underset{\sim}{\dot{\boldsymbol{\chi}}}^* \cdot \underset{\sim}{\boldsymbol{\chi}}^{*-1}\right) dV \\
p^{(e)}\left(\underline{\boldsymbol{v}}^*, \underset{\sim}{\dot{\boldsymbol{\chi}}}^* \cdot \underset{\sim}{\boldsymbol{\chi}}^{*-1}\right) &= \underline{\boldsymbol{f}} \cdot \underline{\boldsymbol{v}}^* + \underset{\sim}{\boldsymbol{p}} : \left(\underset{\sim}{\dot{\boldsymbol{\chi}}}^* \cdot \underset{\sim}{\boldsymbol{\chi}}^{*-1}\right)
\end{aligned}
$$

including simple body forces $\underline{f}$ and double body forces $\underset{\sim}{p}$. More general double and triple volume forces could also be incorporated according to Germain (1973b).

The principle of virtual power is now stated in the static case,

$$\forall \underline{v}^*, \forall \underset{\sim}{\chi}^*, \forall \mathcal{D} \subset \mathcal{B}, \mathcal{P}^{(i)} \left(\underline{v}^*, \dot{\underset{\sim}{\chi}}^* \cdot \underset{\sim}{\chi}^{*-1} \right) = \mathcal{P}^{(c)} \left(\underline{v}^*, \dot{\underset{\sim}{\chi}}^* \cdot \underset{\sim}{\chi}^{*-1} \right) + \mathcal{P}^{(e)} \left(\underline{v}^*, \dot{\underset{\sim}{\chi}}^* \cdot \underset{\sim}{\chi}^{*-1} \right)$$

This variational formulation leads to.

$$\begin{aligned} &\int_{\partial \mathcal{D}} \underline{v}^* \cdot \left(\underset{\sim}{\sigma} + \underset{\sim}{s} \right) \cdot \underline{n} \, ds + \int_{\partial \mathcal{D}} \left(\dot{\underset{\sim}{\chi}}^* \cdot \underset{\sim}{\chi}^{*-1} \right) : \underset{\approx}{M} \cdot \underline{n} \, dS \\ &- \int_{\mathcal{D}} \underline{v}^* \cdot \left(\left(\underset{\sim}{\sigma} + \underset{\sim}{s} \right) \cdot \nabla_x + \underline{f} \right) dV \\ &- \int_{\mathcal{D}} \left(\dot{\underset{\sim}{\chi}}^* \cdot \underset{\sim}{\chi}^{*-1} \right) : \left(\underset{\approx}{M} \cdot \nabla_x + \underset{\sim}{s} + \underset{\sim}{p} \right) dV = 0 \end{aligned}$$

which delivers the field equations of the problem (Kirchner and Steinmann 2005; Lazar and Maugin 2007; Hirschberger et al. 2007):

- balance of momentum equation (static case)

$$\left(\underset{\sim}{\sigma} + \underset{\sim}{s} \right) \cdot \nabla_x + \underline{f} = 0, \quad \forall \underline{x} \in \mathcal{B} \tag{8}$$

- balance of generalized moment of momentum equation (static case)

$$\underset{\approx}{M} \cdot \nabla_x + \underset{\sim}{s} + \underset{\sim}{p} = 0, \ \forall \underline{x} \in \mathcal{B} \tag{9}$$

- boundary conditions

$$\left(\underset{\sim}{\sigma} + \underset{\sim}{s} \right) \cdot \underline{n} = \underline{t}, \quad \forall \underline{x} \in \partial \mathcal{B} \tag{10}$$

$$\underset{\approx}{M} \cdot \underline{n} = \underset{\sim}{m}, \quad \forall \underline{x} \in \partial \mathcal{B} \tag{11}$$

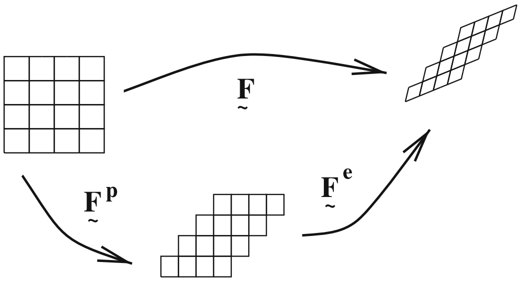

Fig. 2 Multiplicative decomposition of the deformation gradient

Elastoviscoplasticity of Micromorphic Media

This section is dedicated to the formulation of constitutive equations for micromorphic media. The general case of hyperelastic-viscoplastic materials is considered.

Elastic–Plastic Decomposition of the Generalised Strain Measures

According to Eringen (1999), the following Lagrangean strain measures are adopted:

$$STRAIN = \left\{ \underset{\sim}{\boldsymbol{C}} := \underset{\sim}{\boldsymbol{F}}^T \cdot \underset{\sim}{\boldsymbol{F}},\ \underset{\sim}{\boldsymbol{\Upsilon}} := \underset{\sim}{\boldsymbol{\chi}}^{-1} . \underset{\sim}{\boldsymbol{F}},\ \underset{\approx}{\boldsymbol{K}} := \underset{\sim}{\boldsymbol{\chi}}^{-1} \cdot \left(\underset{\sim}{\boldsymbol{\chi}} \otimes \nabla_X \right) \right\}$$

i.e. the Cauchy–Green strain tensor, the relative deformation and the microdeformation gradient.

In the presence of plastic deformation, the question arises of splitting the previous Lagrangean strain measures into elastic and plastic contributions. Following Mandel (1973), a multiplicative decomposition of the deformation gradient is postulated:

$$\underset{\sim}{\boldsymbol{F}} = \underset{\sim}{\boldsymbol{F}}^e \cdot \underset{\sim}{\boldsymbol{F}}^p = \underset{\sim}{\boldsymbol{R}}^e \cdot \underset{\sim}{\boldsymbol{U}}^e \cdot \underset{\sim}{\boldsymbol{F}}^p \tag{12}$$

which defines an intermediate local configuration at each material point, see Fig. 2. Uniqueness of the decomposition requires the suitable definition of directors. Such directors are available in any micromorphic theory.

A multiplicative decomposition of the microdeformation is also considered:

$$\underset{\sim}{\boldsymbol{\chi}} = \underset{\sim}{\boldsymbol{\chi}}^e \cdot \underset{\sim}{\boldsymbol{\chi}}^p = \underset{\sim}{\boldsymbol{R}}^{e\sharp} \cdot \underset{\sim}{\boldsymbol{U}}^{e\sharp} \cdot \underset{\sim}{\boldsymbol{\chi}}^p \tag{13}$$

according to Forest and Sievert (2003, 2006). The uniqueness of the decomposition also requires the suitable definition of directors. As an example, lattice directions in a single crystal are physically relevant directors for an elastoviscoplasticity micromorphic theory, see Aslan et al. (2011). Finally, a partition rule must also be proposed for the third strain measure, namely the microdeformation gradient. Sansour (1998a, b) introduced an additive decomposition of curvature:

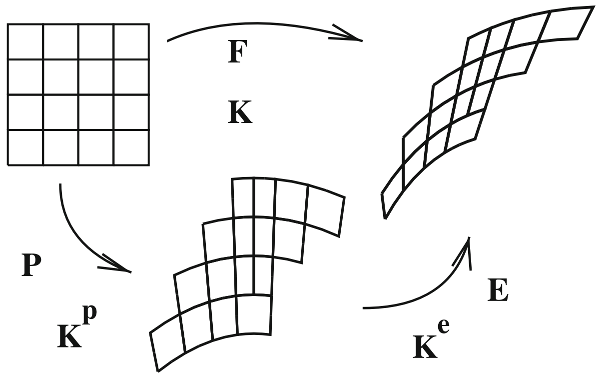

Fig. 3 Definition of an intermediate local configuration for micromorphic elastoplasticity

$$\underset{\approx}{\boldsymbol{K}} = \underset{\approx}{\boldsymbol{K}}^e + \underset{\approx}{\boldsymbol{K}}^p \tag{14}$$

A quasi–additive decomposition was proposed by Forest and Sievert (2003) with the objective of defining an intermediate local configuration for which all generalised stress tensor are simultaneously released, as it will become apparent in the next section:

$$\underset{\approx}{\boldsymbol{K}} = \underset{\sim}{\boldsymbol{\chi}}^{p-1} \cdot \underset{\simeq}{\boldsymbol{K}}^e : \left(\underset{\sim}{\boldsymbol{\chi}}^p \boxtimes \underset{\sim}{\boldsymbol{F}}^p\right) + \underset{\approx}{\boldsymbol{K}}^p \tag{15}$$

Constitutive Equations

The continuum thermodynamic formulation is essentially unchanged in the presence of additional degrees of freedom provided that all functionals are properly extended to the new sets of variables. The local equation of energy balance is written in its usual form:

$$\rho\dot{\epsilon} = p^{(i)} - \underline{\boldsymbol{q}} \cdot \nabla + r \tag{16}$$

where ε is the specific internal energy density, and $p^{(i)}$ is the power density of internal forces according to Eq. (7). The heat flux vector is $\underline{\boldsymbol{q}}$ and r is a heat source term (Fig. 3). The local form of the second principle of thermodynamics is written as.

$$\rho\dot{\eta} + \left(\frac{\underline{\boldsymbol{q}}}{T}\right).\nabla - \frac{r}{T} \geq 0$$

where η is the specific entropy density. Introducing the Helmholtz free energy function the ψ, second law becomes

$$p^{(i)} - \rho\dot{\Psi} - \eta\dot{T} - \frac{\underline{\boldsymbol{q}}}{T} \cdot (\nabla T) \geq 0$$

The state variables of the elastoviscoplastic micromorphic material are all the elastic strain measures and a set of internal variables q. The free energy density is a function of the state variables:

$$\Psi\left(\underset{\sim}{\boldsymbol{C}}^e := \underset{\sim}{\boldsymbol{F}}^{eT}.\underset{\sim}{\boldsymbol{F}}^e,\ \underset{\sim}{\boldsymbol{\Upsilon}}^e := \underset{\sim}{\boldsymbol{\chi}}^{e-1}\cdot\underset{\sim}{\boldsymbol{F}}^e,\ \underset{\approx}{\boldsymbol{K}}^e, q\right)$$

The exploitation of the entropy inequality leads to the definition of the hyperelastic state laws in the form:

$$\underset{\sim}{\boldsymbol{\sigma}} = 2\underset{\sim}{\boldsymbol{F}}^e\cdot\rho\frac{\partial\Psi}{\partial\underset{\sim}{\boldsymbol{C}}^e}\cdot\underset{\sim}{\boldsymbol{F}}^{eT},\ \underset{\sim}{\boldsymbol{s}} = \underset{\sim}{\boldsymbol{R}}^{e\sharp}\cdot\underset{\sim}{\boldsymbol{U}}^{e\sharp-1}\cdot\rho\frac{\partial\Psi}{\partial\underset{\sim}{\boldsymbol{\Upsilon}}^e}\cdot\underset{\sim}{\boldsymbol{F}}^{eT}$$

$$\underset{\approx}{\boldsymbol{M}} = \underset{\sim}{\boldsymbol{\chi}}^{-T}\cdot\rho\frac{\partial\Psi}{\partial\underset{\approx}{\boldsymbol{K}}^e} : \left(\underset{\sim}{\boldsymbol{\chi}}^T \boxtimes \underset{\sim}{\boldsymbol{F}}^T\right) \tag{17}$$

while the entropy density is still given by $\eta = -\frac{\partial\Psi}{\partial T}$. The thermodynamic force associated with the internal variable q is

$$R = -\rho\frac{\partial\Psi}{\partial q}$$

The hyperelasticity law (17) for the double stress tensor was derived for the additive decomposition (14). The quasi–additive decomposition (15) leads to an hyperelastic constitutive equation for the conjugate stress $\underset{\approx}{\boldsymbol{M}}$ in the current configuration, that has also the same form as for pure hyperelastic behaviour. One finds:

$$\underset{\approx}{\boldsymbol{M}} = \underset{\sim}{\boldsymbol{\chi}}^{e-T}\cdot\rho\frac{\partial\Psi}{\partial\underset{\approx}{\boldsymbol{K}}^e} : \left(\underset{\sim}{\boldsymbol{\chi}}^{eT} \boxtimes \underset{\sim}{\boldsymbol{F}}^{eT}\right) \tag{18}$$

The residual intrinsic dissipation is

$$D = \underset{\sim}{\Sigma} : \left(\dot{\underset{\sim}{F}}^p.\underset{\sim}{F}^{p-1}\right) + \underset{\sim}{\boldsymbol{\mathcal{S}}} : \left(\dot{\underset{\sim}{\boldsymbol{\chi}}}^p.\underset{\sim}{\boldsymbol{\chi}}^{p-1}\right) + \underset{\approx}{\boldsymbol{\mathcal{M}}}\vdots\dot{\underset{\approx}{\boldsymbol{K}}}^p + R\dot{q} \geq 0$$

where generalised Mandel stress tensors have been defined.

$$\underset{\sim}{\Sigma} = \underset{\sim}{\boldsymbol{F}}^{eT}\cdot\left(\underset{\sim}{\boldsymbol{\sigma}} + \underset{\sim}{\boldsymbol{s}}\right)\cdot\underset{\sim}{\boldsymbol{F}}^{e-T},\ \underset{\sim}{\boldsymbol{\mathcal{S}}} = -\underset{\sim}{\boldsymbol{U}}^{e\sharp}\cdot\underset{\sim}{\boldsymbol{R}}^{e\sharp T}\cdot\underset{\sim}{\boldsymbol{s}}\cdot\underset{\sim}{\boldsymbol{R}}^{e\sharp}\cdot\underset{\sim}{\boldsymbol{U}}^{e\sharp^{-1}} \tag{19}$$

$$\underset{\approx}{\boldsymbol{\mathcal{M}}} = \underset{\sim}{\boldsymbol{\chi}}^T\cdot\underset{\simeq}{\boldsymbol{S}} : \left(\underset{\sim}{\boldsymbol{\chi}}^{-T} \boxtimes \underset{\sim}{\boldsymbol{F}}^{-T}\right) \tag{20}$$

At this stage, one may define a dissipation potential, function of the Mandel stress tensors, from which the viscoplastic flow rule and the evolution equations for the internal variables are derived.

$$\Omega\left(\underset{\sim}{\boldsymbol{\Sigma}}, \underset{\sim}{\boldsymbol{\mathcal{S}}}, \underset{\approx}{\boldsymbol{\mathcal{M}}}, R\right)$$

$$\dot{\underset{\sim}{\boldsymbol{F}}}^p.\underset{\sim}{\boldsymbol{F}}^{p-1} = \frac{\partial\Omega}{\partial\underset{\sim}{\boldsymbol{\Sigma}}}, \dot{\underset{\sim}{\boldsymbol{\chi}}}^p.\underset{\sim}{\boldsymbol{\chi}}^{p-1} = \frac{\partial\Omega}{\partial\underset{\sim}{\boldsymbol{\mathcal{S}}}}, \dot{\underset{\approx}{\boldsymbol{K}}}^p = \frac{\partial\Omega}{\partial\underset{\approx}{\boldsymbol{\mathcal{M}}}}, \dot{q} = \frac{\partial\Omega}{\partial R}$$

The convexity of the dissipation potential with respect to its arguments ensures the positivity of the dissipation rate at each instant.

Explicit constitutive equations can be found in Forest and Sievert (2003), Grammenoudis and Tsakmakis (2009), Grammenoudis et al. (2009), Regueiro (2010), and Sansour et al. (2010). Examples of application of elastoplastic micromorphic media can be found in Dillard et al. (2006) for plasticity and failure of metallic foams.

The Micromorphic Approach to Various Gradient Field Theories

The previous micromorphic model can be extended to other types of additional degrees of freedom. This leads to a systematic approach for the construction of generalized continuum models with enriched kinematics. The method is presented first within the small deformation framework. The general formulation is discussed in the next section.

Thermomechanics with Additional Degrees of Freedom

One starts from an elastoviscoplasticity model formulation within the framework of the classical Cauchy continuum and classical continuum thermodynamics according to Germain et al. (1983) and Maugin (1999). The material behaviour is characterized by the reference sets of degrees of freedom and state variables.

$$DOF0 = \{\underline{\boldsymbol{u}}\}\,,\ STATE0 = \left\{\underset{\sim}{\varepsilon}, T, q\right\} \tag{21}$$

which the free energy density function ψ may depend on. The small strain tensor is denoted by $\underset{\sim}{\varepsilon}$ whereas q represents the whole set of internal variables of arbitrary tensorial order accounting for nonlinear processes at work inside the material volume element, like isotropic and kinematic hardening variables. The absolute temperature is T.

Additional degrees of freedom ϕ_χ are then introduced in the previous original model. They may be of any tensorial order and of different physical nature (deformation, plasticity or damage variable). The notation χ indicates that these variables eventually represent some microstructural features of the material so that we will call them micromorphic variables or microvariables (*microdeformation, microdamage...*). The *DOF* and *STATE* spaces are extended as follows:

$$DOF = \{\underline{\boldsymbol{u}},\ \phi_\chi\}\ ,\ STATE = \left\{\underset{\sim}{\varepsilon}, T, q,\ \ \phi_\chi, \nabla\phi_\chi\right\} \tag{22}$$

Depending on the physical nature of ϕ_χ, it may or may not be a state variable. For instance, if the microvariable is a microrotation as in the Cosserat model, it is not a state variable for objectivity reasons and will appear in *STATE* only in combination with the macrorotation. In contrast, if the microvariable is a microplastic equivalent strain, as in Aifantis model, it then explicitly appears in the state space.

The virtual power of internal forces is then extended to the power done by the micromorphic variable and its first gradient:

$$\begin{aligned} \mathcal{P}^{(i)}\left(\underline{\boldsymbol{v}}^*, \dot{\phi}_\chi^*\right) &= -\int_{\mathcal{D}} p^{(i)}\left(\underline{\boldsymbol{v}}^*, \dot{\phi}_\chi^*\right) dV \\ p^{(i)}\left(\underline{\boldsymbol{v}}^*, \dot{\phi}_\chi^*\right) &= \underset{\sim}{\boldsymbol{\sigma}} : \nabla\underline{\boldsymbol{v}}^* + a\dot{\phi}_\chi^* + \underline{\boldsymbol{b}} \cdot \nabla\dot{\phi}_\chi^* \end{aligned} \tag{23}$$

where $\mathcal{D}$ is a subdomain of the current configuration of the body. The Cauchy stress is $\underset{\sim}{\boldsymbol{\sigma}}$ and a and $\underline{\boldsymbol{b}}$ are generalized stresses associated with the micromorphic variable and its first gradient. Similarly, the power of contact forces must be extended as follows:

$$\mathcal{P}^{(c)}\left(\underline{\boldsymbol{v}}^*, \dot{\phi}_\chi^*\right) = \int_{\mathcal{D}} p^{(c)}\left(\underline{\boldsymbol{v}}^*, \dot{\phi}_\chi^*\right) dV,\quad p^{(c)}\left(\underline{\boldsymbol{v}}^*, \dot{\phi}_\chi^*\right) = \underline{\boldsymbol{t}}.\underline{\boldsymbol{v}}^* + a^c\dot{\phi}_\chi^* \tag{24}$$

where $\underline{\boldsymbol{t}}$ is the traction vector and a^c a generalized traction. For conciseness, we do not extend the power of forces acting at a distance and keep the classical form:

$$\mathcal{P}^{(e)}\left(\underline{\boldsymbol{v}}^*, \dot{\phi}_\chi^*\right) = \int_{\mathcal{D}} p^{(e)}\left(\underline{\boldsymbol{v}}^*, \dot{\phi}_\chi^*\right) dV,\quad p^{(e)}\left(\underline{\boldsymbol{v}}^*, \dot{\phi}_\chi^*\right) = \rho\underline{\boldsymbol{f}}.\underline{\boldsymbol{v}}^* \tag{25}$$

where $\rho\underline{\boldsymbol{f}}$ accounts for given simple body forces. Following Germain (1973a), given body couples and double forces working with the gradient of the velocity field could also be introduced in the theory. The generalized principle of virtual power with respect to the velocity and micromorphic variable fields, is presented here in the static case only:

$$\mathcal{P}^{(i)}\left(\underline{\boldsymbol{v}}^*, \dot{\phi}_\chi^*\right) + \mathcal{P}^{(e)}\left(\underline{\boldsymbol{v}}^*, \dot{\phi}_\chi^*\right) + \mathcal{P}^{(c)}\left(\underline{\boldsymbol{v}}^*, \dot{\phi}_\chi^*\right) = 0,\ \forall\mathcal{D}\subset\Omega,\ \forall\underline{\boldsymbol{v}}^*, \dot{\phi}_\chi^* \tag{26}$$

The method of virtual power according to Maugin (1980) is used then to derive the standard local balance of momentum equation:

$$\text{div}\ \underset{\sim}{\boldsymbol{\sigma}} + \rho\underline{\boldsymbol{f}} = 0,\ \forall\underline{\boldsymbol{x}}\in\Omega \tag{27}$$

and the generalized balance of micromorphic momentum equation:

$$\text{div}\ \underline{\boldsymbol{b}} - a = 0,\ \forall\underline{\boldsymbol{x}}\in\Omega \tag{28}$$

The method also delivers the associated boundary conditions for the simple and generalized tractions:

$$\underline{\boldsymbol{t}} = \underset{\sim}{\boldsymbol{\sigma}}.\underline{\boldsymbol{n}},\ a^c = \underline{\boldsymbol{b}}.\underline{\boldsymbol{n}},\ \forall \underline{\boldsymbol{x}} \in \partial\mathcal{D} \tag{29}$$

The local balance of energy is also enhanced by the generalized micromorphic power already included in the power of internal forces (23):

$$\rho\dot{\epsilon} = p^{(i)} - \text{div }\underline{\boldsymbol{q}} + \rho r \tag{30}$$

where ϵ is the specific internal energy, $\underline{\boldsymbol{q}}$ the heat flux vector and r denotes external heat sources. The entropy principle takes the usual local form:

$$-\rho\left(\dot{\psi} + \eta\dot{T}\right) + p^{(i)} - \frac{\underline{\boldsymbol{q}}}{T}.\nabla T \geq 0 \tag{31}$$

where it is assumed that the entropy production vector is still equal to the heat vector divided by temperature, as in classical thermomechanics according to Coleman and Noll (1963). Again, the enhancement of the theory goes through the enriched power density of internal forces (7). The entropy principle is exploited according to classical continuum thermodynamics to derive the state laws. At this stage it is necessary to be more specific on the dependence of the state functions ψ, η, $\underset{\sim}{\boldsymbol{\sigma}}$,$a$, $\underline{\boldsymbol{b}}$ on state variables and to distinguish between dissipative and non–dissipative mechanisms. The introduction of dissipative mechanisms may require an increase in the number of state variables. These different situations are considered in the following subsections.

Non–dissipative Contribution of Generalized Stresses

Dissipative events are assumed here to enter the model only via the classical mechanical part. Total strain is split into elastic and plastic parts:

$$\underset{\sim}{\boldsymbol{\varepsilon}} = \underset{\sim}{\boldsymbol{\varepsilon}}^e + \underset{\sim}{\boldsymbol{\varepsilon}}^p \tag{32}$$

The following constitutive functional dependencies are then introduced

$$\begin{aligned} &\psi = \widehat{\psi}\left(\underset{\sim}{\boldsymbol{\varepsilon}}^e, T, q, \phi_\chi, \nabla\phi_\chi\right),\ \underset{\sim}{\boldsymbol{\sigma}} = \widehat{\underset{\sim}{\boldsymbol{\sigma}}}\left(\underset{\sim}{\boldsymbol{\varepsilon}}^e, T, q, \phi_\chi, \nabla\phi_\chi\right),\\ &\eta = \widehat{\eta}\left(\underset{\sim}{\boldsymbol{\varepsilon}}^e, T, q, \phi_\chi, \nabla\phi_\chi\right)\ \ a = \widehat{a}\left(\underset{\sim}{\boldsymbol{\varepsilon}}^e, T, q, \phi_\chi, \nabla\phi_\chi\right),\\ &\underline{\boldsymbol{b}} = \widehat{\underline{\boldsymbol{b}}}\left(\underset{\sim}{\boldsymbol{\varepsilon}}^e, T, q, \phi_\chi, \nabla\phi_\chi\right) \end{aligned} \tag{33}$$

The entropy inequality (31) can be expanded as:

$$\begin{aligned}\left(\underset{\sim}{\boldsymbol{\sigma}} - \rho\frac{\partial\widehat{\psi}}{\partial\underset{\sim}{\boldsymbol{\varepsilon}}^e}\right) : \dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^e + \rho\left(\eta + \frac{\partial\widehat{\psi}}{\partial T}\right)\dot{T} + \left(a - \rho\frac{\partial\widehat{\psi}}{\partial\phi_\chi}\right)\dot{\phi}_\chi \\ + \left(\underline{\boldsymbol{b}} - \rho\frac{\partial\widehat{\psi}}{\partial\nabla\phi_\chi}\right).\nabla\dot{\phi}_\chi + \underset{\sim}{\boldsymbol{\sigma}} : \dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p - \rho\frac{\partial\widehat{\psi}}{\partial q}\dot{q} - \frac{\underline{\boldsymbol{q}}}{T}.\nabla T \geq 0\end{aligned} \tag{34}$$

Assuming that no dissipation is associated with the four first terms of the previous inequality, the following state laws are found.

$$\underset{\sim}{\boldsymbol{\sigma}} = \rho\frac{\partial\widehat{\psi}}{\partial\underset{\sim}{\boldsymbol{\varepsilon}}^e},\ \eta = -\frac{\partial\widehat{\psi}}{\partial T},\ R = -\rho\frac{\partial\widehat{\psi}}{\partial q} \tag{35}$$

$$a = \rho\frac{\partial\widehat{\psi}}{\partial\phi_\chi}, \quad \underline{\boldsymbol{b}} = \rho\frac{\partial\widehat{\psi}}{\partial\nabla\phi_\chi} \tag{36}$$

and the residual dissipation is.

$$D^{res} = W^p + R\dot{q} - \frac{\underline{q}}{T}\cdot\nabla T \geq 0 \tag{37}$$

where W^p represents the (visco–)plastic power and R the thermodynamic force associated with the internal variable q. The existence of a convex dissipation potential, $\Omega\left(\underset{\sim}{\boldsymbol{\sigma}}, R\right)$ depending on the thermodynamic forces can then be assumed from which the evolution rules for internal variables are derived, that identically fulfill the entropy inequality, as usually done in classical continuum thermomechanics (Germain et al. 1983):

$$\dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p = \frac{\partial\Omega}{\partial\underset{\sim}{\boldsymbol{\sigma}}}, \quad \dot{q} = \frac{\partial\Omega}{\partial R} \tag{38}$$

Micromorphic Model

After presenting the general approach, we readily give the most simple example which provides a direct connection to several existing generalized continuum models. An element ϕ is selected in the STATE0 set, see Eq. (21) or among other variables present in the original model. Cases are first considered where ϕ and ϕ_χ are observer invariant quantities. The free energy density function ψ is chosen as a function of the generalized relative strain variable e defined as:

$$e = \phi - \phi_\chi \tag{39}$$

thus introducing a coupling between macro and micromorphic variables. Assuming isotropic material behavior for brevity, the additional contributions to the free energy can be taken as quadratic functions of e and $\nabla\phi_\chi$:

$$\psi\left(\underset{\sim}{\boldsymbol{\varepsilon}}, T, q, \phi_\chi, \nabla\phi\chi\right) = \psi^{(1)}\left(\underset{\sim}{\boldsymbol{\varepsilon}}, T, q\right) + \psi^{(2)}\left(e = \phi - \phi_\chi, \nabla\phi_\chi, T\right), \quad \text{with} \tag{40}$$

$$\rho\psi^{(2)}\left(e, \nabla\phi_\chi, T\right) = \frac{1}{2}H_\chi(\phi - \phi_\chi)^2 + \frac{1}{2}A\nabla\phi_\chi \cdot \nabla\phi_\chi \tag{41}$$

where H_χ and A are the additional moduli introduced by the micrmorphic model. After inserting the state laws (36)

$$a = \rho\frac{\partial\psi}{\partial\phi_\chi} = -H_\chi\left(\phi - \phi_\chi\right), \ \underline{\boldsymbol{b}} = \rho\frac{\partial\psi}{\partial\nabla\phi_\chi} = A\nabla\phi_\chi \tag{42}$$

into the additional balance equation (28), the following partial differential equation is obtained, at least for a homogeneous material under isothermal conditions:

$$\phi = \phi_\chi - \frac{A}{H_\chi}\Delta\phi_\chi \tag{43}$$

where Δ is the Laplace operator. This type of equation is encountered at several places in the mechanics of generalized continua especially in the linear micromorphic theory (Mindlin 1964; Eringen 1999; Dillard et al. 2006) and in the so–called implicit gradient theory of plasticity and damage (Peerlings et al. 2001, 2004; Engelen et al. 2003). Note however that this equation corresponds to a special quadratic potential and represents the simplest micromorphic extension of the classical theory. It involves a characteristic length scale defined by:

$$l_c^2 = \frac{A}{H_\chi} \tag{44}$$

This length is real for positive values of the ratio A/H_χ. The additional material parameters H_χ and A are assumed to be positive in this work. This does not exclude a softening material behaviour that can be induced by the proper evolution of the internal variables (including $\phi = q$ itself).

Viscous Generalized Stress and Phase Field Model

Generalized stresses can also be associated with dissipation by introducing the viscous part a^v of a:

$$\underset{\sim}{\boldsymbol{\varepsilon}} = \underset{\sim}{\boldsymbol{\varepsilon}}^e + \underset{\sim}{\boldsymbol{\varepsilon}}^p,\ a = a^e + a^\upsilon \tag{45}$$

The entropy inequality (31) now becomes

$$\left(\underset{\sim}{\boldsymbol{\sigma}} - \rho\frac{\partial\widehat{\psi}}{\partial\underset{\sim}{\boldsymbol{\varepsilon}}^e}\right) : \underset{\sim}{\dot{\boldsymbol{\varepsilon}}}^e + \rho\left(\eta + \frac{\partial\widehat{\psi}}{\partial T}\right)\dot{T} + \left(a^e - \rho\frac{\partial\widehat{\psi}}{\partial\phi_\chi}\right)\dot{\phi}_\chi + \left(\underline{\boldsymbol{b}} - \rho\frac{\partial\widehat{\psi}}{\partial\nabla\phi_\chi}\right).\nabla\dot{\phi}_\chi$$
$$+\underset{\sim}{\boldsymbol{\sigma}} : \underset{\sim}{\dot{\boldsymbol{\varepsilon}}}^p - \rho\frac{\partial\widehat{\psi}}{\partial q}\dot{q} + a^\upsilon\dot{\phi}_\chi - \frac{\underline{\boldsymbol{q}}}{T}.\nabla T \geq 0 \tag{46}$$

Assuming that no dissipation is associated with the four first terms of the previous inequality, the following state laws are found

$$\underset{\sim}{\boldsymbol{\sigma}} = \rho\frac{\partial\widehat{\psi}}{\partial\underset{\sim}{\boldsymbol{\varepsilon}}^e},\ \eta = -\frac{\partial\widehat{\psi}}{\partial T},\ R = -\rho\frac{\partial\widehat{\psi}}{\partial q} \tag{47}$$

$$a^e = \rho\frac{\partial\widehat{\psi}}{\partial\phi_\chi},\ \underline{\boldsymbol{b}} = \rho\frac{\partial\widehat{\psi}}{\partial\nabla\phi_\chi} \tag{48}$$

and the residual dissipation is

$$D^{res} = \underset{\sim}{\boldsymbol{\sigma}} : \underset{\sim}{\dot{\boldsymbol{\varepsilon}}}^p + R\dot{q} + a^\upsilon\dot{\phi}_\chi - \frac{\underline{\boldsymbol{q}}}{T}.\nabla T \geq 0 \tag{49}$$

Evolution rules for viscoplastic strain, internal variables, and the additional degrees of freedom can be derived from a dissipation potential $\Omega\left(\underset{\sim}{\boldsymbol{\sigma}}, R, a^\upsilon\right)$:

$$\underset{\sim}{\dot{\boldsymbol{\varepsilon}}}^p = \frac{\partial\Omega}{\partial\underset{\sim}{\boldsymbol{\sigma}}},\ \dot{q} = \frac{\partial\Omega}{\partial R},\ \dot{\phi}_\chi = \frac{\partial\Omega}{\partial a^\upsilon} \tag{50}$$

Convexity of the dissipation potential then ensures positivity of dissipation rate for any process.

Note that no dissipative part has been assigned to the generalized stress $\underline{\boldsymbol{b}}$ since then exploitation of second principle does not seem to be straightforward. Instead, the total gradient $\nabla\phi_\chi$ can be split into elastic and plastic parts, as it will be done in section "Elastic-Plastic Decomposition of Generalized Strains".

Phase Field Model

The dissipation potential can be decomposed into the various contributions due to all thermodynamic forces. Let us assume for instance that the contribution of the viscous generalized stress a^υ is quadratic:

$$\Omega\left(\underset{\sim}{\boldsymbol{\sigma}}, R, a^{\upsilon}\right) = \Omega_1\left(\underset{\sim}{\boldsymbol{\sigma}}, R\right) + \Omega_2\left(a^{\upsilon}\right), \; \Omega_2\left(a^{\upsilon}\right) = \frac{1}{2\beta} a^{\upsilon 2} \tag{51}$$

The use of the flow rule (50) and of the additional balance equation (28) then leads to

$$\beta\dot{\phi} = a^{\upsilon} = a - a^e = a - \rho\frac{\partial\widehat{\psi}}{\partial\phi_\chi} = \text{div}\left(\rho\frac{\partial\widehat{\psi}}{\partial\nabla\phi_\chi}\right) - \rho\frac{\partial\widehat{\psi}}{\partial\phi_\chi} \tag{52}$$

One recognizes the Landau–Ginzburg equation that arises in phase field theories. The previous derivation of Landau–Ginzburg equation is due to Gurtin (1996), see also Ammar et al. (2009). The coupling with mechanics is straightforward according to this procedure and more general dissipative mechanisms can be put forward, see Rancourt et al. (2016).

Elastic–Plastic Decomposition of Generalized Strains

Instead of the previous decomposition of generalized stresses, the introduction of additional dissipative mechanisms can rely on the split of all strain measures into elastic and plastic parts:

$$\underset{\sim}{\boldsymbol{\varepsilon}} = \underset{\sim}{\boldsymbol{\varepsilon}}^e + \underset{\sim}{\boldsymbol{\varepsilon}}^p, \; \phi_\chi = \phi_\chi^e + \phi_\chi^p, \; \underline{\boldsymbol{\kappa}} = \nabla\phi_\chi = \underline{\boldsymbol{\kappa}}^e + \underline{\boldsymbol{\kappa}}^p \tag{53}$$

The objectivity of ϕ_χ is required for such a unique decomposition to be defined. We do not require here that

$$\underline{\boldsymbol{\kappa}}^e = \nabla\phi_\chi^e, \; \underline{\boldsymbol{\kappa}}^p = \nabla\phi_\chi^p \tag{54}$$

although such a model also is possible, as illustrated by the *gradient of strain theory* put forward in Forest and Sievert (2003). The Clausius–Duhem inequality then writes

$$\begin{aligned} &\left(\underset{\sim}{\boldsymbol{\sigma}} - \rho\tfrac{\partial\widehat{\psi}}{\partial\underset{\sim}{\boldsymbol{\varepsilon}}^e}\right) : \underset{\sim}{\dot{\boldsymbol{\varepsilon}}}^e + \rho\left(\eta + \tfrac{\partial\widehat{\psi}}{\partial T}\right)\dot{T} + \left(a - \rho\tfrac{\partial\widehat{\psi}}{\partial\phi_\chi}\right)\dot{\phi}_\chi^e + \left(\underline{\boldsymbol{b}} - \rho\tfrac{\partial\widehat{\psi}}{\partial\nabla\phi_\chi}\right).\underline{\dot{\boldsymbol{\kappa}}}^e \\ &+\underset{\sim}{\boldsymbol{\sigma}} : \underset{\sim}{\dot{\boldsymbol{\varepsilon}}}^p - \rho\tfrac{\partial\widehat{\psi}}{\partial q}\dot{q} + a\dot{\phi}_\chi^p + \underline{\boldsymbol{b}}.\underline{\dot{\boldsymbol{\kappa}}}^p - \tfrac{\boldsymbol{q}}{T}.\nabla T \geq 0 \end{aligned} \tag{55}$$

Assuming that no dissipation is associated with the four first terms of the previous inequality, the following state laws are found

$$\underset{\sim}{\boldsymbol{\sigma}} = \rho\frac{\partial\widehat{\psi}}{\partial\underset{\sim}{\boldsymbol{\varepsilon}}^e}, \; \eta = -\frac{\partial\widehat{\psi}}{\partial T}, \; R = -\rho\frac{\partial\widehat{\psi}}{\partial q} \tag{56}$$

$$a = \rho \frac{\partial \widehat{\psi}}{\partial \phi_\chi^e}, \; \underline{\boldsymbol{b}} = \rho \frac{\partial \widehat{\psi}}{\partial \underline{\boldsymbol{\kappa}}^e} \tag{57}$$

and the residual dissipation is

$$D^{res} = \underset{\sim}{\boldsymbol{\sigma}} : \dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p + R\dot{q} + a\dot{\phi}_\chi^p + \underline{\boldsymbol{b}}.\dot{\underline{\boldsymbol{\kappa}}}^p - \frac{\underline{\boldsymbol{q}}}{T}.\nabla T \geq 0 \tag{58}$$

Evolution rules for viscoplastic strain, internal variables, and the additional degrees of freedom can be derived from a dissipation potential $\Omega\left(\underset{\sim}{\boldsymbol{\sigma}}, R, a, \underline{\boldsymbol{b}}\right)$:

$$\dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p = \frac{\partial \Omega}{\partial \underset{\sim}{\boldsymbol{\sigma}}}, \quad \dot{q} = \frac{\partial \Omega}{\partial R}, \quad \dot{\phi}_\chi^p = \frac{\partial \Omega}{\partial a}, \quad \dot{\underline{\boldsymbol{\kappa}}}^p = \frac{\partial \Omega}{\partial \underline{\boldsymbol{b}}} \tag{59}$$

As a result of the additional balance equation (28) combined with the previous state laws, the type of derived partial differential equation can be made more specific by adopting a quadratic free energy potential for $\underline{\boldsymbol{b}}$ (modulus A) and a quadratic dissipation potential with respect to a (parameter β). We obtain:

$$\beta\dot{\phi}_\chi = a + \beta\dot{\phi}_\chi^e = \text{div } A\underline{\boldsymbol{\kappa}} - \text{div } A\underline{\boldsymbol{\kappa}}^p + \beta\dot{\phi}_\chi^e \tag{60}$$

Decompositions of stresses and strains can also be mixed, for instance in the following way:

$$\underset{\sim}{\boldsymbol{\varepsilon}} = \underset{\sim}{\boldsymbol{\varepsilon}}^e + \underset{\sim}{\boldsymbol{\varepsilon}}^p, \quad a = a^e + a^v, \quad \underline{\boldsymbol{\kappa}} = \nabla\phi_\chi = \underline{\boldsymbol{\kappa}}^e + \underline{\boldsymbol{\kappa}}^p \tag{61}$$

based on which a constitutive theory can be built.

Application to a Continuum Damage Model for Single Crystals and Its Regularization

The micromorphic approach is now illustrated in the case of a constitutive model for damaging viscoplastic single crystals. The objective is to simulate crack initiation and propagation. The micromorphic model is used in order to obtain a regularized continuum damage formulation with a view to simulating mesh-independent crack propagation in single crystals.

Constitutive Equations

In the proposed crystal plasticity model taken from Marchal, et al. (2006a), viscoplasticity and damage are coupled by introducing an additional damage strain variable $\underset{\sim}{\boldsymbol{\varepsilon}}^d$, into the strain rate partition equation:

$$\dot{\underset{\sim}{\varepsilon}} = \dot{\underset{\sim}{\varepsilon}}^e + \dot{\underset{\sim}{\varepsilon}}^p + \dot{\underset{\sim}{\varepsilon}}^d \tag{62}$$

where $\dot{\underset{\sim}{\varepsilon}}^e$ and $\dot{\underset{\sim}{\varepsilon}}^p$ are the elastic and the plastic strain rates, respectively. The flow rule for plastic part is written at the slip system level by means of the orientation tensor $\underset{\sim}{\boldsymbol{m}}^s$:

$$\underset{\sim}{\boldsymbol{m}}^s = \frac{1}{2}\left(\underline{\boldsymbol{n}}^s \otimes \underline{\boldsymbol{l}}^s + \underline{\boldsymbol{l}}^s \otimes \underline{\boldsymbol{n}}^s\right) \tag{63}$$

where $\underline{\boldsymbol{n}}^s$ is the normal to the plane of slip system s and $\underline{\boldsymbol{l}}^s$ stands for the corresponding slip direction. Then, plastic strain rate reads:

$$\dot{\underset{\sim}{\varepsilon}}^p = \sum_{s=1}^{N_{slip}} \dot{\gamma}^s \underset{\sim}{\boldsymbol{m}}^s \tag{64}$$

where N_{slip} is the total number of slip systems. The flow rule on slip system s is a classical Norton rule with threshold:

$$\dot{\gamma}^s = \left\langle \frac{|\tau^s - x^s| - r^s}{K} \right\rangle^n sign\,(\tau^s - x^s) \tag{65}$$

where r^s and x^s are the variables for isotropic and kinematic hardening respectively and K and n are viscosity material parameters to be identified from experimental curves.

Material separation is assumed to take place w.r.t. specific crystallographic planes, like cleavage planes in single crystals. The word *cleavage* is written in a more general sense that its original meaning in physical metallurgy associated with brittle fracture of non–f.c.c. crystals. In the continuum mechanical model, *cleavage* means cracking along a specific crystallographic plane as it is often observed in low cycle fatigue of f.c.c. crystals like single crystal nickel–base superalloys. The damage strain $\dot{\underset{\sim}{\varepsilon}}^d$ is decomposed in the following crystallographic contributions:

$$\dot{\underset{\sim}{\varepsilon}}^d = \sum_{s=1}^{N_{damage}} \dot{\delta}_c^s \underline{\boldsymbol{n}}_d^s \otimes \underline{\boldsymbol{n}}_d^s + \dot{\delta}_1^s \underline{\boldsymbol{n}}_d^s \overset{sym}{\otimes} \underline{\boldsymbol{l}}_{d_1}^s + \dot{\delta}_2^s \underline{\boldsymbol{n}}_d^s \overset{sym}{\otimes} \underline{\boldsymbol{l}}_{d_2}^s \tag{66}$$

where $\dot{\delta}^s, \dot{\delta}_1^s$ and $\dot{\delta}_2^s$ are the strain rates for mode I, mode II and mode III crack growth, respectively and N_{damage}^d stands for the number of damage planes which are fixed crystallographic planes depending on the crystal structure. Cleavage damage is represented by the opening $\dot{\delta}_c^s$ of crystallographic cleavage planes with the normal vector $\underline{\boldsymbol{n}}^s$. Additional damage systems must be introduced for the in-plane accommodation along orthogonal directions $\underline{\boldsymbol{l}}_1^s$ and $\underline{\boldsymbol{l}}_2^s$, once cleavage has started (Fig. 4). Three damage criteria are associated to one cleavage and two accommodation systems:

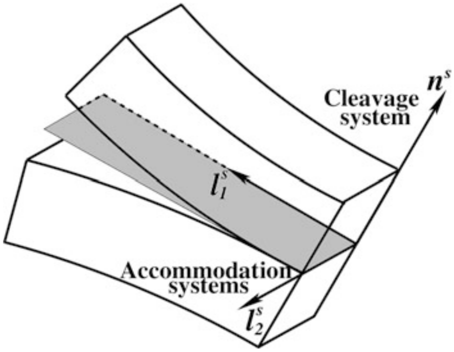

Fig. 4 Illustration of the cleavage and two accommodation systems to be associated to the crystallographic planes

$$f_c^s = \left| \underline{\boldsymbol{n}}_d^s \cdot \underset{\sim}{\boldsymbol{\sigma}} \cdot \underline{\boldsymbol{n}}_d^s \right| - Y_c^s \tag{67}$$

$$f_i^s = \left| \underline{\boldsymbol{n}}_d^s \cdot \underset{\sim}{\boldsymbol{\sigma}} \cdot \underline{\boldsymbol{l}}_{d_i}^s \right| - Y_i^s \;\; (i = 1,\;\; 2) \tag{68}$$

The critical normal stress Y^s for damage decreases as δ increases:

$$Y_c^s = Y_0^s + H\delta_c^s,\; Y_i^s = Y_0^s + H\delta_i^s \tag{69}$$

where Y_0^s is the initial damage stress (usually coupled to plasticity) and H is a negative modulus which controls material softening due to damage. Finally, evolution of damage is given by the following equations;

$$\dot{\delta}_c^s = \left\langle \frac{f_c^s}{K_d} \right\rangle^{n_d} \operatorname{sign}\left(\underline{\boldsymbol{n}}_d^s \cdot \underset{\sim}{\boldsymbol{\sigma}} \cdot \underline{\boldsymbol{n}}_d^s \right) \tag{70}$$

$$\dot{\delta}_i^s = \left\langle \frac{f_i^s}{K_d} \right\rangle^{n_d} \operatorname{sign}\left(\underline{\boldsymbol{n}}_d^s \cdot \underset{\sim}{\boldsymbol{\sigma}} \cdot \underline{\boldsymbol{l}}_{d_i}^s \right) \tag{71}$$

where K_d and n_d are material parameters.

These equations hold for all conditions except when the crack is closed $\left(\delta_c^s < 0\right)$and compressive forces are applied $\left(\underline{n}_d^s \cdot \underset{\sim}{\boldsymbol{\sigma}} \cdot \underline{n}_d^s < 0 \right)$. In this case, damage evolution stops $\left(\dot{\delta}_c^s = \dot{\delta}_i^s = 0 \right)$, corresponding to the unilateral damage conditions.

Note that the damage variables δ introduced in the model differ from the usual corresponding variables of standard continuum damage mechanics that vary from 0 to 1. In contrast, the δs are strain–like quantities that can ever increase.

Coupling between plasticity and damage is generated through initial damage stress Y_0 in (69) which is controlled by cumulative slip variable γ_{cum}:

$$\dot{\gamma}_{cum} = \sum_{s=1}^{N_{slips}} |\dot{\gamma}^s| \tag{72}$$

Then, Y_0 takes the form:

$$Y_0^s = \sigma_n^c \, e^{-d\gamma_{cum}} + \sigma_{ult} \tag{73}$$

This formulation suggests an exponential decaying regime from a preferably high initial cleavage stress value σ_n^c, to an ultimate stress, σ_{ult} which is close but not equal to zero for numerical reasons and d is a material constant.

This model, complemented by the suitable constitutive equations for viscoplastic strain, has been used for the simulation of crack growth under complex cyclic loading at high temperature (Marchal et al. 2006a). Significant mesh dependency of results was found Marchal et al. (2006b).

In the present work, the model is further developed by switching from classical to microdamage continuum in order to assess the regularization capabilities of a higher order theory. The coupling of the model with microdamage theory is achieved by introducing a cumulative damage variable calculated from the damage systems and a new threshold function $Y_0(\delta, \gamma_{cum})$:

$$\dot{\delta}_{cum} = \sum_{s=1}^{N_{planes}} \dot{\delta}^s, \quad \text{where } \dot{\delta}^s = \left|\dot{\delta}_c^s\right| + \left|\dot{\delta}_1^s\right| + \left|\dot{\delta}_2^s\right| \tag{74}$$

$$Y_0 = \sigma_n^c \, e^{-d\gamma_{cum} - H\delta_{cum}} + \sigma_{ult} \tag{75}$$

where the modulus H accounts for damage induced softening and σ_{ult} is a ultimate stress.

Microdamage Continuum

In microdamage theory, the introduced microvariable is a scalar microdamage field δ_χ:

$$DOF = \{\underline{\boldsymbol{u}}, \delta_\chi\} \; STRAIN = \left\{\underset{\sim}{\boldsymbol{\varepsilon}}, \delta_\chi, \nabla\delta_\chi\right\} \tag{76}$$

The power of internal forces is extended as

$$p^{(i)} = \underset{\sim}{\boldsymbol{\sigma}} : \underset{\sim}{\dot{\boldsymbol{\varepsilon}}} + a\dot{\delta}_\chi + \underline{\boldsymbol{b}} \cdot \nabla\dot{\delta}_\chi \tag{77}$$

where generalized stresses $a, \underline{\boldsymbol{b}}$ have been introduced. The generalized balance equations are:

$$\operatorname{div} \underset{\sim}{\boldsymbol{\sigma}} = 0, \; a = \operatorname{div} \underline{\boldsymbol{b}} \tag{78}$$

The free energy density is taken as a quadratic potential in the elastic strain, damage δ, relative damage $\delta - \delta_\chi$ and microdamage gradient $\nabla\delta_\chi$:

$$\rho\psi = \frac{1}{2}\underset{\sim}{\boldsymbol{\varepsilon}}^e : \underset{\approx}{\boldsymbol{c}} : \underset{\sim}{\boldsymbol{\varepsilon}}^e + \frac{1}{2}\sum_{s=1}^{N_{damage}} H\delta_s^2 + \frac{1}{2}H_\chi\left(\delta - \delta_\chi\right)^2 + \frac{1}{2}A\nabla\delta_\chi \cdot \nabla\delta_\chi \tag{79}$$

where H, H_χ and A are scalar material constants. The tensor of elastic moduli is called $\underset{\approx}{\boldsymbol{c}}$. Then, the elastic response of the material becomes:

$$\underset{\sim}{\boldsymbol{\sigma}} = \rho\frac{\partial\psi}{\partial\underset{\sim}{\boldsymbol{\varepsilon}}^e} = \underset{\approx}{\boldsymbol{c}} : \underset{\sim}{\boldsymbol{\varepsilon}}^e \tag{80}$$

The generalized stresses read:

$$a = \rho\frac{\partial\psi}{\partial\delta_\chi} = -H_\chi\left(\delta - \delta_\chi\right), \; \underline{\boldsymbol{b}} = A\nabla\delta_\chi \tag{81}$$

and the driving force for damage can be derived as:

$$Y^s = \rho\frac{\partial\psi}{\partial\delta^s} = H\delta^s + H_\chi\left(\delta^s - \delta_\chi\right) \tag{82}$$

The damage criterion now is:

$$f^s = \left|\underline{\boldsymbol{n}}^s \cdot \underset{\sim}{\boldsymbol{\sigma}} \cdot \underline{\boldsymbol{n}}^s\right| - Y_0 - Y^s = 0 \tag{83}$$

Substituting the linear constitutive equations for generalized stresses into the additional balance equation (78), assuming homogeneous material properties, leads to the following partial differential equation for the microdamage

$$\delta_\chi - \frac{A}{H_\chi}\boldsymbol{\Delta}\delta_\chi = \delta \tag{84}$$

where the macrodamage δ acts as a source term. Exactly this type of Helmholtz equation has been postulated in the so–called implicit gradient theory of plasticity and damage (Peerlings et al. 2001, 2004; Engelen et al. 2003; Germain et al. 2007), where the microvariables are called non local variables and where the generalized stresses a and $\underline{\boldsymbol{b}}$ are not explicitly introduced (see Forest (2009) and Dillard et al. (2006) for the analogy between this latter approach and the micromorphic theory).

The solution of the problem of failure of a 1D bar in tension/compression was treated by Aslan and Forest (2009). The characteristic size of the damage zone was

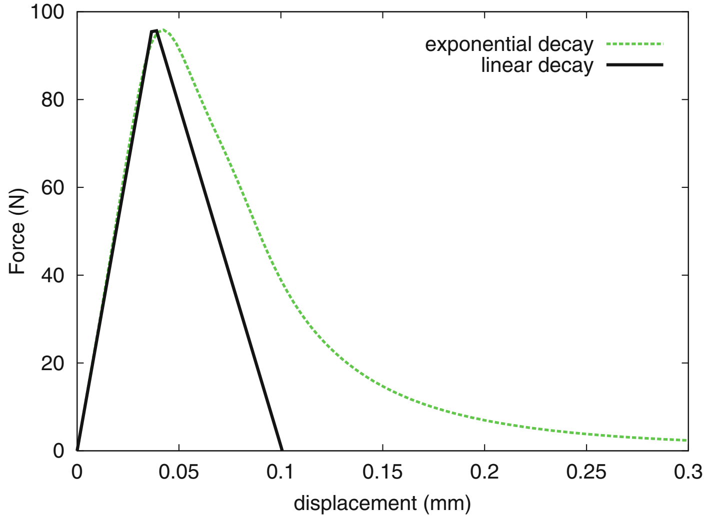

Fig. 5 Comparison between force vs. displacement diagram of a 1D softening rod for linear and exponential decay

shown to be controlled by the characteristic length corresponding to the inverse of

$$\omega = \sqrt{\left|H\ H_\chi / A\left(H + H_\chi\right)\right|} \tag{85}$$

In comparison with the standard strain gradient approaches (Peerlings et al. 2001; Germain et al. 2007), microdamage theory eliminates the final fracture problem without any modification to the damage function, since there exists no direct coupling between the force stress $\underset{\sim}{\boldsymbol{\sigma}}$ and the generalized stresses, a and $\underline{\boldsymbol{b}}$. For a better representation of a cracked element, an exponential drop is used for both the damage threshold Y_0 and the modulus A , since the element should become unable to store energy due to the generalized stresses when broken (see Fig. 5):

$$Y_0 = \sigma_n^c\, e^{-H\delta} + \sigma_{ult},\ \underline{\boldsymbol{b}} = Ae^{-H\delta}\nabla^\chi\delta \tag{86}$$

Numerical Examples

As a 2D example, a single crystal CT-like specimen under tension is analysed. The corresponding FE mesh is given in Fig. 6. Analyses are performed for two different crack widths, obtained by furnishing different material parameters which control the characteristic length (Fig. 7). The propagation of a crack, corresponding stress fields and the comparison with classical elastic solutions are given in Fig. 8. This comparison shows that the microdamage model is able to reproduce the stress

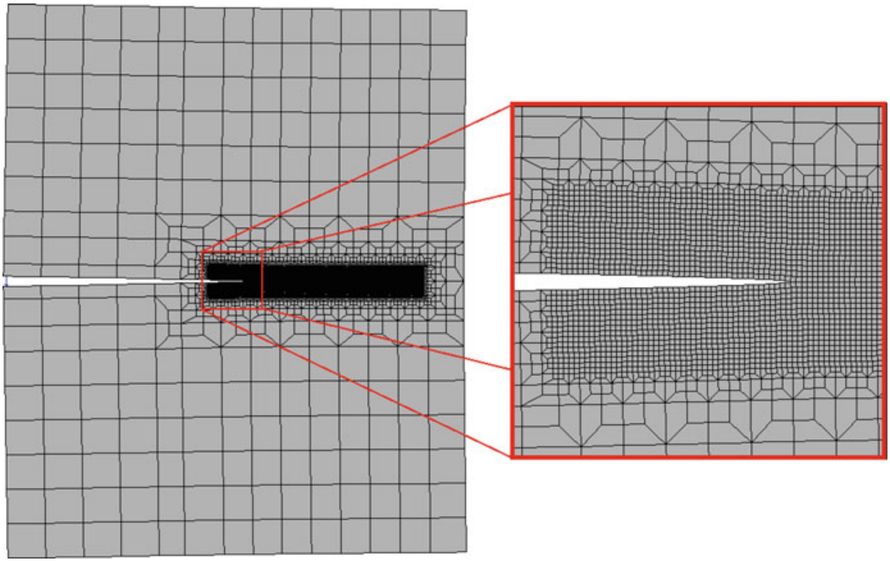

Fig. 6 FE mesh of a CT-like specimen

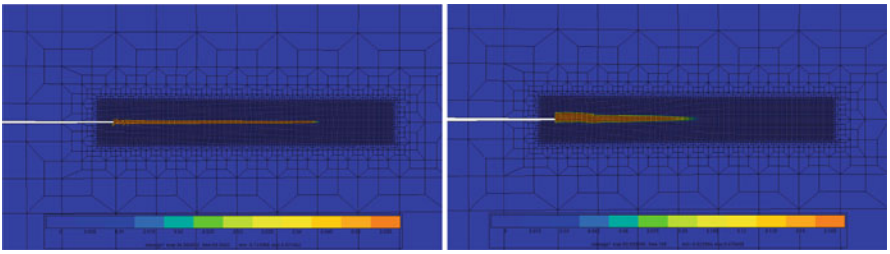

Fig. 7 Crack growth in a 2D single crystal CT-like specimen with a single cleavage plane aligned through the horizontal axis under vertical tension. Field variable δ (Left) $A = 100$ MPa.mm^2, $H = -20000$ MPa, $H_\chi = 30{,}000$ MPa, (Right) $A = 150$ MPa.mm^2, $H =$ -10000 MPa, $H_\chi = 30000$ MPa

concentration at the crack tip except very close to the crack tip where finite stress values are predicted.

The Micromorphic Approach at Finite Deformations

Extensions of the previous systematic micromorphic approach are presented for finite deformations along the lines of Forest (2016). They generalize some aspects of the finite deformation micromorphic theory presented in section "The Micromorphic Theory After Eringen and Mindlin and Its Extension to Plasticity".

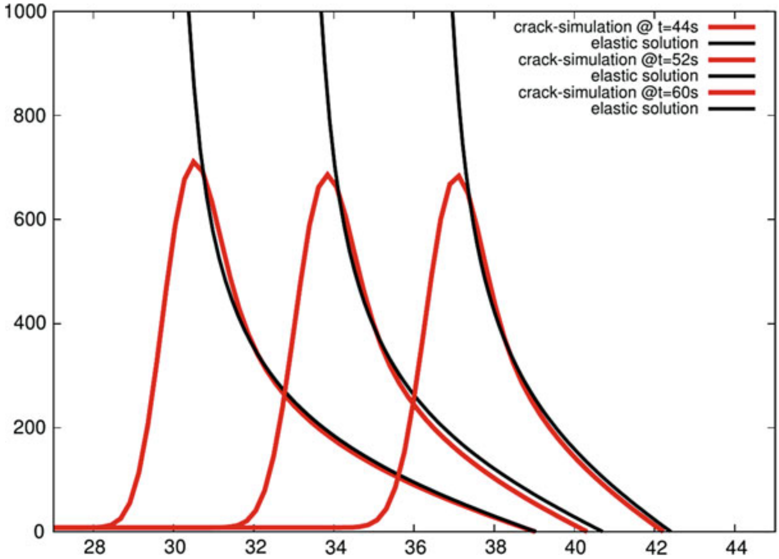

Fig. 8 Mode I stress profile (in MPa) vs. position along the ligament (in mm) at three distinct crack propagation steps: micromorphic solution compared to the linear elastic one

Micromorphic *and Gradient Hyperelasticity*

Nonlinearity arises not only from nonlinear material response but also from the consideration of finite deformations. The impact of finite strains on regularisation operators is largely unexplored. It is first illustrated in the pure hyperelastic case, leaving aside for a moment the inclusion of plastic effects. It is recalled that the Lagrangian coordinates of the material points are denoted by $\underline{X}$ on the reference configuration $_0$, whereas their positions in the current configuration are called $\underline{\boldsymbol{x}}$. The gradient operators with respect to Lagrangian (reference) and Eulerian (current) coordinates are denoted by ∇_X and ∇_x, respectively. The deformation gradient is $\underset{\sim}{\boldsymbol{F}} = \underset{\sim}{\boldsymbol{1}} + \underline{\boldsymbol{u}} \otimes \nabla_X$ where the displacement vector is the function $\underline{\boldsymbol{u}}\,(\underline{X}, t)$. The initial and current mass density functions are $\rho_0\,(\underline{X}, t)$ and $p\,(\underline{\boldsymbol{x}}, t)$, respectively.

It is appropriate here to recall the Eulerian balance equation and boundary conditions in the case of a scalar micromorphic degree of freedom, in the absence of body forces:

$$\operatorname{div}\,\underset{\sim}{\boldsymbol{\sigma}} = 0,\; a = \operatorname{div}\,\underline{\boldsymbol{b}},\; \forall \underline{\boldsymbol{x}} \in \Omega,\; \underline{\boldsymbol{t}} = \underset{\sim}{\boldsymbol{\sigma}} \cdot \underset{\sim}{\boldsymbol{n}},\; a_c = \underline{\boldsymbol{b}} \cdot \underline{\boldsymbol{n}},\; \forall \underline{\boldsymbol{x}} \in \partial\Omega \tag{87}$$

Finite Microstrain Tensor Model

The microstrain model was introduced in Table 1 as a reduced micromorphic model where Eringen's microdeformation tensor is taken as symmetric, as proposed in Forest and Sievert (2006). The additional degrees of freedom are the six components of a microstrain tensor $\underset{\sim}{\boldsymbol{\chi}}$, a second order symmetric tensor associated with the right Cauchy-Green strain measure of the micromorphic deformation. Accordingly, the microstrain is regarded here as a Lagrangian variable. The Lagrangian power density of internal forces takes the form:

$$p_0^{(i)} = \frac{1}{2}\underset{\sim}{\boldsymbol{\Pi}} : \dot{\underset{\sim}{\boldsymbol{C}}} + \underset{\sim}{\boldsymbol{a}}_0 : \dot{\underset{\sim}{\boldsymbol{\chi}}} + \underset{\approx}{\boldsymbol{b}}_0 \vdots \left(\dot{\underset{\sim}{\boldsymbol{\chi}}} \otimes \nabla_X\right) = J p^{(i)} \tag{88}$$

$p^{(i)}$ being the Eulerian internal power density and the Jacobian $J = \det \underset{\sim}{\boldsymbol{F}}$. The right Cauchy-Green tensor is $\underset{\sim}{\boldsymbol{C}} = \underset{\sim}{\boldsymbol{F}}^T \cdot \underset{\sim}{\boldsymbol{F}}$ and $\underset{\sim}{\boldsymbol{\Pi}}$ is the Piola stress tensor. The method of virtual power can be used to show that the generalized Lagrangian stress tensors, a second and a third rank tensor, fulfill the following balance equation:

$$\text{Div } \underset{\approx}{\boldsymbol{b}}_0 = \underset{\sim}{\boldsymbol{a}}_0 \tag{89}$$

in addition to the balance of momentum equation

$$\text{Div } \underset{\sim}{\boldsymbol{F}} \cdot \underset{\sim}{\boldsymbol{\Pi}} = 0 \tag{90}$$

in the absence of body and inertial forces, the divergence operator Div being computed with respect to Lagrangian coordinates. The corresponding Eulerian forms of the balance equations are

$$\text{div } \underset{\sim}{\boldsymbol{\sigma}} = 0, \quad \text{div } \underset{\approx}{\boldsymbol{b}} = \underset{\sim}{\boldsymbol{a}} \tag{91}$$

with the Neumann boundary conditions for tractions and double tractions: $\underline{\boldsymbol{t}} = \underset{\sim}{\boldsymbol{\sigma}} \cdot \underline{\boldsymbol{n}}, \underset{\sim}{\boldsymbol{b}} = \underset{\approx}{\boldsymbol{b}} \cdot \underline{\boldsymbol{n}}$, involving the normal vector of the current surface element. The relations between the Lagrangian and Eulerian generalized stress tensors

$$\underset{\sim}{\boldsymbol{a}}_0 = J\underset{\sim}{\boldsymbol{a}}, \quad \underset{\approx}{\boldsymbol{b}}_0 = J\underset{\approx}{\boldsymbol{b}} \cdot \underset{\sim}{\boldsymbol{F}}^{-T} = J\underset{\sim}{\boldsymbol{F}}^{-1} \cdot \underline{\boldsymbol{b}} \tag{92}$$

The hyperelastic free energy density function is $\psi_0\left(\underset{\sim}{\boldsymbol{C}}, \underset{\sim}{\boldsymbol{\chi}}, \underset{\sim}{\boldsymbol{\chi}} \otimes \nabla_X\right)$ and the stress–strain relations read:

$$\underset{\sim}{\boldsymbol{\Pi}} = 2\rho_0 \frac{\partial \psi_0}{\partial \underset{\sim}{\boldsymbol{C}}}, \quad \underset{\sim}{\boldsymbol{a}}_0 = \rho_0 \frac{\partial \psi_0}{\partial \underset{\sim}{\boldsymbol{\chi}}}, \quad \underset{\approx}{\boldsymbol{b}}_0 = \rho_0 \frac{\partial \psi_0}{\partial \underset{\sim}{\boldsymbol{\chi}} \otimes \nabla_X} \tag{93}$$

The relative strain tensor $\underset{\sim}{\boldsymbol{e}} = \underset{\sim}{\boldsymbol{C}} - \underset{\sim}{\boldsymbol{\chi}}$ was defined in Forest and Sievert (2006) and measures the difference between macro and micro–strain. As an example, the following potential is proposed:

$$\rho_0 \psi_0 = \rho_0 \psi_{ref} \left(\underset{\sim}{\boldsymbol{C}} \right) + \frac{1}{2} H_\chi \left(\underset{\sim}{\boldsymbol{C}} - \underset{\sim}{\boldsymbol{\chi}} \right)^2 + \frac{1}{2} \underset{\sim}{\boldsymbol{\chi}} \otimes \nabla_X \vdots \underset{\approx}{\underset{\sim}{A}} \vdots \underset{\sim}{\boldsymbol{\chi}} \otimes \nabla_X \tag{94}$$

where a penalty modulus H_χ is introduced and where ψ_{ref} is a standard hyperelastic strain energy potential (neo–Hookean, etc.). The higher order term involves a sixth–rank tensor of elasticity moduli which is symmetric and assumed definite positive, see Auffray et al. (2013). The stress–strain relations (93) become:

$$\underset{\sim}{\boldsymbol{\Pi}} = 2\rho_0 \frac{\partial \psi_{ref}}{\partial \underset{\sim}{\boldsymbol{C}}} + H_\chi \left(\underset{\sim}{\boldsymbol{C}} - \underset{\sim}{\boldsymbol{\chi}} \right), \underset{\sim}{\boldsymbol{a}}_0 = -H_\chi \left(\underset{\sim}{\boldsymbol{C}} - \underset{\sim}{\boldsymbol{\chi}} \right), \ \underset{\approx}{\boldsymbol{b}}_0 = \underset{\approx}{\underset{\sim}{A}} \vdots \left(\underset{\sim}{\boldsymbol{\chi}} \otimes \nabla_X \right) \tag{95}$$

Note that the classical hyperelastic relation is complemented by a coupling term involving the microstrain tensor. Taking the balance equation (90) into account, the set of p.d.e. for the microstrain components is found to be:

$$\underset{\sim}{\boldsymbol{C}} = \underset{\sim}{\boldsymbol{\chi}} - \frac{1}{H_\chi} \mathrm{Div} \left(\underset{\approx}{\underset{\sim}{A}} \vdots \left(\underset{\sim}{\boldsymbol{\chi}} \otimes \nabla_X \right) \right) \tag{96}$$

The associated regularisation operator is now given in the simplified case where the sixth rank tensor of higher order moduli is assumed to be the identity multiplied by the single modulus A:

$$\mathrm{Op} = \underset{\sim}{1} - \frac{A}{H_\chi} \Delta_X \tag{97}$$

It involves the Lagrangian Laplace operator $\Delta_X(\bullet) = (\bullet)_{,KK}$ in a Cartesian frame where capital indices refer to Lagrange coordinates and the comma to partial derivation. It is linear w.r.t. Lagrangian coordinates but the full problem is of course highly non linear. The associated Eulerian partial differential operator is nonlinear in the form: $\chi_{IJ,KK} = \chi_{IJ,Kl} F_{kK} F_{lK}$, where small index letters refer to the current coordinate system.

An Eulerian formulation of the proposed constitutive equations is possible. It will be illustrated in the next section in the case of a scalar microstrain variable for the sake of brevity.

Equivalent Microstrain Model at Finite Deformation

According to the micromorphic approach, it is not necessary to consider the full microstrain tensor. Instead, the set of degrees of freedom of a reduced micromorphic model can be the usual displacement vector and a scalar microstrain variable $\chi(\underline{\boldsymbol{X}}, t)$. The latter variable is assumed to be a Lagrangian quantity, invariant w.r.t. change of observer. The free energy density is a function of the following argument $\psi_0 \left(\underset{\sim}{\boldsymbol{C}}, \chi, \nabla_x \chi \right)$. The corresponding hyperelastic state laws fulfilling the Clausius–Duhem inequality take the form:

$$\underset{\sim}{\mathbf{\Pi}} = 2\rho_0 \frac{\partial \psi_0}{\partial \underset{\sim}{\boldsymbol{C}}}, \quad a_0 = \rho_0 \frac{\partial \psi_0}{\partial \chi}, \quad \underline{\boldsymbol{b}}_0 = \rho_0 \frac{\partial \psi_0}{\partial \nabla_x \chi} \tag{98}$$

As an example, the following Lagrangian potential is proposed:

$$\rho_0 \psi_0 = \rho_0 \psi_{ref} \left(\underset{\sim}{\boldsymbol{C}} \right) + \frac{1}{2} H_\chi \left(C_{eq} - \chi \right)^2 + \frac{1}{2} \nabla_x \chi \cdot \underset{\sim}{\boldsymbol{A}} \cdot \nabla_X \chi \tag{99}$$

The microstrain is compared to some equivalent strain measure, C_{eq}, function of the invariants of $\underset{\sim}{\boldsymbol{C}}$. The stress–strain relations (98) become:

$$\underset{\sim}{\mathbf{\Pi}} = 2\rho_0 \frac{\partial \psi_{ref}}{\partial \underset{\sim}{\boldsymbol{C}}} + H_\chi \left(C_{eq} - \chi \right) \frac{\partial C_{eq}}{\partial \underset{\sim}{\boldsymbol{C}}}, \quad a_0 = -H_\chi \left(C_{eq} - \chi \right), \quad \underline{\boldsymbol{b}}_0 = \underset{\sim}{A} \cdot \nabla_X \chi \tag{100}$$

Taking the balance equation $a_0 = \nabla_X \cdot \underline{\boldsymbol{b}}_0$ into account, the p.d.e. governing χ is:

$$C_{eq} = \chi - \frac{1}{H_\chi} \mathrm{Div} \left(\underset{\sim}{\boldsymbol{A}} \cdot \nabla_X \chi \right) \tag{101}$$

Let us mention the corresponding isotropic form, when $\underset{\sim}{\boldsymbol{A}} = A \underset{\sim}{1}$:

$$C_{eq} = \chi - \frac{A}{H_\chi} \Delta_X \chi \tag{102}$$

where Δ_X is the Laplace operator with respect to Lagrangian coordinates. An example of equivalent strain measure which the microstrain is compared with is

$$C_{eq} = \sqrt{\underset{\sim}{\boldsymbol{C}} : \underset{\sim}{\boldsymbol{C}}}, \quad \frac{\partial C_{eq}}{\partial \underset{\sim}{\boldsymbol{C}}} = \underset{\sim}{\boldsymbol{C}} / C_{eq} \tag{103}$$

The penalty modulus H_χ constrains the microstrain degree of freedom to remain close to the equivalent strain measure C_{eq}. In the limit of large values of H_χ, the generalized stress a_0 can be regarded as a Lagrange multiplier enforcing the constraint. As a result, the microstrain gradient becomes equal to the equivalent strain gradient and the micromorphic model degenerates into a gradient model, see Forest and Sab (2017) for a detailed description of internal constraints in micromorphic theories.

The formulation of a constitutive law based on Eulerian strain measures, $\underset{\sim}{\boldsymbol{B}} = \underset{\sim}{\boldsymbol{F}} \cdot \underset{\sim}{\boldsymbol{F}}^T$ and $\nabla_x \chi$, is now envisaged. It relies on the choice of a free energy potential $\rho\psi \left(\underset{\sim}{\boldsymbol{B}}, \chi, \nabla_x \chi \right)$. Galilean invariance of the constitutive law requires that this function fulfills the following conditions:

$$\psi\left(\underset{\sim}{\boldsymbol{Q}}\cdot\underset{\sim}{\boldsymbol{B}}\cdot\underset{\sim}{\boldsymbol{Q}}^T,\chi,\underset{\sim}{\boldsymbol{Q}}\cdot\nabla_x\chi\right)=\psi\left(\underset{\sim}{\boldsymbol{B}},\chi,\cdot\nabla_x\chi\right) \tag{104}$$

for all constant orthogonal transformations $\underset{\sim}{\boldsymbol{Q}}$. This amounts to stating isotropy of the function ψ. Representation theorems are available for such functions, $\psi\left(\underset{\sim}{\boldsymbol{B}},\chi,\nabla_x\chi\right)=\psi\left(B_1,B_2,B_3,\chi,\|\nabla_x\ \chi\|\right)$, where the B_i are the eigenvalues of $\underset{\sim}{\boldsymbol{B}}$. The Cauchy stress tensor, $\underset{\sim}{\boldsymbol{\sigma}}$, is known then to commute with $\underset{\sim}{\boldsymbol{B}}$ such that:

$$\underset{\sim}{\boldsymbol{\sigma}}:\underset{\sim}{\boldsymbol{D}}=\left(\underset{\sim}{\boldsymbol{\sigma}}\cdot\underset{\sim}{\boldsymbol{B}}^{-1}\right):\left(\underset{\sim}{\dot{\boldsymbol{F}}}\cdot\underset{\sim}{\boldsymbol{F}}^T\right)=\left(\underset{\sim}{\boldsymbol{B}}^{-1}\cdot\underset{\sim}{\boldsymbol{\sigma}}\right):\left(\underset{\sim}{\boldsymbol{F}}\cdot\underset{\sim}{\dot{\boldsymbol{F}}}^T\right)=\frac{1}{2}\left(\underset{\sim}{\boldsymbol{B}}^{-1}\cdot\underset{\sim}{\boldsymbol{\sigma}}\right):\underset{\sim}{\dot{\boldsymbol{B}}} \tag{105}$$

where the strain rate tensor, $\underset{\sim}{\boldsymbol{D}}$, is the symmetric part of the velocity gradient,$\underset{\sim}{\dot{\boldsymbol{F}}}\cdot\underset{\sim}{\boldsymbol{F}}^{-1}$. The hyperelastic state laws then take the form:

$$\underset{\sim}{\boldsymbol{\sigma}}=2\underset{\sim}{\boldsymbol{B}}\cdot\rho\frac{\partial\psi}{\partial\underset{\sim}{\boldsymbol{B}}},\quad a=\rho\frac{\partial\psi}{\partial\chi},\quad \underline{\boldsymbol{b}}=\rho\frac{\partial\psi}{\partial\nabla_X\chi} \tag{106}$$

As an example, the following Eulerian potential is proposed:

$$\rho\psi=\rho\psi_{ref}\left(\underset{\sim}{\boldsymbol{B}}\right)+\frac{1}{2}H_\chi\left(B_{eq}-\chi\right)^2+\frac{1}{2}A\nabla_x\chi:\nabla_x\chi, \tag{107}$$

where ψ_{ref} refers to a standard isotropic elasticity potential from the classical finite elasticity theory. Note that $B_{eq}=C_{eq}$ since $\underset{\sim}{\boldsymbol{B}}$ and $\underset{\sim}{\boldsymbol{C}}$ share the same eigenvalues. The state laws (106) become:

$$\underset{\sim}{\boldsymbol{\sigma}}=2\rho\frac{\partial\psi_{ref}}{\partial\underset{\sim}{\boldsymbol{B}}}+H_\chi\left(B_{eq}-\chi\right)\frac{\partial B_{eq}}{\partial\underset{\sim}{\boldsymbol{B}}},\qquad a=-H_\chi\left(B_{eq}-\chi\right),\quad \underline{\boldsymbol{b}}=A\nabla_x\chi \tag{108}$$

The Eulerian regularisation operator follows from (91):

$$B_{eq}=\chi-\frac{A}{H_\chi}\Delta_x\chi \tag{109}$$

where Δ_x is the Laplace operator with respect to the Eulerian coordinates.

It is essential to notice that the isotropic regularisation operators (102) and (109) are distinct. For, if the Lagrangian higher order elastic law is linear with respect to the constitutive quantities involved, the deduced Eulerian law is NOT linear:

$$\underline{\boldsymbol{b}}_0=A\nabla_X\chi\quad\Rightarrow\quad\underline{\boldsymbol{b}}=J^{-1}\underset{\sim}{\boldsymbol{F}}\cdot\underline{\boldsymbol{b}}_0=AJ^{-1}\underset{\sim}{\boldsymbol{B}}\cdot\nabla_x\chi \tag{110}$$

so that the Eulerian regularisation operator will not be linear, i.e. different from (109).

Finite Deformation Micromorphic Elastoviscoplasticity Using an Additive Decomposition of a Lagrangian Strain

The most straightforward extension of the previous framework to viscoplasticity is to introduce a finite plastic strain measure in the decomposition of a Lagrangian total strain tensor. Such Lagrangian formulations of elastoviscoplasticity involve the additive decomposition of some Lagrangian total strain measure into elastic and viscoplastic parts:

$$\underset{\sim}{\boldsymbol{E}}_h = h\left(\underset{\sim}{\boldsymbol{C}}\right) = \underset{\sim}{\boldsymbol{E}}_h^e + \underset{\sim}{\boldsymbol{E}}_h^p \tag{111}$$

Many choices are possible for the invertible function h with restrictions ensuring that $\underset{\sim}{\boldsymbol{E}}_h$ is a strain measure (symmetric, vanishing for rigid body motions, differentiable at 0 so that the tangent is the usual small strain tensor $\underset{\sim}{\boldsymbol{\varepsilon}}$, used before, see Besson et al. 2009). Seth-Hill's strain measures are obtained for power law functions such that: $\underset{\sim}{\boldsymbol{E}}_m = \frac{1}{m}\left(\underset{\sim}{\boldsymbol{C}}^{\frac{m}{2}} - \underset{\sim}{\boldsymbol{1}}\right)$, for $m > 0$, $\underset{\sim}{\boldsymbol{E}}_0 = \log \underset{\sim}{\boldsymbol{C}}^{\frac{1}{2}}$, the latter being the Lagrangian logarithmic strain. The case $m = 2$ corresponds to the Green–Lagrange strain measure for which this finite deformation theory was first formulated by Green and Naghdi (see Lee and Germain 1972; Bertram 2012) for the pros and the cons of various such formulations. This Lagrangian formulation is preferable to Eulerian ones based on corresponding Eulerian strain measures in order not to limit the approach to isotropic material behaviour (Simo and Miehe 1992). The additive decomposition of the Lagrangian logarithmic strain is put forward in the computational plasticity strategies developed in Miehe et al. (2002) and Helfer (2015). However, there is generally no physical motivation for the selection of one or another Lagrangian strain measure within this framework. In addition, this approach favours one particular reference configuration for which the corresponding strain is decomposed into elastic and plastic parts, again without clear physical argument. Changes of reference configuration lead to complex hardly interpretable transformation rules for the plastic strain variables, see Shotov and Ihlemann (2014) for a comparison of finite deformation constitutive laws with respect to this issue. Note also that limitations arise from using a symmetric plastic strain variable $\underset{\sim}{\boldsymbol{E}}_h^p$ especially when plastic spin relations are needed for anisotropic materials (Bertram 2012). This framework was applied to micromorphic and gradient plasticity and damage theories in Geers (2004), Horak and Jirasek (2013), and Miehe (2014).

A Lagrangian conjugate stress tensor $\underset{\sim}{\Pi}_h$ is defined for each strain measure $\underset{\sim}{\boldsymbol{E}}_h$ such that

$$\frac{1}{2}\underset{\sim}{\boldsymbol{\Pi}} : \dot{\underset{\sim}{\boldsymbol{C}}} = \underset{\sim}{\boldsymbol{\Pi}}_h : \dot{\underset{\sim}{\boldsymbol{E}}}_h \text{ with } \underset{\sim}{\boldsymbol{\Pi}}_h = \underset{\sim}{\boldsymbol{\Pi}} : \left(\frac{\partial h}{\partial \underset{\sim}{\boldsymbol{C}}}\right)^{-1} \tag{112}$$

The power density of internal forces is:

$$p_0^{(i)} = \underset{\sim}{\boldsymbol{\Pi}}_h : \dot{\underset{\sim}{\boldsymbol{E}}}_h + a_0 \dot{\chi} + \underline{\boldsymbol{b}}_0 \cdot \nabla_X \dot{\chi}$$

and the free energy density function has the following arguments: $\psi_0 \left(\underset{\sim}{\boldsymbol{E}}^e_h, q, \chi, \underline{\boldsymbol{K}} := \nabla X \chi\right)$. The dissipation inequality then reads:

$$\begin{aligned} &\left(\underset{\sim}{\boldsymbol{\Pi}}_h - \rho_0 \frac{\partial \psi_0}{\partial \underset{\sim}{\boldsymbol{E}}^e_h}\right) : \dot{\underset{\sim}{\boldsymbol{E}}}^e_h + \left(a - \rho_0 \frac{\partial \psi_0}{\partial \chi}\right) \dot{\chi} + \left(\underline{\boldsymbol{b}}_0 - \rho_0 \frac{\partial \psi_0}{\partial \nabla_X \chi}\right) \cdot \dot{\underline{\boldsymbol{K}}} \\ &+ \underset{\sim}{\boldsymbol{\Pi}}_h : \dot{\underset{\sim}{\boldsymbol{E}}}^p_h - \rho_0 \frac{\partial \psi_0}{\partial q} \dot{q} \geq 0 \end{aligned} \tag{113}$$

from which the following state laws are selected:

$$\underset{\sim}{\boldsymbol{\Pi}}_h = \rho_0 \frac{\partial \psi_0}{\partial \underset{\sim}{\boldsymbol{E}}^e_h}, \quad a = \rho_0 \frac{\partial \psi_0}{\partial \text{ffl}}, \underline{\boldsymbol{b}}_0 = \rho_0 \frac{\partial \psi_0}{\partial \nabla_X \chi}, \quad R = \rho_0 \frac{\partial \psi_0}{\partial q} \tag{114}$$

The flow and hardening rules can be determined from the suitable choice of dissipation potential $\Omega\left(\underset{\sim}{\boldsymbol{\Pi}}_h, R\right)$:

$$\dot{\underset{\sim}{\boldsymbol{E}}}^p_h = \frac{\partial \Omega}{\partial \underset{\sim}{\boldsymbol{\Pi}}_h}, \quad \dot{q} = -\frac{\partial \Omega}{\partial R} \tag{115}$$

The existence of such a dissipation potential is not necessary but assumed in the whole chapter for convenience. Alternative methods of exploitation of the dissipation principle exist for micromorphic continua, for instance based on the extended Liu procedure (Ván et al. 2014; Berezovski et al. 2014).

Two straightforward extensions of the micromorphic approach to finite strain viscoplasticity based on an additive decomposition of a Lagrangian strain measure are presented:

$$\rho_0 \psi_0 = \frac{1}{2} \underset{\sim}{\boldsymbol{E}}^e_h : \underset{\approx}{\boldsymbol{c}} : \underset{\sim}{\boldsymbol{E}}^e_h + \rho_0 \psi_q(q) + H_\chi \left(E_{heq} - \chi\right)^2 + \rho_0 \psi_\nabla \left(\underline{\boldsymbol{K}}\right) \tag{116}$$

where E_{heq} is an equivalent total strain measure, or, alternatively,

$$\rho_0\psi_0 = \frac{1}{2}\underset{\sim}{\boldsymbol{E}}^e_h : \underset{\approx}{\boldsymbol{c}} : \underset{\sim}{\boldsymbol{E}}^e_h + \rho_0\psi_q(q) + H_\chi(p-\chi)^2 + \rho_0\psi_\nabla\left(\underline{\boldsymbol{K}}\right) \tag{117}$$

where $\dot{p} = \sqrt{2/3\dot{\underset{\sim}{\boldsymbol{E}}}^p_h : \dot{\underset{\sim}{\boldsymbol{E}}}^p_h}$ is the cumulative plastic strain in the present context. These choices respectively provide the following regularisation partial differential equations:

$$E_{heq} = \chi - \mathrm{Div}\rho_0\frac{\partial\psi_0}{\partial\underline{\boldsymbol{K}}}, \text{ or } \quad p = \chi - \mathrm{Div}\ \rho_0\frac{\partial\psi_0}{\partial\underline{\boldsymbol{K}}} \tag{118}$$

If, $\underline{\boldsymbol{b}}_0 = A\underline{\boldsymbol{K}}$, then the regularisation operator involves the Lagrangian Laplace operator Δ_X in the same way as in Eq. (97).

Finite Deformation Micromorphic Viscoplasticity Using Local Objective Frames

An alternative and frequently used method to formulate anisotropic elastoviscoplastic constitutive equations at finite deformations that identically fulfill the condition of Euclidean invariance (also called material frame indifference, see Besson et al. 2009), is to resort to local objective rotating frames, as initially proposed by Dogui and Sidoroff (1985) and Sidoroff and Dogui (2001). A local objective rotating frame is defined by the rotation field $\underset{\sim}{\boldsymbol{Q}}\ (\underline{x}, t)$, objective w.r.t. to further change of observer, and taking different values at different material points and different times:

$$\underline{\boldsymbol{x}}^\dagger = \underset{\sim}{\boldsymbol{Q}}^T\left(\underline{\boldsymbol{x}}, t\right)\cdot\underline{\boldsymbol{x}} \tag{119}$$

It is based on the idea that there exists for each material point a privileged observer w.r.t. which the constitutive law takes a simple form. The method is described in details in Besson et al. (2009) and is used in many commercial finite element codes with the standard choices: corotational frame, such that $\underset{\sim}{\boldsymbol{W}}^\dagger = \dot{\underset{\sim}{\boldsymbol{Q}}}\cdot\underset{\sim}{\boldsymbol{Q}}^T = \underset{\sim}{\boldsymbol{W}}, \underset{\sim}{\boldsymbol{W}}$ being the skew–symmetric part of the velocity gradient, and polar frame, such that $\underset{\sim}{\boldsymbol{Q}}\left(\underline{\boldsymbol{x}}, t\right) = \underset{\sim}{\boldsymbol{R}}\left(\underline{\boldsymbol{x}}, t\right), \underset{\sim}{\boldsymbol{R}}$ being the rotation part in the polar decomposition of the deformation gradient $\underset{\sim}{\boldsymbol{F}}$. The main drawback of this method is the absence of thermodynamic background since, depending on specific constitutive choices within this framework, a free energy function of the strain may not exist.

This method is now applied to a micromorphic model including a scalar additional d.o.f. χ. Extension to higher order tensor–valued additional degrees of freedom is straightforward. The field equations are still given by (87). The stresses w.r.t. the local objective frames are:

$$\underset{\sim}{\boldsymbol{\sigma}}^{\dagger} := \underset{\sim}{\boldsymbol{Q}}^{T} \cdot \underset{\sim}{\boldsymbol{\sigma}} \cdot \underset{\sim}{\boldsymbol{Q}}, \;\; a^{\dagger} = a, \;\; \underline{\boldsymbol{b}}^{\dagger} = \underset{\sim}{\boldsymbol{Q}}^{T} \cdot \underline{\boldsymbol{b}} \tag{120}$$

Time–derivation of these relations shows that the rotated stress derivatives are given by

$$\underset{\sim}{\boldsymbol{Q}} \cdot \dot{\underset{\sim}{\boldsymbol{\sigma}}}^{\dagger} \cdot \underset{\sim}{\boldsymbol{Q}}^{T} = \dot{\underset{\sim}{\boldsymbol{\sigma}}} + \underset{\sim}{\boldsymbol{\sigma}} \cdot \underset{\sim}{\boldsymbol{W}}^{\dagger} - \underset{\sim}{\boldsymbol{W}}^{\dagger} \cdot \underset{\sim}{\boldsymbol{\sigma}}, \quad \underset{\sim}{\boldsymbol{Q}} \cdot \dot{\underline{\boldsymbol{b}}}^{\dagger} = \dot{\underline{\boldsymbol{b}}} - \underset{\sim}{\boldsymbol{W}}^{\dagger} \cdot \underline{\boldsymbol{b}} \tag{121}$$

i.e. objective derivatives of the corresponding Eulerian stress tensors. If the corotational frame is used, the corresponding time derivative is the Jaumann rate, whereas it is the Green–Naghdi stress rate when the polar rotation is used. The same procedure is applied to the strain rates:

$$\underset{\sim}{\boldsymbol{D}}^{\dagger} := \underset{\sim}{\boldsymbol{Q}}^{T} \cdot \underset{\sim}{\boldsymbol{D}} \cdot \underset{\sim}{\boldsymbol{Q}}, \quad \dot{\underline{\boldsymbol{k}}}^{\dagger} = \underset{\sim}{\boldsymbol{Q}}^{T} \cdot \nabla_{x} \dot{\chi} \tag{122}$$

The time integration of the second equation in the rotating frame provides the variable $\underline{\boldsymbol{k}}^{\dagger}$. It must be underlined that $\underline{\boldsymbol{k}}^{\dagger}$ is NOT equal to $\underset{\sim}{\boldsymbol{Q}}^{T} \cdot \nabla_{x} \chi$. It is NOT the exact material time derivative of a constitutive variable, in general. The standard procedure then consists in postulating an additive decomposition of the rotated strain rate into elastic and plastic parts as

$$\underset{\sim}{\boldsymbol{D}}^{\dagger} = \dot{\underset{\sim}{\boldsymbol{e}}}^{e} + \dot{\underset{\sim}{\boldsymbol{e}}}^{p} \tag{123}$$

where the elastic and plastic strain $\underset{\sim}{e}^{e}$ and $\underset{\sim}{e}^{p}$ are solely defined in the rotated frame. Anisotropic elastic laws are assumed to take the form:

$$\underset{\sim}{\boldsymbol{\sigma}}^{\dagger} = \underset{\approx}{\boldsymbol{c}} : \underset{\sim}{\boldsymbol{e}}^{e}, \quad a = -H_{\chi}\left(p - \chi\right), \quad \underline{\boldsymbol{b}}^{\dagger} = \underset{\sim}{\boldsymbol{A}} \cdot \underline{\boldsymbol{b}}^{\dagger} \tag{124}$$

Time–derivation of these equations and consideration of Eq. (120) show that the elasticity laws are in fact hypoelastic and that, generally, there does not exist a free energy density function from which they can be derived (Toll 2011).

The yield function and the flow rule are formulated within the rotated frame:

$$f\left(\underset{\sim}{\boldsymbol{\sigma}}^{\dagger}, R\right) = \sigma^{\dagger}_{eq} - R, \quad \dot{\underset{\sim}{\boldsymbol{e}}}^{p} = \dot{p} \frac{\partial f}{\partial \underset{\sim}{\boldsymbol{\sigma}}^{\dagger}} \tag{125}$$

where normality is assumed for convenience and the viscoplastic multiplier $\dot{p}$ is given by some viscoplastic law. The evolution of internal variables is of the form $\dot{q} = H\left(q, \dot{\underset{\sim}{\boldsymbol{e}}}^{p}\right)$ for suitable functions H. The yield radius is chosen as the following expression inspired by the previous thermodynamically based models: $R = R_{ref}(p) - a = R_{ref} - \text{div}\ \underline{\boldsymbol{b}}$. This extension of the micromorphic approach to finite deformations using rotating frames has been proposed first by Saanouni and

Hamed (2013) and used by these authors for metal forming simulations involving regularised damage laws. As a result, the regularisation operator can be written as:

$$p = \chi - \frac{1}{H_\chi} \operatorname{div} \underline{\boldsymbol{b}} = \chi - \frac{1}{H_\chi} \operatorname{div} \left(\underset{\sim}{\boldsymbol{Q}} \cdot \underline{\boldsymbol{b}}^\dagger \right) = \chi - \frac{1}{H_\chi} \operatorname{div} \left(\underset{\sim}{\boldsymbol{Q}} \cdot \underset{\sim}{\boldsymbol{A}} \cdot \underline{\boldsymbol{k}}^\dagger \right) \quad (126)$$

In the isotropic case, $\underset{\sim}{\boldsymbol{A}} = A \underset{\sim}{\boldsymbol{1}}$, the regularisation operator reduces to

$$p = \chi - \frac{A}{H_\chi} \operatorname{div} \left(\underset{\sim}{\boldsymbol{Q}} \cdot \underline{\boldsymbol{k}}^\dagger \right) \quad (127)$$

It is worth insisting on the fact that, in general, $\underset{\sim}{\boldsymbol{Q}} \cdot \underline{\boldsymbol{k}}^\dagger \neq \nabla_x \chi$. Accordingly, the previous equation does not involve the Laplace operator and the regularisation is therefore nonlinear even with respect to rotated quantities.

Among all choices of rotating frames, the one associated with the logarithmic spin rate tensor (Xiao et al. 1999) was claimed to be the only one such that, when $\underset{\sim}{\dot{\boldsymbol{e}}}^p = 0$, the isotropic hypoelastic strain-stress relation turns out to be hyperelastic. However, this property does not pertain to the general case $\underset{\sim}{\dot{\boldsymbol{e}}}^p \neq 0$, so that this specific choice does not in general provide any explicit form of the regularisation operator.

Alternative constitutive choices are possible for the higher order stresses if Laplacian operators are preferred. They amount to restricting the use of the rotating frame only to the classical elastoviscoplasticity equations and to writing independently, $\underline{\boldsymbol{b}} = A\, \nabla_x \chi$, so that the regularisation operator is expressed in terms of the Eulerian Laplace operator Δ_x, see Eq. (148) in the next section, or $\underline{\boldsymbol{b}}_0 = A\, \nabla_X \chi$ which leads to the Lagrangian Laplace operator Δ_X, see Eq. (143) in the next section.

Note that limitations in the formulation of anisotropic plasticity arise from using symmetric plastic strain variable $\underset{\sim}{\boldsymbol{e}}^p$ and that generalisations are needed in order to introduce necessary plastic spins for materials with microtructures, which is possible within the rotating frame approach (Forest and Pilvin 1999).

Finite Deformation Micromorphic Elastoviscoplasticity Based on the Multiplicative Decomposition

The most appropriate thermodynamically based framework for the formulation of finite deformation anisotropic elastoviscoplasticity relies on the multiplicative decomposition of the deformation gradient, as settled by Mandel (1971). This method is applied here to a generalised continuum model again limited to one scalar degree of freedom, χ, in addition to the displacement vector, $\underline{\boldsymbol{u}}$. The gradients of the degrees of freedom can be computed with respect to the reference or current coordinates:

$$\underset{\sim}{\boldsymbol{F}} = \underset{\sim}{\boldsymbol{1}} + \text{Grad}\, \underline{\boldsymbol{u}} = \underset{\sim}{\boldsymbol{1}} + \underline{\boldsymbol{u}} \otimes \nabla_X \tag{128}$$

$$\underline{\boldsymbol{K}} = \text{Grad}\, \chi = \nabla_X \chi, \quad \underline{\boldsymbol{k}} = \text{grad}\ \chi = \nabla_X \chi = \underset{\sim}{\boldsymbol{F}}^{-T} \cdot \underline{\boldsymbol{K}} = \underline{\boldsymbol{K}} \cdot \underset{\sim}{\boldsymbol{F}}^{-1} \tag{129}$$

The consideration of microdeformation degrees of freedom of higher order is possible without fundamental modification of the approach below, see Forest and Sievert (2006).

A multiplicative decomposition is envisaged in this section, see Eq. (12), in the form:

$$\underset{\sim}{\boldsymbol{F}} = \underset{\sim}{\boldsymbol{F}}^{e} \cdot \underset{\sim}{\boldsymbol{F}}^{p} \tag{130}$$

which assumes the existence of a triad of directors attached to the material point in order to unambiguously define the isoclinic intermediate local configuration, labelled (#) in the sequel, see Mandel (1971, 1973). The directors are usually related to non–material microstructure directions like lattice directions in single crystals or fibre directions in composites. The existence of such directors is required for the formulation of objective anisotropic constitutive equations (Besson et al. 2009). The Jacobians of all contributions in Eq. (130) are denoted by

$$J = \det \underset{\sim}{\boldsymbol{F}},\ J_e = \det \underset{\sim}{\boldsymbol{F}}^{e},\ J_p = \det \underset{\sim}{\boldsymbol{F}}^{p},\ J = J_e J_p,\ \rho_0 = \rho_\sharp J_p = \rho J \tag{131}$$

They are used to relate the mass densities with respect to the three local configurations: In the present section, the microdeformation gradient $\underline{\boldsymbol{K}}$ is not split into elastic and plastic contributions, although it is possible as done in section "Elastic-Plastic Decomposition of the Generalized Strain Measures", at the expense of additional evolution laws to be determined and of drastically different regularisation operators.

The power density of internal and contact forces are

$$p^{(i)} = \underset{\sim}{\boldsymbol{\sigma}} : \dot{\underset{\sim}{\boldsymbol{F}}}\, \underset{\sim}{\boldsymbol{F}}^{-1} + a\dot{\chi} + \underline{\boldsymbol{k}} \cdot \nabla_x \dot{\chi},\ \forall \underline{x} \in \Omega,\ p^{(c)} = \underline{\boldsymbol{t}} \cdot \underline{\boldsymbol{v}} + a_c \dot{\chi},\ \forall \underline{\boldsymbol{x}} \in \partial\Omega \tag{132}$$

The invariance of $p^{(i)}$ with respect to any change of observer requires the Cauchy stress $\underset{\sim}{\boldsymbol{\sigma}}$ to be symmetric. The scalar microstrain is assumed to be invariant. The corresponding balance and boundary conditions are still given by Eq. (87).

Lagrangian Formulation

The Lagrangian free energy density is a function $\psi_0\left(\underset{\sim}{\boldsymbol{C}}^{e}, q, \chi, \underline{\boldsymbol{K}}\right)$, where $\underset{\sim}{\boldsymbol{C}}^{e} = \underset{\sim}{\boldsymbol{F}}^{eT} \cdot \underset{\sim}{\boldsymbol{F}}^{e}$ is the elastic strain and q a set of internal variables accounting for material hardening. Note that the usual elastic strain tensor $\underset{\sim}{\boldsymbol{C}}^{e}$ is defined with respect to the intermediate configuration to comply with standard anisotropic plasticity, whereas $\underline{\boldsymbol{K}}$ is Lagrangian. The presented formulation is therefore not purely Lagrangian but rather mixed. The local Lagrangian form of the entropy inequality is: $\mathcal{D}_0 = p_0^{(i)} - \rho_0\dot{\psi}_0 \geq 0, \quad p_0^{(i)} = Jp^{(i)}$. Accounting for the multiplicative decomposition (130), the power of internal forces is expanded as:

$$
\begin{aligned}
p_0^{(i)} &= J\underset{\sim}{\boldsymbol{\sigma}} : \dot{\underset{\sim}{\boldsymbol{F}}} \cdot \underset{\sim}{\boldsymbol{F}}^{-1} + Ja\dot{\chi} + J\underline{b} \cdot \nabla_x \dot{\chi} = \frac{J_p}{2}\underset{\sim}{\boldsymbol{\Pi}}^e : \dot{\underset{\sim}{\boldsymbol{C}}}^e \\
&+ J_p\underset{\sim}{\boldsymbol{\Pi}}^M : \dot{\underset{\sim}{\boldsymbol{F}}}^p \cdot \underset{\sim}{\boldsymbol{F}}^{p-1} + a_0\dot{\chi} + \underline{\boldsymbol{b}}_0 \cdot \dot{\underline{\boldsymbol{K}}}
\end{aligned}
\tag{133}
$$

where the Piola stress tensor w.r.t. the intermediate configuration and the Mandel stress tensor according to (Haupt 2000) are respectively defined as:

$$
\underset{\sim}{\boldsymbol{\Pi}}^e = J_e\underset{\sim}{\boldsymbol{F}}^{e-1} \cdot \boldsymbol{\sigma} \cdot \underset{\sim}{\boldsymbol{F}}^{e-T}, \quad \underset{\sim}{\boldsymbol{\Pi}}^M = \underset{\sim}{\boldsymbol{C}}^e \cdot \underset{\sim}{\boldsymbol{\Pi}}^e = J_e\underset{\sim}{\boldsymbol{F}}^{eT} \cdot \underset{\sim}{\boldsymbol{\sigma}} \cdot \underset{\sim}{\boldsymbol{F}}^{e-T} \tag{134}
$$

The Lagrangian generalized stresses in (133) are $a_0 = Ja$ and $\underline{\boldsymbol{b}}_0 = J\underset{\sim}{\boldsymbol{F}}^{-1}\underline{\boldsymbol{b}}$. As a result the dissipation rate becomes:

$$
\begin{aligned}
&\left(\frac{J_p}{2}\underset{\sim}{\boldsymbol{\Pi}}^e - \rho_0\frac{\partial\psi_0}{\partial\underset{\sim}{\boldsymbol{C}}^e}\right) : \dot{\underset{\sim}{\boldsymbol{C}}}^e + \left(a_0 - \rho_0\frac{\partial\psi_0}{\partial\chi}\right)\dot{\chi} + \left(\underline{\boldsymbol{b}}_0 - \rho_0\frac{\partial\psi_0}{\partial\underline{\boldsymbol{K}}}\right) \cdot \dot{\underline{\boldsymbol{K}}} \\
&+ J_p\underset{\sim}{\boldsymbol{\Pi}}^M : \dot{\underset{\sim}{\boldsymbol{F}}}^p \cdot \underset{\sim}{\boldsymbol{F}}^{p-1} - \rho_0\frac{\partial\psi_0}{\partial q}\dot{q} \geq 0
\end{aligned}
\tag{135}
$$

The following state and evolution laws ensure the positivity of $\mathcal{D}_0$:

$$
\underset{\sim}{\boldsymbol{\Pi}}^e = 2\rho_\sharp\frac{\partial\psi_0}{\partial\underset{\sim}{\boldsymbol{C}}^e},\ a_0 = \rho_0\frac{\partial\psi_0}{\partial\chi},\ \underline{\boldsymbol{b}}_0 = \rho_0\frac{\partial\psi_0}{\partial\underline{\boldsymbol{K}}},\ R = \rho_0\frac{\partial\psi_0}{\partial q} \tag{136}
$$

$$
\dot{\underset{\sim}{\boldsymbol{F}}}^p \cdot \dot{\underset{\sim}{\boldsymbol{F}}}^{p-1} = \frac{\partial\Omega}{\partial\underset{\sim}{\boldsymbol{\Pi}}^M}\left(\underset{\sim}{\boldsymbol{\Pi}}^M, R\right),\ \dot{q} = -\frac{\partial\Omega}{\partial R}\left(\underset{\sim}{\boldsymbol{\Pi}}^M, R\right), \tag{137}
$$

Following Mandel (1971), a dissipation potential $\Omega\left(\underset{\sim}{\boldsymbol{\Pi}}^M, R\right)$ function of the driving forces for plasticity, is introduced to formulate the flow and hardening variable evolution rule. If the dissipation function is convex w.r.t. $\underset{\sim}{\boldsymbol{\Pi}}^M$ and concave w.r.t. R, the positivity of dissipation is ensured. Specific expressions for within the context of viscoplasticity can be found in Besson et al. (2009).

As an example, the following free energy potential is proposed:

$$
\rho_0\psi_0 = \frac{1}{2}J_p\underset{\sim}{\boldsymbol{E}}^e : \underset{\approx}{\boldsymbol{c}} : \underset{\sim}{\boldsymbol{E}}^e + \rho_0\psi_q(q) + \frac{1}{2}H_\chi(p-\chi)^2 + \rho_0\psi_{\nabla_X}(\underline{\boldsymbol{K}}) \tag{138}
$$

where ψ_q is the appropriate free energy contribution associated with usual work-hardening (not specified here) and $2\underset{\sim}{\boldsymbol{E}}^e = \underset{\sim}{\boldsymbol{C}}^e - \underset{\sim}{\boldsymbol{1}}$ is the Green-Lagrange elastic strain measure. The microstrain variable is compared to the cumulative plastic strain variable p defined as:

$$\dot{p} = \sqrt{\frac{2}{3}\left(\underset{\sim}{\dot{\boldsymbol{P}}} \cdot \underset{\sim}{\boldsymbol{P}}^{-1}\right) : \left(\underset{\sim}{\dot{\boldsymbol{P}}} \cdot \underset{\sim}{\boldsymbol{P}}^{-1}\right)} \tag{139}$$

According to the state laws (136), we obtain

$$\underset{\sim}{\boldsymbol{\Pi}}^e = \rho_\sharp \frac{\partial \psi_0}{\partial \underset{\sim}{\boldsymbol{E}}^e} = \underset{\approx}{\boldsymbol{c}} : \underset{\sim}{\boldsymbol{E}}^e,\ a_0 = -H_\chi\,(p - \chi)\,,\ \underline{\boldsymbol{b}}_0 = \rho_0 \frac{\partial \psi_{\nabla X}}{\partial \underline{\boldsymbol{K}}} \tag{140}$$

The regularisation operator then follows from the combination of the previous constitutive equations with the balance equation (89):

$$p = \chi - \frac{1}{H_\chi}\mathrm{Div}\left(\rho_0 \frac{\partial \psi_{\nabla X}}{\partial \underline{\boldsymbol{K}}}\right) \tag{141}$$

The specific choice $\rho_0 \psi_{\nabla X}\,(\underline{\boldsymbol{K}}) = \frac{1}{2}\underline{\boldsymbol{K}} \cdot \underset{\sim}{\boldsymbol{A}} \cdot \underline{\boldsymbol{K}}$ leads to a regularisation operator that is linear with respect to Lagrangian coordinates:

$$p = \chi - \frac{1}{H_\chi}\mathrm{Div}\left(\underset{\sim}{\boldsymbol{A}} \cdot \mathrm{Grad}\ \chi\right) \tag{142}$$

which involves the Laplacian operator Δ_X in the isotropic case, i.e. $\underset{\sim}{\boldsymbol{A}} = A\underset{\sim}{\boldsymbol{1}}$,

$$\mathrm{Op} = 1 - \frac{A}{H_\chi}\Delta_X \tag{143}$$

The impact on hardening can be seen by choosing, as an example, $q = p$, according to (139), as an internal variable in (138). The dissipation potential can be chosen in such a way that the residual dissipation takes the form

$$\mathcal{D}_0 = J_p \underset{\sim}{\boldsymbol{\Pi}}^M : \left(\underset{\sim}{\dot{\boldsymbol{F}}}^p \cdot \underset{\sim}{\boldsymbol{F}}^{p-1}\right) - \rho_0 \frac{\partial \Psi_q}{\partial p}\dot{p} = J_p \Pi^M_{eq}\dot{p} - R\dot{p} \geq 0 \tag{144}$$

with

$$\dot{p} = \frac{\partial \Omega}{\partial f},\ f\left(\underset{\sim}{\boldsymbol{\Pi}}^M, R\right) = J_p \Pi^M_{eq} - R$$

where f is the yield function. As a result, the yield stress R is given by the following enhanced hardening law:

$$R = \rho_0 \frac{\partial \psi_0}{\partial p} = \rho_0 \frac{\partial \psi_q}{\partial p} + H_\chi (p - \chi) = \rho_0 \frac{\partial \psi_q}{\partial p} - \text{Div}\left(\rho_0 \frac{\partial \psi_{\nabla \chi}}{\partial \underline{\boldsymbol{K}}}\right) \tag{145}$$

Eulerian Formulation

In the Eulerian formulation, the free energy density is taken as a function of $\underline{\boldsymbol{k}}$ instead of $\underline{\boldsymbol{K}}$, according to (129): $\psi\left(\underset{\sim}{\boldsymbol{C}}^e, \chi, \underline{\boldsymbol{k}}\right)$, so that the state laws for generalised stresses become:

$$a = \rho \frac{\partial \psi}{\partial \chi}, \; \underline{\boldsymbol{b}} = \rho \frac{\partial \psi}{\partial \underline{\boldsymbol{k}}} \tag{146}$$

The arguments of the free energy mix the invariant quantities $\underset{\sim}{\boldsymbol{C}}^e$, χ and the observer–dependent variable $\underline{\boldsymbol{k}}$. Galilean invariance then requires ψ to be isotropic with respect to $\underline{\boldsymbol{k}}$.

The constitutive choice (138) is now replaced by

$$\rho\psi = \frac{1}{2} J \underset{\sim}{\boldsymbol{E}}^e : \underset{\approx}{c} : \underset{\sim}{\boldsymbol{E}}^e + \rho\psi_q(q) + \frac{1}{2} H_\chi (p - \chi)^2 + \rho\psi_\nabla (\underline{\boldsymbol{k}}) \tag{147}$$

A quadratic potential ψ_∇ is necessarily of the form $A\|\underline{\boldsymbol{k}}\|^2/2$, for objectivity reasons, so that the regularisation operator involves the Laplace operator Δ_x w.r.t. Eulerian coordinates:

$$\text{Op} = 1 - \frac{A}{H_\chi} \Delta_x \tag{148}$$

If the same viscoplastic yield function $f\left(\underset{\sim}{\boldsymbol{\Pi}}^M, R\right)$ as in the previous subsection is adopted, the hardening rule is enhanced as follows:

$$R = \rho \frac{\partial \psi}{\partial p} = \rho \frac{\partial \psi_q}{\partial p} + H_\chi (p - \chi) = \rho \frac{\partial \psi_q}{\partial p} - \text{div } \underline{\boldsymbol{b}} = \rho \frac{\partial \psi_q}{\partial p} - A \Delta_x \chi \tag{149}$$

This therefore yields a finite strain generalisation of Aifantis strain gradient plasticity model (Aifantis 1987; Forest and Aifantis 2010).

Formulation Using the Local Intermediate Configuration Only

In the two previous formulations, Lagrangian or Eulerian generalized strain variables were mixed with the elastic strain variable $\underset{\sim}{\boldsymbol{C}}^e$ attached to the intermediate local configuration, as the arguments of the free energy function. It is possible to assign the free energy function with a consistent set of arguments solely attached to the intermediate configuration. For that purpose, a generalised strain $\underline{\boldsymbol{K}}^\sharp$ and generalised stresses $a_\sharp, \underline{\boldsymbol{b}}^\sharp$ are now defined on the intermediate local configuration:

$$\underline{\boldsymbol{K}}^{\sharp} = \underline{\boldsymbol{k}} \cdot \underset{\sim}{\boldsymbol{F}}^{e} = \underset{\sim}{\boldsymbol{F}}^{eT} \cdot \underline{\boldsymbol{k}} = \underline{\boldsymbol{K}} \cdot \underset{\sim}{\boldsymbol{F}}^{p-1} = \underset{\sim}{\boldsymbol{F}}^{p-T} \cdot \underline{\boldsymbol{K}} \tag{150}$$

$$a_{\sharp} = J_e a = J_p^{-1} a_0, \;\; \underline{\boldsymbol{b}}^{\sharp} = J_p^{-1} \underset{\sim}{\boldsymbol{F}}^{p} \cdot \underline{\boldsymbol{b}}_0 = J_e \underset{\sim}{\boldsymbol{F}}^{e-1} \cdot \underline{\boldsymbol{b}} \tag{151}$$

The power density of internal forces expressed w.r.t. the intermediate local configuration then takes the form:

$$p_{\sharp}^{(i)} = J_p^{-1} p_0^{(i)} = \frac{1}{2} \underset{\sim}{\boldsymbol{\Pi}}^{e} : \dot{\underset{\sim}{\boldsymbol{C}}}^{e} + \left(\underset{\sim}{\boldsymbol{\Pi}}^{M} + \underline{\boldsymbol{K}}^{\sharp} \otimes \underline{\boldsymbol{b}}^{\sharp} \right) : \dot{\underset{\sim}{\boldsymbol{F}}}^{p} \cdot \underset{\sim}{\boldsymbol{F}}^{p-1} + a_{\sharp} \dot{\chi} + \underline{\boldsymbol{b}}_{\sharp} \cdot \dot{\underline{\boldsymbol{K}}}^{\sharp} \tag{152}$$

where $p_0^{(i)}$ is still given by Eq. (133). To establish this expression, the following relation was used:

$$\dot{\underline{\boldsymbol{K}}} = \underset{\sim}{\boldsymbol{F}}^{pT} \cdot \dot{\underline{\boldsymbol{K}}}^{\sharp} + \dot{\underset{\sim}{\boldsymbol{F}}}^{pT} \cdot \underline{\boldsymbol{K}}^{\sharp} \tag{153}$$

The dissipation rate density measured w.r.t. the intermediate local configuration is then:

$$\mathcal{D}_{\sharp} = p_{\sharp}^{(i)} - \rho_{\sharp} \dot{\psi}_{\sharp} \geq 0 \tag{154}$$

The free energy density function is chosen as $\psi_{\sharp} \left(\underset{\sim}{\boldsymbol{C}}^{e}, q, \chi, \underline{\boldsymbol{K}}^{\sharp} \right)$. As such, it is invariant w.r.t. change of observer. The Clausius–Duhem inequality is now derived as

$$\begin{aligned} &\left(\tfrac{1}{2} \underset{\sim}{\boldsymbol{\Pi}}^{e} - \rho_{\sharp} \tfrac{\partial \psi_{\sharp}}{\partial \underset{\sim}{\boldsymbol{C}}^{e}} \right) : \dot{\underset{\sim}{\boldsymbol{C}}}^{e} + \left(a_{\sharp} - \rho_{\sharp} \tfrac{\partial \psi_{\sharp}}{\partial \chi} \right) \dot{\chi} + \left(\underline{\boldsymbol{b}}^{\sharp} - \rho_{\sharp} \tfrac{\partial \psi_{\sharp}}{\partial \underline{\boldsymbol{K}}^{\sharp}} \right) \cdot \dot{\underline{\boldsymbol{K}}}^{\sharp} \\ &+ \left(\underset{\sim}{\boldsymbol{\Pi}}^{M} + \underline{\boldsymbol{K}}^{\sharp} \otimes \underline{\boldsymbol{b}}^{\sharp} \right) : \dot{\underset{\sim}{\boldsymbol{F}}}^{p} \cdot \underset{\sim}{\boldsymbol{F}}^{p-1} - \rho_{\sharp} \tfrac{\partial \psi_{\sharp}}{\partial q} \dot{q} \geq 0 \end{aligned} \tag{155}$$

This expression reveals the existence of a generalised Mandel tensor, $\underset{\sim}{\boldsymbol{\Pi}}^{M} + \underline{\boldsymbol{K}}^{\sharp} \otimes \underset{\sim}{\boldsymbol{b}}^{\sharp}$, conjugate to the plastic deformation rate, that is a function of the classical Mandel stress and of microdeformation related stress and strain. Positivity of dissipation is ensured by the choice of the following state laws and plastic flow and hardening rules:

$$\underset{\sim}{\boldsymbol{\Pi}}^{e} = 2 \rho_{\sharp} \frac{\partial \psi_{\sharp}}{\partial \underset{\sim}{\boldsymbol{C}}^{e}}, \; a_{\sharp} = \rho_{\sharp} \frac{\partial \psi_{\sharp}}{\partial \chi}, \; \underline{\boldsymbol{b}}^{\sharp} = \rho_{\sharp} \frac{\partial \psi_{\sharp}}{\partial \underline{\boldsymbol{K}}^{\sharp}}, \; R = \rho_{\sharp} \frac{\partial \psi_{\sharp}}{\partial q} \tag{156}$$

$$\dot{\underset{\sim}{\boldsymbol{F}}}^{p} \cdot \underset{\sim}{\boldsymbol{F}}^{p-1} = \frac{\partial \Omega}{\partial \left(\underset{\sim}{\boldsymbol{\Pi}}^{M} + \underline{\boldsymbol{K}}^{\sharp} \otimes \underline{\boldsymbol{b}}^{\sharp} \right)}, \; \dot{q} = - \frac{\partial \Omega}{\partial R} \tag{157}$$

provided that the dissipation potential $\Omega\left(\underset{\sim}{\boldsymbol{\Pi}}^M + \underline{\boldsymbol{K}}^\sharp \otimes \underline{\boldsymbol{b}}^\sharp, R\right)$ displays suitable convexity properties with respect to both arguments.

The yield criterion is taken as a function $f\left(\underset{\sim}{\boldsymbol{\Pi}}^M + \underline{\boldsymbol{K}}^\sharp \otimes \underline{\boldsymbol{b}}^\sharp, R\right) = \Pi^M_{eq} - R$ where the Π^M_{eq} is an equivalent stress measure based on the generalized Mandel stress tensor. Choosing $q = p$, where p is still given by Eq. (139), the residual dissipation takes the form:

$$\mathcal{D}_\sharp = \left(\underset{\sim}{\boldsymbol{\Pi}}^M + \underline{\boldsymbol{K}}^\sharp \otimes \underline{\boldsymbol{b}}^\sharp\right) : \underset{\sim}{\dot{\boldsymbol{F}}}^p \cdot \underset{\sim}{\boldsymbol{F}}^{p-1} - R\dot{p} = \Pi^M_{eq}\dot{p} - R\dot{p} \tag{158}$$

Note that the contribution $\underline{\boldsymbol{K}}^\sharp \otimes \underline{\boldsymbol{b}}^\sharp$in the generalised Mandel stress acts as a size–dependent kinematic hardening component which comes in addition to isotropic hardening represented by R. This is a specific feature of the model formulation w.r.t. the intermediate configuration.

As an example, a typical form of the free energy density function based on constitutive variables defined on the intermediate configuration, and hyperelastic laws are:

$$\rho_\sharp\psi_\sharp = \frac{1}{2}\underset{\sim}{\boldsymbol{E}}^e : \underset{\approx}{\boldsymbol{c}} : \underset{\sim}{\boldsymbol{E}}^e + \rho_\sharp\psi_\sharp(q) + \frac{1}{2}H_\chi(p-\chi)^2 + \frac{1}{2}\rho_\sharp\psi_{\sharp\nabla}\left(\underline{\boldsymbol{K}}^\sharp\right) \tag{159}$$

$$\underset{\sim}{\boldsymbol{\Pi}}^e = \underset{\approx}{\boldsymbol{c}} : \underset{\sim}{\boldsymbol{E}}^e,\ a_\sharp = -H_\chi\left(p-\chi\right),\ \underline{\boldsymbol{b}}^\sharp = \rho_\sharp\frac{\partial\psi_\sharp}{\partial\underline{\boldsymbol{K}}^\sharp} \tag{160}$$

These generalized stresses can be inserted into the balance equation

$$a_0 = \mathrm{Div}\ \underline{\boldsymbol{b}}_0\ \Rightarrow J_p a_\sharp = \mathrm{Div}\left(J_p\underset{\sim}{\boldsymbol{F}}^{p-1}\cdot\underline{\boldsymbol{b}}^\sharp\right) \tag{161}$$

This provides the form of the regularisation operator:

$$p = \chi - \frac{1}{J_p H_\chi}\mathrm{Div}\ \left(J_p\underset{\sim}{\boldsymbol{F}}^{p-1}\rho_\sharp\frac{\partial\psi_\sharp}{\partial\underline{\boldsymbol{K}}^\sharp}\right) \tag{162}$$

A quadratic dependence of the contribution $\rho_\sharp\psi_{\sharp\nabla} = 1/2\underline{\boldsymbol{K}}^\sharp\cdot\underset{\sim}{\boldsymbol{A}}\cdot\underline{\boldsymbol{K}}^\sharp$ leads to the following linear relationship between $\underline{\boldsymbol{b}}^\sharp$ and$\underline{\boldsymbol{K}}^\sharp$:

$$\underline{\boldsymbol{b}}^\sharp = A\underline{\boldsymbol{K}}^\sharp\ \ \Rightarrow \underline{\boldsymbol{b}} = J_e^{-1}A\underset{\sim}{\boldsymbol{B}}^e\cdot\underline{\boldsymbol{k}}\ \ \text{and}\ \ \underline{\boldsymbol{b}}^\sharp = J_p A\underset{\sim}{\boldsymbol{C}}^{p-1}\cdot\underline{\boldsymbol{k}} \tag{163}$$

with $\underset{\sim}{\boldsymbol{B}}^e = \underset{\sim}{\boldsymbol{F}}^e\cdot\underset{\sim}{\boldsymbol{F}}^{eT}$ and $\underset{\sim}{\boldsymbol{C}}^p = \underset{\sim}{\boldsymbol{F}}^{pT}\cdot\underset{\sim}{\boldsymbol{F}}^p$. However, in that case, the regularisation operator (161) is nonlinear and does not involve a Laplace operator, even in the isotropic case $\underset{\sim}{\boldsymbol{A}} = A\underset{\sim}{\boldsymbol{1}}$. As a result, the hyperelastic relationships for the higher order stresses are not linear w.r.t. to the associated strain gradient measures.

Conclusion

Eringen and Mindlin's micromorphic theory offers real opportunities for the modeling of size effects in the mechanics of materials. Elastic-viscoplastic constitutive laws have been formulated at finite deformations. They remain to be further specialized and calibrated with respect to size effects observed in metal and polymer plasticity. Successful applications deal for example with the ductile fracture of metallic alloys (Enakoutsa and Leblond 2009; Hutter 2017b) and porous metals (Dillard et al. 2006). Micromorphic elasticity has been recently revisited and further developed to account for the dispersion of elastic waves in architectured and metamaterials (Neff et al. 2014; Rosi and Auffray 2016; Madeo et al. 2016). Intensive work is needed to establish connections between the micromorphic continuum theories and the actual underlying microstructure (Forest and Trinh 2011; Hutter 2017a; Biswas and Poh 2017).

The proposed systematic treatment of the thermomechanics of continua with additional degrees of freedom leads to model formulations ranging from micromorphic to phase field models. In particular, a general framework for the introduction of dissipative processes associated with the additional degrees of freedom has been proposed. If internal constraints are enforced on the relation between macro and microvariables in the micromorphic approach, standard second gradient and strain gradient plasticity models can be retrieved.

As a variant of micromorphic continuum, microdamage continuum and its regularization capabilities for the modelling of crack propagation in single crystals have been studied. First, a crystallographic constitutive model which accounts for continuum damage with respect to fracture planes has been presented. Then, the theory has been extended from classical continuum to microdamage continuum. It has been shown that the approach can be a good candidate for solving mesh dependency and the prediction of final fracture in anistropic media. Analytical fits and numerical results showed that the theory is well suited for FEA and possesses a great potential for the future modelling aspects. Comparison with available data on crack growth especially cyclic loading in nickel-based superalloys, will be decisive to conclude on the ability of the approach to reach realistic prediction of component failure.

The presented extensions to finite deformations show that the regularisation operator cannot be postulated in an intuitive way. It is rather the result of a constitutive choice regarding the dependence of the free energy function on the gradient term. Purely Lagrangian and Eulerian formulations are straightforward and lead to Helmholtz–like operators w.r.t. Lagrangian of Eulerian coordinates. Two alternative standard procedures of extension of classical constitutive laws to large strains, widely used in commercial finite element codes, have been combined with the micromorphic approach. In particular, the choice of local objective rotating frames leads to new nonlinear regularisation operators that are not of the Helmholtz type. Three distinct operators were proposed within the context of the multiplicative decomposition of the deformation gradient. A new feature is that a free energy density function depending on variables solely defined with respect to the intermediate isoclinic configuration leads to the existence of additional kinematic hardening induced by the gradient of a scalar micromorphic degree of freedom.

Note that the results obtained for the micromorphic theory with additional degrees of freedom are also valid for gradient theories (gradient plasticity or gradient damage) once an internal constraint is imposed linking the additional degrees of freedom to strain or internal variables. This amounts to selecting high values of parameter H_χ or introducing corresponding Lagrange multipliers. The analysis was essentially limited to scalar micromorphic degrees of freedom for the sake of simplicity, even though tensorial examples were also given. Scalar plastic microstrain approaches suffer from limitations like indeterminacy of flow direction at cusps of the cumulative plastic strain in bending for instance, see Peerlings (2007), Poh et al. (2011), and Wulfinghoff et al. (2014). Those limitations can be removed by the use of tensorial micromorphic variable (microstrain or microdeformation tensors). The micromorphic approach is not limited to the gradient of strain–like, damage or phase field variables. It can also be applied to other internal variables, as demonstrated for hardening variables in Dorgan and Voyiadjis (2003) and Saanouni and Hamed (2013).

It remains that the regularisation properties of the derived nonlinear operators are essentially unknown, except through examples existing in the mentioned literature. For instance, the Eulerian and Lagrangian variants of the Helmholtz-type equation for scalar micromorphic strain variables have been assessed by Wcislo et al. (2013) giving the advantage to the latter, based on finite element simulations of specific situations. The regularising properties of more general operators should be explored in the future from the mathematical and computational perspectives in order to select the most relevant constitutive choices that may depend on the type of material classes.

It may be surprising that the constitutive theory underlying the construction of regularisation operators for plasticity and damage, mainly relies on the enhancement of the free energy density function instead of the dissipative laws. It is in fact widely recognised that plastic strain gradients, e.g. associated with the multiplication of geometrically necessary dislocations, lead to energy storage that can be released by further deformation or heat treatments. However, the limitation to the enhancement of free energy potential is mainly due to the simplicity of the theoretical treatment and to the computational efficiency of the operators derived in that way. Dissipative higher order contributions remain to be explored from the viewpoint of regularisation, as started in Forest (2009). Constitutive models of that kind are already available for plasticity, damage and fracture (Amor et al. 2009; Pham et al. 2011; Vignollet et al. 2014; Miehe et al. 2016).

Acknowledgments The first author thanks Prof. O. Aslan for his contribution to the presented micro-morphic damage theory. These contributions are duly cited in the references quoted in the text and listed below.

References

E. Aifantis, On the microstructural origin of certain inelastic models. J. Eng. Mater. Technol. **106**, 326–330 (1984)

E. Aifantis, The physics of plastic deformation. Int. J. Plast. **3**, 211–248 (1987)

K. Ammar, B. Appolaire, G. Cailletaud, F. Feyel, F. Forest, Finite element formulation of a phase field model based on the concept of generalized stresses. Comput. Mater. Sci. **45**, 800–805 (2009)

H. Amor, J.J. Marigo, C. Maurini, Regularized formulation of the variational brittle fracture with unilateral contact: Numerical experiments. J. Mech. Phys. Solids **57**, 1209–1229 (2009)

O. Aslan, S. Forest, Crack growth modelling in single crystals based on higher order continua. Comput. Mater. Sci. **45**, 756–761 (2009)

O. Aslan, S. Forest, The micromorphic versus phase field approach to gradient plasticity and damage with application to cracking in metal single crystals, in *Multiscale Methods in Computational Mechanics*, ed. by R. de Borst, E. Ramm. Lecture Notes in Applied and Computational Mechanics, vol. 55, (Springer, New York, 2011), pp. 135–154

O. Aslan, N.M. Cordero, A. Gaubert, S. Forest, Micromorphic approach to single crystal plasticity and damage. Int. J. Eng. Sci. **49**, 1311–1325 (2011)

N. Auffray, H. Le Quang, Q. He, Matrix representations for 3D strain-gradient elasticity. J. Mech. Phys. Solids **61**, 1202–1223 (2013)

A. Berezovski, J. Engelbrecht, P. Van, Weakly nonlocal thermoelasticity for microstructured solids: Microdeformation and microtemperature. Arch. Appl. Mech. **84**, 1249–1261 (2014)

A. Bertram, *Elasticity and Plasticity of Large Deformations* (Springer, Heidelberg, 2012)

Besson, J., Cailletaud, G., Chaboche, J.-L., Forest, S., Bletry, M., *Non–linear Mechanics of Materials*. Solid Mechanics and Its Applications, vol. 167 (Springer, Dordrecht, 2009), 433 p. ISBN:978-90-481-3355-0

R. Biswas, L. Poh, A micromorphic computational homogenization framework for heterogeneous materials. J. Mech. Phys. Solids **102**, 187–208 (2017)

G. Cailletaud, S. Forest, D. Jeulin, F. Feyel, I. Galliet, V. Mounoury, S. Quilici, Some elements of microstructural mechanics. Comput. Mater. Sci. **27**, 351–374 (2003)

A. L. Cauchy, Recherches sur l'équilibre et le mouvement intérieur des corps solides ou fluides, élastiques ou non élastiques. Bulletin de la Société Philomatique, 9–13 (1822)

B. Coleman, W. Noll, The thermodynamics of elastic materials with heat conduction and viscosity. Arch. Ration. Mech. Anal. **13**, 167–178 (1963)

T. Dillard, S. Forest, P. Ienny, Micromorphic continuum modelling of the deformation and fracture behaviour of nickel foams. Eur. J. Mech. A/Solids **25**, 526–549 (2006)

A. Dogui, F. Sidoroff, Kinematic hardening in large elastoplastic strain. Eng. Fract. Mech. **21**, 685–695 (1985)

J. Dorgan, G. Voyiadjis, Nonlocal dislocation based plasticity incorporating gradients of hardening. Mech. Mater. **35**, 721–732 (2003)

K. Enakoutsa, J. Leblond, Numerical implementation and assessment of the glpd micromorphic model of ductile rupture. Eur. J. Mech. A/Solids **28**, 445–460 (2009)

R. Engelen, M. Geers, F. Baaijens, Nonlocal implicit gradient-enhanced elasto-plasticity for the modelling of softening behaviour. Int. J. Plast. **19**, 403–433 (2003)

A. Eringen, Theory of thermo-microstretch elastic solids. Int. J. Eng. Sci. **28**, 1291–1301 (1990)

A. Eringen, *Microcontinuum Field Theories* (Springer, New York, 1999)

A. Eringen, E. Suhubi, Nonlinear theory of simple microelastic solids. Int. J. Eng. Sci. **2**(189–203), 389–404 (1964)

S. Forest, The micromorphic approach for gradient elasticity, viscoplasticity and damage. ASCE J. Eng. Mech. **135**, 117–131 (2009)

S. Forest, Micromorphic media, in *Generalized Continua – From the Theory to Engineering Applications. CISM International Centre for Mechanical Sciences*, ed. by H. Altenbach, V. Eremeyev. Courses and Lectures No. 541 (Springer, New York, 2012), pp. 249–300

S. Forest, Nonlinear regularisation operators as derived from the micromorphic approach to gradient elasticity, viscoplasticity and damage. Proc. R. Soc. A **472**, 20150755 (2016)
S. Forest, E.C. Aifantis, Some links between recent gradient thermo-elasto-plasticity theories and the thermomechanics of generalized continua. Int. J. Solids Struct. **47**, 3367–3376 (2010)
S. Forest, A. Bertram, Formulations of strain gradient plasticity, in *Mechanics of Generalized Continua*, ed. by H. Altenbach, G.A. Maugin, V. Erofeev. Advanced Structured Materials, vol. 7 (Springer, Berlin/Heidelberg, 2011), pp. 137–150
S. Forest, P. Pilvin, Modelling finite deformation of polycrystals using local objective frames. Z. Angew. Math. Mech. **79**, S199–S202 (1999)
S. Forest, K. Sab, Finite deformation second order micromorphic theory and its relations to strain and stress gradient models. Math. Mech. Solids (2017). https://doi.org/10.1177/1081286517720844
S. Forest, R. Sievert, Elastoviscoplastic constitutive frameworks for generalized continua. Acta Mech. **160**, 71–111 (2003)
S. Forest, R. Sievert, Nonlinear microstrain theories. Int. J. Solids Struct. **43**, 7224–7245 (2006)
S. Forest, D.K. Trinh, Generalized continua and non–homogeneous boundary conditions in homogenization methods. ZAMM **91**, 90–109 (2011)
M.G.D. Geers, Finite strain logarithmic hyperelasto-plasticity with softening: A strongly non-local implicit gradient framework. Comput. Methods Appl. Mech. Eng. **193**, 3377–3401 (2004)
M. Geers, R.D. Borst, W. Brekelmans, R. Peerlings, On the use of local strain fields for the determination of the intrinsic length scale. Journal de Physique IV **8**(Pr8), 167–174 (1998)
P. Germain, La méthode des puissances virtuelles en mécanique des milieux continus, premiere partie : théorie du second gradient. J. de Mécanique **12**, 235–274 (1973a)
P. Germain, The method of virtual power in continuum mechanics. Part 2: microstructure. SIAM J. Appl. Math. **25**, 556–575 (1973b)
P. Germain, Q. Nguyen, P. Suquet, Continuum thermodynamics. J. Appl. Mech. **50**, 1010–1020 (1983)
N. Germain, J. Besson, F. Feyel, Simulation of laminate composites degradation using mesoscopic non–local damage model and non-local layered shell element. Model. Simul. Mater. Sci. Eng. **15**, S425–S434 (2007)
M. Goodman, S. Cowin, A continuum theory for granular materials. Arch. Ration. Mech. Anal. **44**, 249–266 (1972)
P. Grammenoudis, C. Tsakmakis, Micromorphic continuum part I: Strain and stress tensors and their associated rates. Int. J. Non–Linear Mech. **44**, 943–956 (2009)
P. Grammenoudis, C. Tsakmakis, D. Hofer, Micromorphic continuum part II: Finite deformation plasticity coupled with damage. Int. J. Non–Linear Mech. **44**, 957–974 (2009)
M. Gurtin, Generalized Ginzburg–landau and Cahn–Hilliard equations based on a microforce balance. Physica D **92**, 178–192 (1996)
P. Haupt, *Continuum Mechanics and Theory of Materials* (Springer, Berlin, 2000)
T. Helfer, Extension of monodimensional fuel performance codes to finite strain analysis using a Lagrangian logarithmic strain framework. Nucl. Eng. Des. **288**, 75–81 (2015)
C. Hirschberger, P. Steinmann, Classification of concepts in thermodynamically consistent generalized plasticity. ASCE J. Eng. Mech. **135**, 156–170 (2009)
C. Hirschberger, E. Kuhl, P. Steinmann, On deformational and configurational mechanics of micromorphic hyperelasticity - theory and computation. Comput. Methods Appl. Mech. Eng. **196**, 4027–4044 (2007)
M. Horak, M. Jirasek, An extension of small-strain models to the large-strain range based on an additive decomposition of a logarithmic strain. Programs Algorithms Numer. Math. **16**, 88–93 (2013)
G. Hütter, Homogenization of a cauchy continuum towards a micromorphic continuum. J. Mech. Phys. Solids **99**, 394–408 (2017a)
G. Hütter, A micromechanical gradient extension of Gurson's model of ductile damage within the theory of microdilatational media. Int. J. Solids Struct. **110-111**, 15–23 (2017b)
C. Kafadar, A. Eringen, Micropolar media: I the classical theory. Int. J. Eng. Sci. **9**, 271–305 (1971)

N. Kirchner, P. Steinmann, A unifying treatise on variational principles for gradient and micromorphic continua. Philos. Mag. **85**, 3875–3895 (2005)

M. Lazar, G. Maugin, On microcontinuum field theories: The eshelby stress tensor and incompatibility conditions. Philos. Mag. **87**, 3853–3870 (2007)

E.H. Lee, P. Germain, Elastic–plastic theory at finite strain, in *Problems of Plasticity*, ed. by A. Sawczuk (Ed), (Noordhoff International Publishing, 1972), pp. 117–133

A. Madeo, G. Barbagallo, M.V. d'Agostino, L. Placidi, P. Neff, First evidence of non-locality in real band-gap metamaterials: Determining parameters in the relaxed micromorphic model. Proc. R. Soc. Lond. A **472**, 20160169 (2016)

J. Mandel, *Plasticité Classique et Viscoplasticité*, CISM Courses and Lectures No. 97, Udine (Springer, Berlin, 1971)

J. Mandel, Equations constitutives et directeurs dans les milieux plastiques et viscoplastiques. Int. J. Solids Struct. **9**, 725–740 (1973)

N. Marchal, S. Flouriot, S. Forest, L. Remy, Crack–tip stress–strain fields in single crystal nickel–base superalloys at high temperature under cyclic loading. Comput. Mater. Sci. **37**, 42–50 (2006a)

N. Marchal, S. Forest, L. Rémy, S. Duvinage, Simulation of fatigue crack growth in single crystal superalloys using local approach to fracture, in *Local Approach to Fracture, 9th European Mechanics of Materials Conference, Euromech–Mecamat*. ed. by J. Besson, D. Moinereau, D. Steglich (Presses de l'Ecole des Mines de Paris, Moret–sur–Loing, 2006b), pp. 353–358

G. Maugin, The method of virtual power in continuum mechanics: Application to coupled fields. Acta Mech. **35**, 1–70 (1980)

G. Maugin, Internal variables and dissipative structures. J. Non–Equilib. Thermo-dyn. **15**, 173–192 (1990)

G. Maugin, *Thermomechanics of Nonlinear Irreversible Behaviors* (World Scientific, Singapore/River Edge, 1999)

C. Miehe, Variational gradient plasticity at finite strains. Part I: Mixed potentials for the evolution and update problems of gradient-extended dissipative solids. Comput. Methods Appl. Mech. Eng. **268**, 677–703 (2014)

C. Miehe, N. Apel, M. Lambrecht, Anisotropic additive plasticity in the logarithmic strain space: Modular kinematic formulation and implementation based on incremental minimization principles for standard materials. Comp. Methods Appl. Mech. Eng **191**, 5383–5425 (2002)

C. Miehe, S. Teichtmeister, F. Aldakheel, Phase-field modelling of ductile fracture: a variational gradient-extended plasticity-damage theory and its micromorphic regularization. Philos. Trans. R. Soc. Lond. A **374**(2066) (2016)

R. Mindlin, Micro–structure in linear elasticity. Arch. Ration. Mech. Anal. **16**, 51–78 (1964)

R. Mindlin, Second gradient of strain and surface–tension in linear elasticity. Int. J. Solids Struct. **1**, 417–438 (1965)

H. Mühlhaus, *Continuum Models for Materials with Microstructure* (Wiley, Chichester, 1995)

P. Neff, I. Ghiba, A. Madeo, L. Placidi, G. Rosi, A unifying perspective: The relaxed linear micromorphic continuum. Contin. Mech. Thermodyn. **26**, 639–681 (2014)

R. Peerlings, On the role of moving elastic-plastic boundaries in strain gradient plasticity. Model. Simul. Mater. Sci. Eng. **15**, S109–S120 (2007)

R. Peerlings, M. Geers, R. de Borst, W. Brekelmans, A critical comparison of nonlocal and gradient–enhanced softening continua. Int. J. Solids Struct. **38**, 7723–7746 (2001)

R. Peerlings, T. Massart, M. Geers, A thermodynamically motivated implicit gradient damage framework and its application to brick masonry cracking. Comput. Methods Appl. Mech. Eng. **193**, 3403–3417 (2004)

K. Pham, H. Amor, J.J. Marigo, C. Maurini, Gradient damage models and their use to approximate brittle fracture. Int. J. Damage Mech. **20**, 618–652 (2011)

L. Poh, R. Peerlings, M. Geers, S. Swaddiwudhipong, An implicit tensorial gradient plasticity model - formulation and comparison with a scalar gradient model. Int. J. Solids Struct. **48**, 2595–2604 (2011)

V.D. Rancourt, B. Appolaire, S. Forest, K. Ammar, Homogenization of viscoplastic constitutive laws within a phase field approach. J. Mech. Phys. Solids **88**, 35–48 (2016)

R. Regueiro, On finite strain micromorphic elastoplasticity. Int. J. Solids Struct. **47**, 786–800 (2010)

G. Rosi, N. Auffray, Anisotropic and dispersive wave propagation within strain-gradient framework. Wave Motion **63**, 120–134 (2016)

K. Saanouni, M. Hamed, Micromorphic approach for finite gradient-elastoplasticity fully coupled with ductile damage: Formulation and computational aspects. Int. J. Solids Struct. **50**, 2289–2309 (2013)

C. Sansour, A theory of the elastic–viscoplastic cosserat continuum. Arch. Mech. **50**, 577–597 (1998a)

C. Sansour, A unified concept of elastic–viscoplastic Cosserat and micromorphic continua. Journal de Physique IV **8**(Pr8), 341–348 (1998b)

C. Sansour, S. Skatulla, H. Zbib, A formulation for the micromorphic continuum at finite inelastic strains. Int. J. Solids Struct. **47**, 1546–1554 (2010)

A.V. Shotov, J. Ihlemann, Analysis of some basic approaches to finite strain elasto-plasticity in view of reference change. Int. J. Plast. **63**, 183–197 (2014)

F. Sidoroff, A. Dogui, Some issues about anisotropic elastic-plastic models at finite strain. Int. J. Solids Struct. **38**, 9569–9578 (2001)

J.C. Simo, C. Miehe, Associative coupled thermoplasticity at finite strains: Formulation, numerical analysis and implementation. Comput. Methods Appl. Mech. Eng. **98**, 41–104 (1992)

H. Steeb, S. Diebels, A thermodynamic–consistent model describing growth and remodeling phenomena. Comput. Mater. Sci. **28**, 597–607 (2003)

S. Toll, The dissipation inequality in hypoplasticity. Acta Mech. **221**, 39–47 (2011)

C. Truesdell, W. Noll, The non-linear field theories of mechanics, in *Handbuch der Physik*, ed. by S. Flügge, reedition (Springer, Berlin/Heidelberg, 1965)

C. Truesdell, R. Toupin, The classical field theories, in *Handbuch der Physik*, ed. by S. Flügge, vol. 3 (Springer, Berlin, 1960), pp. 226–793

P. Ván, A. Berezovski, C. Papenfuss, Thermodynamic approach to generalized continua. Contin. Mech. Thermodyn. **26**, 403–420 (2014)

J. Vignollet, S. May, R.D. Borst, C. Verhoosel, Phase–field models for brittle and cohesive fracture. Meccanica **49**, 2587–2601 (2014)

B. Wcislo, L. Pamin, K. Kowalczyk-Gajewska, Gradient-enhanced damage model for large deformations of elastic-plastic materials. Arch. Mech. **65**, 407–428 (2013)

S. Wulfinghoff, E. Bayerschen, T. Böhlke, Conceptual difficulties in plasticity including the gradient of one scalar plastic field variable. PAMM. Proc. Appl. Math. Mech. **14**, 317–318 (2014)

H. Xiao, O.T. Bruhns, A. Meyers, Existence and uniqueness of the integrable–exactly hypoelastic equation and its significance to finite elasticity. Acta Mech. **138**, 31–50 (1999)