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Elastoviscoplastic constitutive frameworks for generalized continua

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Summary. A unifying thermomechanical constitutive framework for generalized continua including additional degrees of freedom or/and the second gradient of displacement is presented. Based on the analysis of the dissipation, state laws, flow rules and evolution equations are proposed for Cosserat, strain gradient and micromorphic continua. The case of the gradient of internal variable approach is also incorporated by regarding the nonlocal internal variable as an actual additional degree of freedom. The consistency of the continuum thermodynamical framework is ensured by the introduction of a viscoplastic pseudo–potential of dissipation, thus extending the classical class of so–called standard material models to generalized continua.

Variants of the higher order and higher grade theories are also reported based on the explicit introduction of the plastic strain tensor as additional degree of freedom. Within this new class of models, called here *gradient of strain* models, one recognizes the fact that, in a second grade theory for instance, the plastic part of the strain gradient can be identified with the gradient of plastic strain.

Simple examples dealing with bending and shearing of Cosserat or second grade media are given to illustrate two types of extensions of classical J_2-plasticity : single-criterion and multi-mechanism generalized elastoplasticity.

Finally, formulations at finite deformation of the proposed models are provided focusing on proper decompositions of Cosserat curvature, strain gradient and gradient of micromorphic deformation into elastic and viscoplastic parts.

1 Introduction

1.1 Scope of this work

The sixties have definitely been the Golden Age of the mechanics of generalized continua with the milestones [1]–[3], even though its origins go back to Cauchy, Voigt, Boltzmann and the Cosserat brothers. The authors were mainly interested in the development of *higher-grade* media, including higher derivatives of the displacement fields (second-grade medium for instance, [4]), and *higher-order* continua that incorporate additional degrees of freedom (independent rotation for the Cosserat continuum, and a full micro-deformation tensor for the micromorphic medium, see [2], [3]). They provided us with a rigorous and almost exhaustive corpus of balance and constitutive equations for generalized continua. They were however mainly concerned with elasticity, hyperelasticity or even viscoelasticity. The renewal of the mechanics of generalized continua started in the eighties and is still continuing. It has been stimulated by the development of computational tools but also of experimental field measurement methods well–suited for the study of localization phenomena [5]. It brings with it a real need for nonlinear elastoplastic or elastoviscoplastic constitutive equations for generalized continua. Nonlinear models for higher order and higher grade continua remain scattered in the literature, and a comprehensive treatise similar to Eringen's books dedicated to balances and linear constitutive equations [6], [7] is still lacking. Such models are often available since the early seventies, especially for Cosserat elastoplasticity [8]. Due to the success of the internal variable approach in the nonlinear constitutive modelling for classical continuum mechanics, a competing theory appeared in the early eighties, namely the gradient of internal variable approach. The introduction in the constitutive framework of the gradient of one properly selected internal variable (cumulative plastic strain, dislocation density, [9]) can be sufficient to account for some observed nonlocal effects. Even though it may look more simple, the gradient of internal variable approach belongs to the mechanics of generalized continua and requires additional boundary conditions similarly to higher-order and higher-grade theories [10].

When a significant gradient of macroscopic loading is already present over the size of the relevant substructure (e.g., particles, grains, holes etc.) then the influence of that gradient of loading on the material behavior within a macroscopic continuum element, which has to cover the heterogeneity of the material, should be taken into account (e.g., the influence of the accommodation of plastic deformation in the microstructure at localization phenomena). But then, the deformation field within the macroscopic continuum element can be described no more in the classical way, i.e. by a linear approximation on the basis of the macroscopic displacement only. A systematic way to enrich the classical continuum is the introduction of additional degrees of freedom, i.e., internal deformations (as, for instance, lattice rotation or geometrical internal variables), or higher-order gradients of the degrees of freedom, e.g. the second gradient of displacement.

The present work aims at providing a unifying thermomechanical framework for the development of elastoviscoplastic constitutive equations for weakly nonlocal theories [11], namely higher order and higher grade media. The case of fully nonlocal media involving an integral formulation of the constitutive equations is not envisaged here. One first has to decide whether a given variable, relevant for the material modelling, has to be treated as a hidden internal variable or an actual degree of freedom. In the early stage of continuum crystal plasticity for instance, this question arose for crystal lattice rotation. In [12], the author clearly asks the question and recommends to deal with lattice rotation as a hidden triad of directors. This was sufficient for the modelling of homogeneous or slightly heterogeneous deformation of large single crystals. However, to account for size effects observed in crystal plasticity (grain or particle size effects, strain localization phenomena), it is appropriate to consider lattice rotation as an actual degree of freedom of the crystal, the gradient of which, lattice curvature, is associated with couple–stresses [13], [14]. An alternative is to resort to strain gradient plasticity in its second grade form [15] or gradient of internal variable form [16].

The continuum thermodynamical framework settled in [17] is extended to Cosserat, second grade and micromorphic continua, and also to the gradient of internal variable approach (Sects. 2–5). The notion of the pseudo-potential of dissipation that defines the special class of so–called standard materials is introduced for elastoviscoplastic generalized continua in a straightforward manner. This is the meaning to be attributed to the adjective *standard* used throughout this work. Balance of momentum equations are recalled for each continuum in order to clearly see the role of the higher order stress tensors that arise in the theories.

Explicit examples are provided in the case of Cosserat and strain gradient models enlightening the role of the characteristic lengths associated with the elastic and plastic deformation modes. Examples for the gradient of internal variable approach are more numerous in literature [9], [16], [18]. The examples deal with extensions of classical J_2-plasticity for each continuum.

The plasticity models available in the literature usually include coupled criteria involving classical and higher-order stress tensors. A multi-mechanism elastoviscoplastic framework is also possible and presented here. Both model classes are compared for the considered simple examples. The detailed (semi)-analytical solutions are given in Appendices C and D.

An intrinsic tensor notation is used throughout this work. It is described in Appendix A.

1.2 Method of derivation

For each continuum, balance equations, state laws and intrinsic dissipation are derived from the successive application of the three following principles:

(i) Principle of virtual power

The method of virtual power has proved to be a powerful tool to derive the field equations and the associated boundary conditions that the unknown fields must fulfill on a body V [19]. V denotes the open body and ∂V its closure. In Sect. 3, the surface ∂V is considered twice continuously differentiable, so that it possesses at each point a normal $\underline{\mathbf{n}}$ and a mean curvature R. The presence of edges and vertices must be treated as shown in [20]. The degrees of freedom of the investigated continuum are the displacement $\underline{\mathbf{u}}$ and additional fields denoted here by the general term γ. The tensorial nature of γ will be given for each continuum. The set of virtual motions $\mathscr{V}^\circ$ contains the corresponding rate fields:

$$\mathscr{V}^\circ = \{\dot{\underline{\mathbf{u}}},\ \dot{\gamma}\}. \tag{1}$$

One defines also the sets $\mathscr{V}$ and $\mathscr{V}^c$ of modelling variables entering the virtual power of internal and contact forces, respectively. For a first grade theory, they read:

$$\mathscr{V} = \{\dot{\underline{\mathbf{u}}},\ \dot{\underline{\mathbf{u}}} \otimes \boldsymbol{\nabla}, \dot{\gamma}, \boldsymbol{\nabla}\dot{\gamma}\}, \quad \mathscr{V}^c = \{\dot{\underline{\mathbf{u}}},\ \dot{\gamma}\}, \tag{2}$$

for an enriched theory of second grade with respect to the displacement:

$$\mathscr{V} = \{\dot{\underline{\mathbf{u}}}, \dot{\underline{\mathbf{u}}} \otimes \boldsymbol{\nabla}, \dot{\underline{\mathbf{u}}} \otimes \boldsymbol{\nabla} \otimes \boldsymbol{\nabla}, \dot{\gamma}, \boldsymbol{\nabla}\dot{\gamma}\}, \quad \mathscr{V}^c = \{\dot{\underline{\mathbf{u}}}, D_n\dot{\underline{\mathbf{u}}}, \dot{\gamma}\}, \tag{3}$$

where D_n is the normal gradient operator [20]. In the static case, the principle of virtual power states that the virtual power of externally acting forces is equal to the virtual power of internally acting forces, for all generalized virtual motions ϑ^* and for all subdomains $\mathscr{D}$ of body V:

$$\mathscr{P}^{(i)}(\vartheta^* \in \mathscr{V}) = \mathscr{P}^{(c)}(\vartheta^* \in \mathscr{V}^c). \tag{4}$$

Body forces are not considered here for conciseness. The virtual powers of internal (i) and contact (c) forces are supposed to admit densities according to:

$$\mathscr{P}^{(i)} = \int_{\mathscr{D}} p^{(i)}\mathrm{d}V, \quad \mathscr{P}^{(c)} = \int_{\partial\mathscr{D}} p^{(c)}\mathrm{d}S. \tag{5}$$

The densities are then taken as linear forms on the appropriate set of generalized virtual motions. Objectivity requirements reduce the number of the generalized virtual motions eventually appearing in the power densities. For each continuum, the final expressions of $p^{(i)}$ and $p^{(c)}$ are given in the corresponding Section. The application of the principle of virtual power and the use of the divergence theorem lead to the balance equations and the associated boundary conditions.

(ii) Energy principle

The specific internal energy ϵ, entropy η and Helmholtz free energy $\Psi = \epsilon - T\eta$ are introduced as functions of state and internal variables. The following local form of the energy principle is retained in this work:

$$\rho\dot{\epsilon} = p^{(i)} - \underline{\mathbf{Q}}.\boldsymbol{\nabla}, \tag{6}$$

where $\underline{\mathbf{Q}}$ is the heat flux vector.

(iii) Entropy principle

The global form of the second principle reads: $\dot{\mathscr{S}} \geq \mathscr{Q}_s$, where $\mathscr{S}$ is the global entropy of the system and $\mathscr{Q}_s$ is the total flux of entropy,

$$\mathscr{S} = \int_V \rho\eta \mathrm{d}V, \quad \mathscr{Q}_s = -\int_{\partial V} \underline{\mathbf{J}}_\eta.\underline{\mathbf{n}}\mathrm{d}S \quad \text{and} \quad \underline{\mathbf{J}}_\eta = \frac{\underline{\mathbf{Q}}}{T}, \tag{7}$$

where $\underline{\mathbf{J}}_\eta$ is the entropy flux vector. No extra–entropy flux is assumed in the present theory, although this can be considered for internal variables influencing the heat conduction as in [10]. For the consideration of degrees of freedom, an energy approach is definitely preferred here (see [21], [22] and [23] for a discussion on this point and the consequences for the heat equation). This question is briefly addressed in Sect. 4.2. The following local form of the entropy inequality is adopted:

$$\rho\dot{\eta} + \underline{\mathbf{J}}_\eta.\boldsymbol{\nabla} \geq 0. \tag{8}$$

Combining (6) and (8), the Clausius–Duhem inequality is obtained:

$$-\rho(\dot{\Psi} + \eta\dot{T}) + p^{(i)} - \frac{1}{T}\underline{\mathbf{Q}}.\boldsymbol{\nabla}T \geq 0. \tag{9}$$

This inequality is used to derive the state laws and the remaining intrinsic dissipation D. The case of second grade thermoelastoviscoplasticity incorporating also the temperature as full degree of freedom, i.e. including also its gradient in the free energy, leads to a modification of the state law for the entropy. This is not addressed here (see [22]).

Since the present work mainly deals with the constitutive framework for material modelling, the derivation of balance equations is not detailed for each continuum. The equations of balance of momentum and of moment of momentum are recalled in Appendix B. For the same reason, the form of the boundary conditions required for the well-posedness of a boundary value problem on V will not be given systematically.

For the clarity of the presentation, the constitutive frameworks are developed within the assumption of small perturbations, in the isothermal case. Section 6 provides the necessary extensions for application to finite deformation.

2 Cosserat media

2.1 Standard Cosserat materials

In a Cosserat medium, each material point is endowed with translational degrees of freedom $\underline{\mathbf{u}}$ and independent rotational degrees of freedom $\underline{\boldsymbol{\phi}}$:

$$\mathscr{V} = \{\underline{\dot{\mathbf{u}}}, \underline{\dot{\mathbf{u}}} \otimes \boldsymbol{\nabla}, \underline{\dot{\boldsymbol{\phi}}}, \underline{\dot{\boldsymbol{\phi}}} \otimes \boldsymbol{\nabla}\}, \quad \mathscr{V}^c = \{\underline{\dot{\mathbf{u}}}, \underline{\dot{\boldsymbol{\phi}}}\}. \tag{10}$$

The corresponding strain measures are the deformation tensor $\underset{\sim}{\mathbf{e}}$ and the curvature tensor $\underset{\sim}{\boldsymbol{\kappa}}$:

$$\underset{\sim}{\mathbf{e}} = \underline{\mathbf{u}} \otimes \boldsymbol{\nabla} + \underset{\approx}{\boldsymbol{\epsilon}}.\underline{\boldsymbol{\phi}}, \quad \underset{\sim}{\boldsymbol{\kappa}} = \underline{\boldsymbol{\phi}} \otimes \boldsymbol{\nabla}. \tag{11}$$

The dual quantities in the virtual power of internal forces are the non–symmetric tensors of force and couple–stresses, $\underset{\sim}{\boldsymbol{\sigma}}$ and $\underset{\sim}{\boldsymbol{\mu}}$, respectively:

$$p^{(i)} = \underset{\sim}{\boldsymbol{\sigma}} : \dot{\underset{\sim}{\mathbf{e}}} + \underset{\sim}{\boldsymbol{\mu}} : \dot{\underset{\sim}{\boldsymbol{\kappa}}}. \tag{12}$$

The total deformation and curvature tensors are decomposed into elastic and plastic parts according to:

$$\underset{\sim}{\mathbf{e}} = \underset{\sim}{\mathbf{e}}^e + \underset{\sim}{\mathbf{e}}^p, \quad \underset{\sim}{\boldsymbol{\kappa}} = \underset{\sim}{\boldsymbol{\kappa}}^e + \underset{\sim}{\boldsymbol{\kappa}}^p. \tag{13}$$

The free energy is taken as a function of elastic deformation and curvature and internal variable(s) q accounting for material hardening: $\Psi(\underset{\sim}{\mathbf{e}}^e, \underset{\sim}{\boldsymbol{\kappa}}^e, q)$. As a result, the intrinsic dissipation takes the form:

$$\begin{aligned} D &= \underset{\sim}{\boldsymbol{\sigma}} : \dot{\underset{\sim}{\mathbf{e}}} + \underset{\sim}{\boldsymbol{\mu}} : \dot{\underset{\sim}{\boldsymbol{\kappa}}} - \rho\dot{\Psi} \\ &= \left(\underset{\sim}{\boldsymbol{\sigma}} - \rho\frac{\partial\Psi}{\partial\underset{\sim}{\mathbf{e}}^e}\right) : \dot{\underset{\sim}{\mathbf{e}}}^e + \left(\underset{\sim}{\boldsymbol{\mu}} - \rho\frac{\partial\Psi}{\partial\underset{\sim}{\boldsymbol{\kappa}}^e}\right) : \dot{\underset{\sim}{\boldsymbol{\kappa}}}^e + \underset{\sim}{\boldsymbol{\sigma}} : \dot{\underset{\sim}{\mathbf{e}}}^p + \underset{\sim}{\boldsymbol{\mu}} : \dot{\underset{\sim}{\boldsymbol{\kappa}}}^p - \rho\frac{\partial\Psi}{\partial q}\dot{q} \end{aligned} \tag{14}$$

from which the state laws are deduced :

$$\underset{\sim}{\boldsymbol{\sigma}} = \rho\frac{\partial\Psi}{\partial\underset{\sim}{\mathbf{e}}^e}, \quad \underset{\sim}{\boldsymbol{\mu}} = \rho\frac{\partial\Psi}{\partial\underset{\sim}{\boldsymbol{\kappa}}^e}, \quad R = \rho\frac{\partial\Psi}{\partial q}. \tag{15}$$

The classical theory of so-called standard materials proposed in [24], [17] and [25] can be extended to Cosserat media by choosing a viscoplastic potential $\Omega(\underset{\sim}{\boldsymbol{\sigma}}, \underset{\sim}{\boldsymbol{\mu}}, R)$, the so-called pseudo–potential of dissipation, such that:

$$\dot{\underset{\sim}{\mathbf{e}}}^p = \frac{\partial\Omega}{\partial\underset{\sim}{\boldsymbol{\sigma}}}, \quad \dot{\underset{\sim}{\boldsymbol{\kappa}}}^p = \frac{\partial\Omega}{\partial\underset{\sim}{\boldsymbol{\mu}}}, \quad \dot{q} = -\frac{\partial\Omega}{\partial R}. \tag{16}$$

The convexity of the potential Ω with respect to its variables then ensures the positivity of intrinsic dissipation. The Legendre–Fenchel transform can be used to define the convex dual potential $\Omega^\star(\dot{\underset{\sim}{\mathbf{e}}}^p, \dot{\underset{\sim}{\boldsymbol{\kappa}}}^p, \dot{q})$ such that:

$$\underset{\sim}{\boldsymbol{\sigma}} = \frac{\partial\Omega^\star}{\partial\dot{\underset{\sim}{\mathbf{e}}}^p}, \quad \underset{\sim}{\boldsymbol{\mu}} = \frac{\partial\Omega^\star}{\partial\dot{\underset{\sim}{\boldsymbol{\kappa}}}^p}, \quad R = -\frac{\partial\Omega^\star}{\partial\dot{q}}. \tag{17}$$

2.2 Single vs. multi–criterion Cosserat plasticity

Two main classes of potentials have been used in the past. In the first class, the potential is a coupled function of force and couple-stresses, whereas in the second class the potential is a sum of two independent functions of force-stress and couple-stress, respectively:

$$\Omega_{tot} = \Omega\left(\underset{\sim}{\boldsymbol{\sigma}}, R\right) + \Omega_c\left(\underset{\sim}{\boldsymbol{\mu}}, R_c\right) \tag{18}$$

in the spirit of [26] and [27]. Both situations can be illustrated for the rate-independent material behavior. The first class of models involves a single yield function $f(\underset{\sim}{\boldsymbol{\sigma}}, \underset{\sim}{\boldsymbol{\mu}}, R)$ and a single plastic multiplier $\dot{p}$:

$$\dot{\underset{\sim}{\mathbf{e}}}^p = \dot{p}\frac{\partial f}{\partial\underset{\sim}{\boldsymbol{\sigma}}}, \quad \dot{\underset{\sim}{\boldsymbol{\kappa}}}^p = \dot{p}\frac{\partial f}{\partial\underset{\sim}{\boldsymbol{\mu}}}, \quad \dot{q} = -\dot{p}\frac{\partial f}{\partial R}. \tag{19}$$

The second class of models requires two yield functions $f(\underset{\sim}{\boldsymbol{\sigma}},R,R_c)$ and $f_c(\underset{\sim}{\boldsymbol{\mu}},R,R_c)$ and two plastic multipliers:

$$\dot{\underset{\sim}{\mathbf{e}}}^p = \dot{p}\frac{\partial f}{\partial \underset{\sim}{\boldsymbol{\sigma}}}, \quad \dot{\underset{\sim}{\boldsymbol{\kappa}}}^p = \dot{\kappa}\frac{\partial f_c}{\partial \underset{\sim}{\boldsymbol{\mu}}}, \quad \dot{q} = -\dot{p}\frac{\partial f}{\partial R}, \quad \dot{q}_c = -\dot{\kappa}\frac{\partial f_c}{\partial R_c}. \tag{20}$$

In the latter case, coupling between deformation and curvature comes from the balance equations and possibly coupled hardening laws. This type of coupling between several hardening variables has been investigated within the framework of multi-mechanism based plasticity theory in [28] for the classical continuum. The treatment of the Cosserat continuum is very similar.

The first trials for an extension of classical von Mises elastoplasticity to the Cosserat continuum are due to [29], [30], [8], [5] and [31], [32]. They belong to the class of single criterion plasticity models. The following form of the extended von Mises criterion encompasses these previous models:

$$f(\underset{\sim}{\boldsymbol{\sigma}},\underset{\sim}{\boldsymbol{\mu}},R) = J_2(\underset{\sim}{\boldsymbol{\sigma}},\underset{\sim}{\boldsymbol{\mu}}) - R(p), \tag{21}$$

$$J_2(\underset{\sim}{\boldsymbol{\sigma}},\underset{\sim}{\boldsymbol{\mu}}) = \sqrt{a_1\underset{\sim}{\boldsymbol{\sigma}}' : \underset{\sim}{\boldsymbol{\sigma}}' + a_2\underset{\sim}{\boldsymbol{\sigma}}' : \underset{\sim}{\boldsymbol{\sigma}}'^T + b_1\underset{\sim}{\boldsymbol{\mu}} : \underset{\sim}{\boldsymbol{\mu}} + b_2\underset{\sim}{\boldsymbol{\mu}} : \underset{\sim}{\boldsymbol{\mu}}^T}, \tag{22}$$

where $\underset{\sim}{\boldsymbol{\sigma}}'$ is the deviatoric part of $\underset{\sim}{\boldsymbol{\sigma}}$, and a_i, b_i are material parameters. The flow rules and plastic multiplier then read:

$$\dot{\underset{\sim}{\mathbf{e}}}^p = \dot{p}\frac{a_1\underset{\sim}{\boldsymbol{\sigma}}' + a_2\underset{\sim}{\boldsymbol{\sigma}}'^T}{J_2(\underset{\sim}{\boldsymbol{\sigma}},\underset{\sim}{\boldsymbol{\mu}})}, \quad \dot{\underset{\sim}{\boldsymbol{\kappa}}}^p = \dot{p}\frac{b_1\underset{\sim}{\boldsymbol{\mu}} + b_2\underset{\sim}{\boldsymbol{\mu}}^T}{J_2(\underset{\sim}{\boldsymbol{\sigma}},\underset{\sim}{\boldsymbol{\mu}})}, \tag{23}$$

$$\dot{p} = \sqrt{\frac{a_1}{a_1^2 - a_2^2}\dot{\underset{\sim}{\mathbf{e}}}^p : \dot{\underset{\sim}{\mathbf{e}}}^p + \frac{a_2}{a_2^2 - a_1^2}\dot{\underset{\sim}{\mathbf{e}}}^p : \dot{\underset{\sim}{\mathbf{e}}}^{pT} + \frac{b_1}{b_1^2 - b_2^2}\dot{\underset{\sim}{\boldsymbol{\kappa}}}^p : \dot{\underset{\sim}{\boldsymbol{\kappa}}}^p + \frac{b_2}{b_2^2 - b_1^2}\dot{\underset{\sim}{\boldsymbol{\kappa}}}^p : \dot{\underset{\sim}{\boldsymbol{\kappa}}}^{pT}}. \tag{24}$$

The use of the consistency condition $\dot{f} = 0$ under plastic loading yields the following expression of the plastic multiplier:

$$\dot{p} = \frac{\underset{\sim}{\mathbf{N}} : \underset{\approx}{\mathbf{E}} : \dot{\underset{\sim}{\mathbf{e}}} + \underset{\sim}{\mathbf{N}}_c : \underset{\approx}{\mathbf{C}} : \dot{\underset{\sim}{\boldsymbol{\kappa}}}}{H + \underset{\sim}{\mathbf{N}} : \underset{\approx}{\mathbf{E}} : \underset{\sim}{\mathbf{N}} + \underset{\sim}{\mathbf{N}}_c : \underset{\approx}{\mathbf{C}} : \underset{\sim}{\mathbf{N}}_c}. \tag{25}$$

This expression involves the normal tensors $\underset{\sim}{\mathbf{N}}$ and $\underset{\sim}{\mathbf{N}}_c$ to the yield surface, the hardening modulus H and the tensors of elastic moduli $\underset{\approx}{\mathbf{E}}$ and $\underset{\approx}{\mathbf{C}}$ for linear elasticity (for a material admitting at least point symmetry):

$$\underset{\sim}{\mathbf{N}}_c = \frac{\partial f}{\partial \underset{\sim}{\boldsymbol{\sigma}}}, \quad \underset{\sim}{\mathbf{N}}_c = \frac{\partial f}{\partial \underset{\sim}{\boldsymbol{\mu}}}, \quad H = \frac{\partial R}{\partial p}, \quad \underset{\approx}{\mathbf{E}} = \frac{\partial^2 \Psi}{\partial \underset{\sim}{\mathbf{e}}^e \partial \underset{\sim}{\mathbf{e}}^e}, \quad \underset{\approx}{\mathbf{C}} = \frac{\partial^2 \Psi}{\partial \underset{\sim}{\boldsymbol{\kappa}}^e \partial \underset{\sim}{\boldsymbol{\kappa}}^e}. \tag{26}$$

The condition of plastic loading for the material point is that the numerator of Eq. (25) is positive, provided that the denominator remains positive, which still allows softening behaviors ($H < 0$).

This is however not the only possible extension of von Mises plasticity since a multi–criterion framework can also be adopted:

$$f(\underset{\sim}{\boldsymbol{\sigma}},R) = J_2(\underset{\sim}{\boldsymbol{\sigma}}) - R(p,\kappa), \quad f_c(\underset{\sim}{\boldsymbol{\mu}},R_c) = J_2(\underset{\sim}{\boldsymbol{\mu}}) - R_c(p,\kappa), \tag{27}$$

$$J_2(\underset{\sim}{\boldsymbol{\sigma}}) = \sqrt{a_1 \underset{\sim}{\boldsymbol{\sigma}}' : \underset{\sim}{\boldsymbol{\sigma}}' + a_2 \underset{\sim}{\boldsymbol{\sigma}}' : \underset{\sim}{\boldsymbol{\sigma}}'^T}, \quad J_2(\underset{\sim}{\boldsymbol{\mu}}) = \sqrt{b_1 \underset{\sim}{\boldsymbol{\mu}} : \underset{\sim}{\boldsymbol{\mu}} + b_2 \underset{\sim}{\boldsymbol{\mu}} : \underset{\sim}{\boldsymbol{\mu}}^T}. \tag{28}$$

There are then two distinct plastic multipliers

$$\dot{p} = \sqrt{\frac{a_1}{a_1^2 - a_2^2} \underset{\sim}{\dot{\mathbf{e}}}^p : \underset{\sim}{\dot{\mathbf{e}}}^p + \frac{a_2}{a_2^2 - a_1^2} \underset{\sim}{\dot{\mathbf{e}}}^p : \underset{\sim}{\dot{\mathbf{e}}}^{pT}}, \quad \dot{\kappa} = \sqrt{\frac{b_1}{b_1^2 - b_2^2} \underset{\sim}{\dot{\boldsymbol{\kappa}}}^p : \underset{\sim}{\dot{\boldsymbol{\kappa}}}^p + \frac{b_2}{b_2^2 - b_1^2} \underset{\sim}{\dot{\boldsymbol{\kappa}}}^p : \underset{\sim}{\dot{\boldsymbol{\kappa}}}^{pT}}. \tag{29}$$

The exploitation of two consistency conditions $\dot{f} = 0$ and $\dot{f_c} = 0$ under plastic loading leads to a system of two equations for the unknowns $\dot{p}, \dot{\kappa}$:

$$(H + \underset{\sim}{\mathbf{N}} : \underset{\approx}{\mathbf{E}} : \underset{\sim}{\mathbf{N}})\dot{p} + H^{pc} = \underset{\sim}{\mathbf{N}} : \underset{\approx}{\mathbf{E}} : \underset{\sim}{\dot{\mathbf{e}}}, \quad H^{pc}\dot{p} + (H^c + \underset{\sim}{\mathbf{N}}_c : \underset{\approx}{\mathbf{C}} : \underset{\sim}{\mathbf{N}}_c)\dot{\kappa} = \underset{\sim}{\mathbf{N}}_c : \underset{\approx}{\mathbf{C}} : \underset{\sim}{\dot{\boldsymbol{\kappa}}}, \tag{30}$$

where a coupling hardening modulus appears: $H^{pc} = \partial R / \partial \kappa = \partial R_c / \partial p$. Whether both plastic mechanisms are active or not is determined by the sign of the solutions $(\dot{p}, \dot{\kappa})$ of the previous system. If the determinant of the system vanishes, the value of the plastic multipliers can remain undeterminate [27]. The choice of one viscoplastic potential for deformation or curvature can be used as a regularization procedure to settle the undeterminacy:

$$\dot{p} = -\frac{\partial \Omega}{\partial R} \quad \text{or} \quad \dot{\kappa} = -\frac{\partial \Omega_c}{\partial R_c}. \tag{31}$$

Such mixed plastic-viscoplastic potentials are already recommended in the classical case [33], [28].

An example of multi–mechanism elastoviscoplastic Cosserat material is the case of Cosserat crystal plasticity described in [13], [34]. Single and multi–criterion plasticity including generalized kinematic hardening variables can be found in [35]. Non–associative flow rules are necessary in the case of geomaterials for which the yield function appearing in Eqs. (19) and (20) must be replaced by a different function of the same arguments. Some extensions of classical compressible plasticity models are reported in [36].

2.3 Application to simple glide and bending

It is important to see the respective role of Cosserat characteristic lengths appearing in the elastic and plastic constitutive equations in some simple situations. The difference between the use of single or multi-mechanism Cosserat elastoplasticity can also be shown. Analytical solutions for an isotropic elastic-ideally plastic Cosserat material involving one or two yield functions can be worked out in the case of the Cosserat glide and bending tests. The considered boundary value problems are depicted on Figs. 1 and 2, respectively. The detailed solutions are provided in Appendices C.1 and C.2. Two characteristic lengths can be defined:

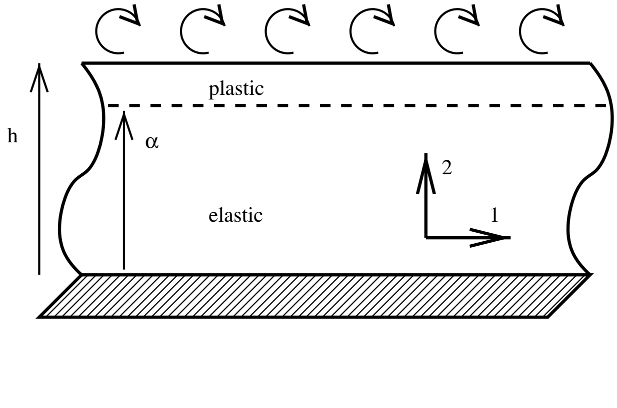

Fig. 1. Simple glide test for a Cosserat infinite layer: elastic and elastoplastic domains, boundary conditions

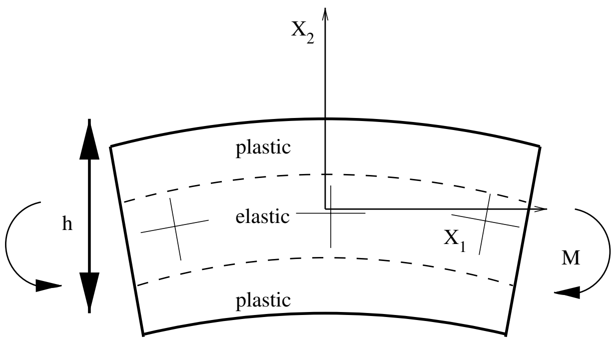

Fig. 2. Simple bending test for a Cosserat material: elastic and elasto-plastic domains, boundary conditions

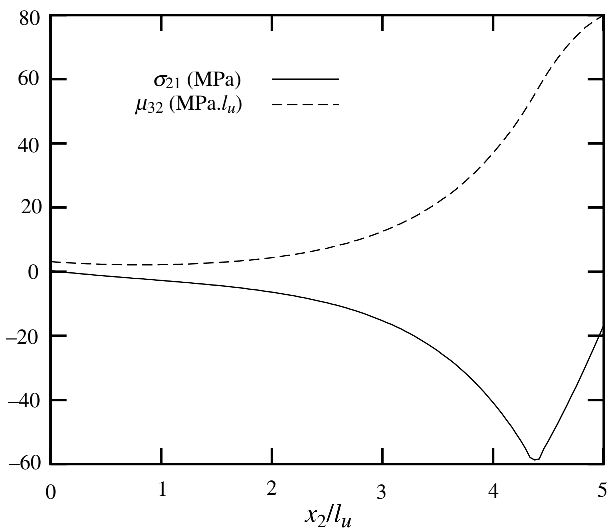

Fig. 3. Simple glide test for a single criterion von Mises elastoplastic Cosserat infinite layer: force stress and couple stress profiles along a vertical line. A micro–rotation $\phi = 0.001$ is prescribed at the top $h = 5l_u$. The material parameters are: E = 200000 MPa, $\nu = 0.3$, $\mu_c = 100000$ MPa, β =76923 MPa.l_u^2, $R_0 = 100$ MPa, $a_1 = 1.5, a_2 = 0, b_1 = 1.5l_u^{-2}$, $b_2 = 0$. The micro–couple prescribed at the top is $\mu_{32}^0 = 80$ MPa.l_u. l_u is a length unit

$$l_e = \sqrt{\frac{\beta}{\mu}}, \quad l_p = \sqrt{\frac{a}{b}}; \tag{32}$$

in the simple case $a_1 = a, a_2 = 0, b_1 = b, b_2 = 0$ (see also Eq. (C.7) for the definition of isotropic Cosserat elastic bending modulus β). In the glide and bend tests, the material can be divided into elastic and plastic zones (Figs. 1 and 2). The characteristic length l_e explicitly appears in the solution in the elastic zone, whereas the solution in the plastic zone is driven by length l_p. Classical solutions are retrieved for vanishing l_e and l_p.

The use of a single coupled yield criterion (22) leads to a non-homogeneous distribution of force and couple stress in the plastic zone for both glide and bending, as can be seen from Figs. 3 and 4. In contrast, if no hardening is introduced, the use of two uncoupled criteria (28) gives rise to constant values of the force and couple–stress components in the plastic zone of the bent beam.

2.4 Cosserat theory with a special dependence on plastic deformation as additional degree of freedom

Looking at the definition of the Cosserat deformation measures (11), it appears that the curvature tensor and the gradient of Cosserat deformation are related by the relation:

$$\underset{\sim}{\mathbf{e}} \otimes \boldsymbol{\nabla} = \underline{\mathbf{u}} \otimes \boldsymbol{\nabla} \otimes \boldsymbol{\nabla} + \underset{\approx}{\boldsymbol{\epsilon}} . \underset{\sim}{\boldsymbol{\kappa}}. \tag{33}$$

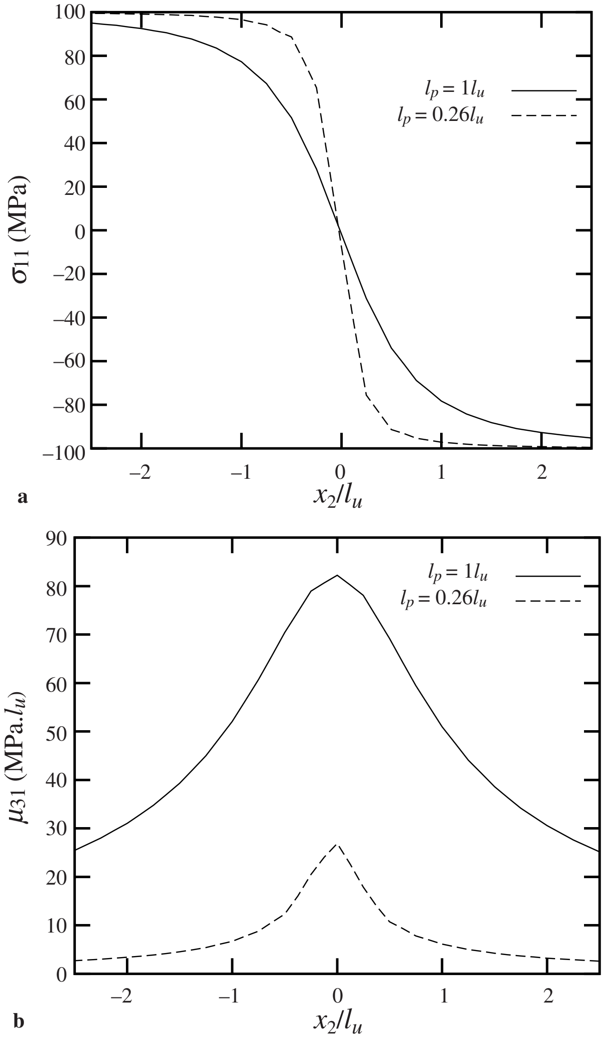

Fig. 4. Simple bending test of an elastoplastic Cosserat material: influence of the characteristic length l_p on the profiles of stress components σ_{11} and μ_{31}, obtained for a fully plastic beam. The parameters are the same as in Fig. 3 except that $b = 15$ when $l_c = 0.26 l_u$. The beam thickness is $h = 5 l_u$

Using the compatibility requirements, the gradient of the mean rotation of the displacement field can be reduced to the gradient of the symmetric part of the displacement gradient [1], [37]:

$$\underline{\mathbf{u}} \otimes \nabla \otimes \nabla = \underset{\sim}{\boldsymbol{\varepsilon}} \otimes \nabla - \underset{\approx}{\underline{\boldsymbol{\epsilon}}} \cdot \underset{\simeq}{\boldsymbol{\epsilon}} : (\nabla \otimes \underset{\sim}{\boldsymbol{\varepsilon}}). \tag{34}$$

As a result, the curvature tensor is uniquely determined by the gradient of Cosserat deformation. This expression reads:

$$\underset{\sim}{\boldsymbol{\kappa}} = \frac{1}{2} \underset{\simeq}{\boldsymbol{\epsilon}} : (\underset{\sim}{\mathbf{e}} \otimes \nabla + \nabla \otimes (\underset{\sim}{\mathbf{e}} + \underset{\sim}{\mathbf{e}}^T)). \tag{35}$$

This equation is explicited with indices in Appendix 8, Eq. (A.14). A five–rank linear operator $\underset{\approx}{\underline{\aleph}}$ can therefore be defined such that:

$$\underset{\sim}{\boldsymbol{\kappa}} = \underset{\approx}{\underline{\aleph}} \vdots (\underset{\sim}{\mathbf{e}} \otimes \nabla). \tag{36}$$

Thus it is natural to identify the elastic and plastic parts (13) of total curvature as follows [38]:

$$\underset{\sim}{\boldsymbol{\kappa}}^e \hat{=} \underset{\approx}{\underline{\aleph}} \vdots (\underset{\sim}{\mathbf{e}}^e \otimes \nabla), \quad \underset{\sim}{\boldsymbol{\kappa}}^p \hat{=} \underset{\approx}{\underline{\aleph}} \vdots (\underset{\sim}{\mathbf{e}}^p \otimes \nabla), \tag{37}$$

so that a specific flow rule for $\dot{\underset{\sim}{\boldsymbol{\kappa}}}^p$ becomes unnecessary, provided that the entire driving force for the evolution of $\underset{\sim}{\mathbf{e}}^p$ is known. This economical model represents an alternative to the standard framework proposed in the previous sections. However it involves by $\underset{\sim}{\mathbf{e}}^e \otimes \boldsymbol{\nabla} = \underset{\sim}{\mathbf{e}} \otimes \boldsymbol{\nabla} - \underset{\sim}{\mathbf{e}}^p \otimes \boldsymbol{\nabla}$ also the second gradient of displacement in the constitutive equations. Therefore, a consistent thermomechanical framework must be formulated, that ensures a positive dissipation rate. For that, the classical Cosserat continuum theory has to be enlarged.

This enriched theory can be seen as that of a Cosserat medium incorporating a part of the gradient of the plastic deformation $\underset{\sim}{\mathbf{e}}^p$ as additional degree of freedom. In this case, the sets of modelling quantities include:

$$\mathscr{V} = \left\{ \dot{\underline{\mathbf{u}}}, \dot{\underline{\mathbf{u}}} \otimes \boldsymbol{\nabla},\ \dot{\underline{\boldsymbol{\phi}}}, \dot{\underline{\boldsymbol{\phi}}} \otimes \boldsymbol{\nabla}, \dot{\underset{\sim}{\mathbf{e}}}^p, \underset{\approx}{\underline{\boldsymbol{\aleph}}} \vdots \left(\dot{\underset{\sim}{\mathbf{e}}}^p \otimes \boldsymbol{\nabla} \right) \right\}, \quad \mathscr{V}^c = \{ \dot{\underline{\mathbf{u}}}, \dot{\underline{\boldsymbol{\phi}}}, \dot{\underset{\sim}{\mathbf{e}}}^p \}. \tag{38}$$

According to the virtual power method [19], the power of internal forces must be a linear form with respect to all modelling variables. This argument leads to the introduction of four generalized stress tensors:

$$p^{(i)} = \underset{\sim}{\boldsymbol{\sigma}} : \dot{\underset{\sim}{\mathbf{e}}} + \underset{\sim}{\boldsymbol{\mu}} : \dot{\underset{\sim}{\boldsymbol{\kappa}}} + \underset{\sim}{\mathbf{A}} : \dot{\underset{\sim}{\mathbf{e}}}^p + (\underset{\sim}{\mathbf{B}} - \underset{\sim}{\boldsymbol{\mu}}) : \left(\underset{\approx}{\underline{\boldsymbol{\aleph}}} \vdots \left(\dot{\underset{\sim}{\mathbf{e}}}^p \otimes \boldsymbol{\nabla} \right) \right). \tag{39}$$

The reason for introducing the difference $(\underset{\sim}{\mathbf{B}} - \underset{\sim}{\boldsymbol{\mu}})$ instead of a single generalized stress tensor is a separation of the roles played by tensors $\underset{\sim}{\boldsymbol{\mu}}$ and $\underset{\sim}{\mathbf{B}}$: they work then on $\underset{\sim}{\boldsymbol{\kappa}}^e$ and $\underset{\sim}{\boldsymbol{\kappa}}^p$, respectively. Tensor $\underset{\sim}{\mathbf{A}}$ is at first introduced for the sake of generality, but it will in fact play an important role in the exploitation of the second principle. The expression (B.1) of the power of contact forces must also be extended:

$$p^{(c)} = \underline{\mathbf{t}}.\dot{\underline{\mathbf{u}}} + \underline{\mathbf{m}}.\dot{\underline{\boldsymbol{\phi}}} + \underset{\sim}{\mathbf{A}}^c : \dot{\underset{\sim}{\mathbf{e}}}^p, \tag{40}$$

where $\underline{\mathbf{t}}, \underline{\mathbf{m}}$ and $\underset{\sim}{\mathbf{A}}^c$ are surface simple, couple and generalized force vectors and tensor. The application of the principle of virtual power (4) leads, on the one hand, to the balance equations (B.2) and boundary conditions (B1.2) and, on the other hand, to two additional conditions:

$$\underset{\sim}{\mathbf{A}} = ((\underset{\sim}{\mathbf{B}} - \underset{\sim}{\boldsymbol{\mu}}) : \underset{\approx}{\underline{\boldsymbol{\aleph}}}).\boldsymbol{\nabla}, \quad \underset{\sim}{\mathbf{A}}^c = ((\underset{\sim}{\mathbf{B}} - \underset{\sim}{\boldsymbol{\mu}}) : \underset{\approx}{\underline{\boldsymbol{\aleph}}}).\underline{\mathbf{n}}. \tag{41}$$

The following explicit dependence of the Helmholtz free energy is chosen: $\Psi(\underset{\sim}{\mathbf{e}}^e, \underset{\sim}{\boldsymbol{\kappa}}^e, \underset{\sim}{\boldsymbol{\kappa}}^p, q)$. The intrinsic dissipation (14) now becomes:

$$\begin{aligned} D = {} & \underset{\sim}{\boldsymbol{\sigma}} : \dot{\underset{\sim}{\mathbf{e}}} + \underset{\sim}{\boldsymbol{\mu}} : \dot{\underset{\sim}{\boldsymbol{\kappa}}} + \underset{\sim}{\mathbf{A}} : \dot{\underset{\sim}{\mathbf{e}}}^p + (\underset{\sim}{\mathbf{B}} - \underset{\sim}{\boldsymbol{\mu}}) : \dot{\underset{\sim}{\boldsymbol{\kappa}}}^p \\ & - \left(\rho \frac{\partial \Psi}{\partial \underset{\sim}{\mathbf{e}}^e} : \dot{\underset{\sim}{\mathbf{e}}}^e + \rho \frac{\partial \Psi}{\partial \underset{\sim}{\boldsymbol{\kappa}}^e} : \dot{\underset{\sim}{\boldsymbol{\kappa}}}^e + \rho \frac{\partial \Psi}{\partial \underset{\sim}{\boldsymbol{\kappa}}^p} : \dot{\underset{\sim}{\boldsymbol{\kappa}}}^p + \rho \frac{\partial \Psi}{\partial q} \dot{q} \right) \end{aligned} \tag{42}$$

from which the state laws are deduced:

$$\underset{\sim}{\boldsymbol{\sigma}} = \rho \frac{\partial \Psi}{\partial \underset{\sim}{\mathbf{e}}^e}, \quad \underset{\sim}{\boldsymbol{\mu}} = \rho \frac{\partial \Psi}{\partial \underset{\sim}{\boldsymbol{\kappa}}^e}, \quad \underset{\sim}{\mathbf{B}} = \rho \frac{\partial \Psi}{\partial \underset{\sim}{\boldsymbol{\kappa}}^p}, \quad R = \rho \frac{\partial \Psi}{\partial q}. \tag{43}$$

The intrinsic dissipation then reduces to:

$$D = \underset{\sim}{\boldsymbol{\tau}}^{eff} : \dot{\underset{\sim}{\mathbf{e}}}^p - R\dot{q}, \quad \text{with} \quad \underset{\sim}{\boldsymbol{\tau}}^{eff} \hat{=} \underset{\sim}{\boldsymbol{\sigma}} + \underset{\sim}{\mathbf{A}} = \underset{\sim}{\boldsymbol{\sigma}} + ((\underset{\sim}{\mathbf{B}} - \underset{\sim}{\boldsymbol{\mu}}) : \underset{\approx}{\underline{\boldsymbol{\aleph}}}).\boldsymbol{\nabla}. \tag{44}$$

The driving force for the activation of plastic flow is therefore not only $\underset{\sim}{\boldsymbol{\sigma}}$ but a sort of effective stress $\underset{\sim}{\boldsymbol{\tau}}^{eff}$. A systematic way of ensuring the positivity of the dissipation is to choose again a pseudo-potential of dissipation $\Omega(\underset{\sim}{\boldsymbol{\tau}}^{eff}, R)$, convex in its arguments:

$$\dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p = \frac{\partial \Omega}{\partial \underset{\sim}{\boldsymbol{\tau}}^{eff}}, \quad \dot{q} = -\frac{\partial \Omega}{\partial R}. \tag{45.1, 2}$$

For rate-independent materials and associative plasticity, the potential is the yield function $f(\underset{\sim}{\boldsymbol{\tau}}^{eff}, R)$. Since the stress tensor $\underset{\sim}{\boldsymbol{\tau}}^{eff}$ contains divergence terms according to (44), the yield condition becomes a partial differential equation. It is similar to the case of material models incorporating gradients of internal variables, analyzed in Sect. 4.

3 Strain gradient models

Early attempts of constitutive frameworks for elastoplastic strain gradient media go back to [39] and [40]. They fall into the class of standard strain gradient materials eventually settled in [15] and [41].

3.1 Standard strain gradient materials

For a second-grade continuum, the sets of modelling quantities in the volume and at a surface are:

$$\mathscr{V} = \{\dot{\underline{\mathbf{u}}},\ \dot{\underline{\mathbf{u}}} \otimes \boldsymbol{\nabla}, \dot{\underline{\mathbf{u}}} \otimes \boldsymbol{\nabla} \otimes \boldsymbol{\nabla}\}, \quad \mathscr{V}^c = \{\dot{\underline{\mathbf{u}}},\ D_n \dot{\underline{\mathbf{u}}}\}. \tag{46}$$

The corresponding strain measures can be taken as

$$\underset{\sim}{\boldsymbol{\varepsilon}} = \tfrac{1}{2}(\boldsymbol{\nabla} \otimes \underline{\mathbf{u}} + \underline{\mathbf{u}} \otimes \boldsymbol{\nabla}), \quad \underset{\approx}{\underline{\mathbf{K}}} = \underset{\sim}{\boldsymbol{\varepsilon}} \otimes \boldsymbol{\nabla}. \tag{47}$$

The dual quantities in the virtual power of internal forces are the simple and double force stress tensors $\underset{\sim}{\boldsymbol{\sigma}}$ and $\underset{\approx}{\underline{\mathbf{S}}}$ (also called hyperstress tensor), respectively:

$$p^{(i)} = \underset{\sim}{\boldsymbol{\sigma}} : \dot{\underset{\sim}{\boldsymbol{\varepsilon}}} + \underset{\approx}{\underline{\mathbf{S}}} \;\vdots\; \dot{\underset{\approx}{\underline{\mathbf{K}}}}. \tag{48}$$

The balance equations resulting from the application of the principle of virtual power are given in Appendix B. An additive partition of both strain and strain gradient into elastic and plastic parts is postulated:

$$\underset{\sim}{\boldsymbol{\varepsilon}} = \underset{\sim}{\boldsymbol{\varepsilon}}^e + \underset{\sim}{\boldsymbol{\varepsilon}}^p, \quad \underset{\approx}{\underline{\mathbf{K}}} = \underset{\approx}{\underline{\mathbf{K}}}^e + \underset{\approx}{\underline{\mathbf{K}}}^p. \tag{49}$$

The free energy is taken as a function of elastic strain and strain gradient, and internal variable(s) q accounting for material hardening : $\Psi(\underset{\sim}{\boldsymbol{\varepsilon}}^e, \underset{\approx}{\underline{\mathbf{K}}}^e, q)$. As a result, the intrinsic dissipation takes the form:

$$D = \underset{\sim}{\boldsymbol{\sigma}} : \dot{\underset{\sim}{\boldsymbol{\varepsilon}}} + \underset{\approx}{\underline{\mathbf{S}}} \vdots \dot{\underset{\approx}{\underline{\mathbf{K}}}} - \rho\dot{\Psi} = \left(\underset{\sim}{\boldsymbol{\sigma}} - \rho\frac{\partial \Psi}{\partial \underset{\sim}{\boldsymbol{\varepsilon}}^e}\right) : \dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^e + \left(\underset{\approx}{\underline{\mathbf{S}}} - \rho\frac{\partial \Psi}{\partial \underset{\approx}{\underline{\mathbf{K}}}^e}\right) \vdots \dot{\underset{\approx}{\underline{\mathbf{K}}}}^e + \underset{\sim}{\boldsymbol{\sigma}} : \dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p + \underset{\approx}{\underline{\mathbf{S}}} \vdots \dot{\underset{\approx}{\underline{\mathbf{K}}}}^p - \rho\frac{\partial \Psi}{\partial q}\dot{q} \tag{50}$$

from which the state laws are deduced:

$$\underset{\sim}{\boldsymbol{\sigma}} = \rho\frac{\partial \Psi}{\partial \underset{\sim}{\boldsymbol{\varepsilon}}^e}, \quad \underset{\approx}{\underline{\mathbf{S}}} = \rho\frac{\partial \Psi}{\partial \underset{\approx}{\underline{\mathbf{K}}}^e}, \quad R = \rho\frac{\partial \Psi}{\partial q}. \tag{51}$$

The classical theory of so-called standard materials can be extended to second grade media by choosing a viscoplastic dissipation potential $\Omega(\underset{\sim}{\boldsymbol{\sigma}}, \underset{\approx}{\mathbf{S}}, R)$ such that:

$$\dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p = \frac{\partial \Omega}{\partial \underset{\sim}{\boldsymbol{\sigma}}}, \quad \dot{\underset{\approx}{\mathbf{K}}}^p = \frac{\partial \Omega}{\partial \underset{\approx}{\mathbf{S}}}, \quad \dot{q} = -\frac{\partial \Omega}{\partial R}. \tag{52}$$

The convexity of the potential ensures again the positivity of intrinsic dissipation.

In [41], a constitutive model for second grade elastoplastic porous materials is presented, as an extension of Gurson potential. In [15], crystal plasticity theory is extended to second grade media. In both cases, a coupled plasticity criterion is introduced, involving both simple force stress and hyperstress tensors. In the same way as for the Cosserat continuum (Sect. 2.2), a multi-criterion framework can also be developed and may have some advantages. The defined potentials (free energy, dissipation potential) are functions of third-rank tensors so that the isotropic functions of such tensors given in [42] can be useful.

3.2 Strain gradient model with the plastic strain as additional degree of freedom

In the previous theory of standard second grade materials, the elastic (resp. plastic) part of the strain gradient is not equal to the gradient of the elastic (resp. plastic) part of strain. This point has been emphasized in [15], [41]. In short, the plastic strain gradient is not the gradient of plastic strain. It is however possible to build a theory, called here the *gradient of strain* model, that respects the constraints:

$$\underset{\approx}{\mathbf{K}}^e \hat{=} \underset{\sim}{\boldsymbol{\varepsilon}}^e \otimes \boldsymbol{\nabla}, \quad \underset{\approx}{\mathbf{K}}^p \hat{=} \underset{\sim}{\boldsymbol{\varepsilon}}^p \otimes \boldsymbol{\nabla}. \tag{53}$$

The main consequence of such an assumption is that no specific flow rule for $\dot{\underset{\approx}{\mathbf{K}}}^p$ is required any more. The theory is therefore formally very similar to the modified Cosserat constitutive theory presented in Sect. 2.4. There are in fact several arguments pleading for an identification of $\underset{\approx}{\mathbf{K}}^e$ and $\underset{\sim}{\boldsymbol{\varepsilon}}^e \otimes \boldsymbol{\nabla}$. First, a vanishing elastic strain energy is not plausible at $\underset{\approx}{\mathbf{K}}^e = 0$ if a non-zero $\underset{\sim}{\boldsymbol{\varepsilon}}^e \otimes \boldsymbol{\nabla}$ still does exist, because some elastic energy should be connected with that gradient of elastic strain. In short:

$$\underset{\sim}{\boldsymbol{\varepsilon}}^e \otimes \boldsymbol{\nabla} \neq 0 \implies \Psi_e \neq 0,$$

where the free energy is assumed to be additively split into two parts, the first one Ψ_e depending homogeneously on elastic quantities, the second one on the remaining internal variables only. On the other hand, Ψ_e can be considered as vanishing if $\underset{\sim}{\boldsymbol{\varepsilon}}^e$ and $\underset{\sim}{\boldsymbol{\varepsilon}}^e \otimes \boldsymbol{\nabla}$ are simultaneously zero:

$$\underset{\sim}{\boldsymbol{\varepsilon}}^e \otimes \boldsymbol{\nabla} = 0 \quad \text{and} \quad \underset{\sim}{\boldsymbol{\varepsilon}}^e = 0 \implies \Psi_e = 0.$$

This motivates the introduction of $\underset{\sim}{\boldsymbol{\varepsilon}}^e \otimes \boldsymbol{\nabla}$ as state variable in the free energy. Let us develop the theory based on this assumption.

Again, the sets $\mathscr{V}$ and $\mathscr{V}^c$ must be enlarged:

$$\mathscr{V} = \{\underline{\dot{\mathbf{u}}}, \underline{\dot{\mathbf{u}}} \otimes \boldsymbol{\nabla}, \underline{\dot{\mathbf{u}}} \otimes \boldsymbol{\nabla} \otimes \boldsymbol{\nabla}, \dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p, \dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p \otimes \boldsymbol{\nabla}\}, \quad \mathscr{V}^c = \{\underline{\dot{\mathbf{u}}}, D_n \underline{\dot{\mathbf{u}}}, \dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p\}. \tag{54}$$

The power densities of internal and contact forces must be taken as general linear forms on the elements of $\mathscr{V}$ and $\mathscr{V}^c$, respectively:

$$p^{(i)} = \underset{\sim}{\boldsymbol{\sigma}} : \dot{\underset{\sim}{\boldsymbol{\varepsilon}}} + \underset{\approx}{\mathbf{S}} \,\vdots\, \dot{\underset{\approx}{\mathbf{K}}} + \underset{\sim}{\mathbf{A}} : \dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p + (\underset{\approx}{\mathbf{B}} - \underset{\approx}{\mathbf{S}}) \,\vdots\, (\dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p \otimes \boldsymbol{\nabla}), \tag{55}$$

$$p^{(c)} = \underline{\mathbf{t}}.\underline{\dot{\mathbf{u}}} + \underline{\mathbf{M}}.D_n\underline{\dot{\mathbf{u}}} + \underset{\sim}{\mathbf{A}}^c : \dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p. \tag{56}$$

The reason for introducing the difference $(\underset{\approx}{\mathbf{B}} - \underset{\approx}{\mathbf{S}})$ instead of a single generalized stress tensor is a separation of the roles played by tensors $\underset{\approx}{\mathbf{S}}$ and $\underset{\approx}{\mathbf{B}}$: they work then on $\underset{\approx}{\mathbf{K}}^e$ and $\underset{\approx}{\mathbf{K}}^p$, respectively. Tensor $\underset{\sim}{\mathbf{A}}$ is at first introduced for the sake of generality, but it will in fact play an important role in the exploitation of the second principle. Simple and double force vectors $\underline{\mathbf{t}}$ and $\underline{\mathbf{M}}$, and surface generalized force tensor $\underset{\sim}{\mathbf{A}}^c$ have been introduced in the power density of the contact forces. The application of the principle of virtual power (4) leads, on the one hand, to the already known balance equations (B.3) and boundary conditions (B.4) and (B.5), and, on the other hand, to two additional conditions:

$$\underset{\sim}{\mathbf{A}} = (\underset{\approx}{\mathbf{B}} - \underset{\approx}{\mathbf{S}}).\boldsymbol{\nabla}, \quad \underset{\sim}{\mathbf{A}}^c = (\underset{\approx}{\mathbf{B}} - \underset{\approx}{\mathbf{S}}).\underline{\mathbf{n}}. \tag{57}$$

The Helmholtz free energy must then be taken explicitly as a function of elastic strain, its gradient and the gradient of plastic deformation: $\Psi(\underset{\sim}{\boldsymbol{\varepsilon}}^e, \underset{\sim}{\boldsymbol{\varepsilon}}^e \otimes \boldsymbol{\nabla}, \underset{\sim}{\boldsymbol{\varepsilon}}^p \otimes \boldsymbol{\nabla}, q)$. The intrinsic dissipation is then evaluated as follows:

$$\begin{aligned} D &= \underset{\sim}{\boldsymbol{\sigma}} : \dot{\underset{\sim}{\boldsymbol{\varepsilon}}} + \underset{\approx}{\mathbf{S}} \,\vdots\, \dot{\underset{\approx}{\mathbf{K}}} + \underset{\sim}{\mathbf{A}} : \dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p + (\underset{\approx}{\mathbf{B}} - \underset{\approx}{\mathbf{S}}) \vdots (\dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p \otimes \boldsymbol{\nabla}) - \rho\dot{\Psi} \\ &= (\underset{\sim}{\boldsymbol{\sigma}} - \rho\frac{\partial\Psi}{\partial\underset{\sim}{\boldsymbol{\varepsilon}}^e}) : \dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^e + (\underset{\approx}{\mathbf{S}} - \rho\frac{\partial\Psi}{\partial\underset{\sim}{\boldsymbol{\varepsilon}}^e \otimes \boldsymbol{\nabla}}) \vdots (\dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^e \otimes \boldsymbol{\nabla}) + (\underset{\approx}{\mathbf{B}} - \rho\frac{\partial\Psi}{\partial\underset{\sim}{\boldsymbol{\varepsilon}}^p \otimes \boldsymbol{\nabla}}) \vdots (\dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p \otimes \boldsymbol{\nabla}) \\ &\quad + (\underset{\sim}{\boldsymbol{\sigma}} + \underset{\sim}{\mathbf{A}}) : \dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p - \rho\frac{\partial\Psi}{\partial q}\dot{q}, \end{aligned} \tag{58}$$

from which the state laws are deduced :

$$\underset{\sim}{\boldsymbol{\sigma}} = \rho\frac{\partial\Psi}{\partial\underset{\sim}{\mathbf{e}}^e}, \quad \underset{\approx}{\mathbf{S}} = \rho\frac{\partial\Psi}{\partial\underset{\approx}{\mathbf{K}}^e}, \quad \underset{\approx}{\mathbf{B}} = \rho\frac{\partial\Psi}{\partial\underset{\approx}{\mathbf{K}}^p}, \quad R = \rho\frac{\partial\Psi}{\partial q}. \tag{59}$$

Finally, the residual intrinsic dissipation amounts to:

$$D = \underset{\sim}{\boldsymbol{\tau}}^{eff} : \dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p - R\dot{q} \quad \text{with} \quad \underset{\sim}{\boldsymbol{\tau}}^{eff} \hat{=} \underset{\sim}{\boldsymbol{\sigma}} + (\underset{\approx}{\mathbf{B}} - \underset{\approx}{\mathbf{S}}).\boldsymbol{\nabla}. \tag{60}$$

The driving force for the activation of plastic flow is therefore the effective stress $\underset{\sim}{\boldsymbol{\tau}}^{eff}$. A general form of the associated constitutive equations, ensuring the positivity of the intrinsic dissipation is given by the choice of a convex potential $\Omega\,(\underset{\sim}{\boldsymbol{\tau}}^{eff}, R)$ (or pseudo-potential of dissipation), such that:

$$\dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p = \frac{\partial\Omega}{\partial\underset{\sim}{\boldsymbol{\tau}}^{eff}}, \quad \dot{q} = -\frac{\partial\Omega}{\partial R}. \tag{61}$$

3.3 Application to simple glide

The strain gradient models presented in the two previous Sections can be compared in the case of simple glide. The same geometry of Fig. 1 is considered again but now for a second-grade material. The situation is very similar to the example given for the Cosserat continuum. The simple and double force vectors $\underline{\mathbf{t}}$ and $\underline{\mathbf{M}}$ are prescribed at the top, whereas the displacement and normal gradient of displacement are given at the bottom. Note that additional boundary conditions are necessary for the second strain gradient model, namely the prescription of $\underset{\sim}{\boldsymbol{\varepsilon}}^p$ or $\underset{\sim}{\mathbf{A}}^c$. A detailed presentation of the boundary value problem and its solution in the elastoplastic case is given in Appendix D.

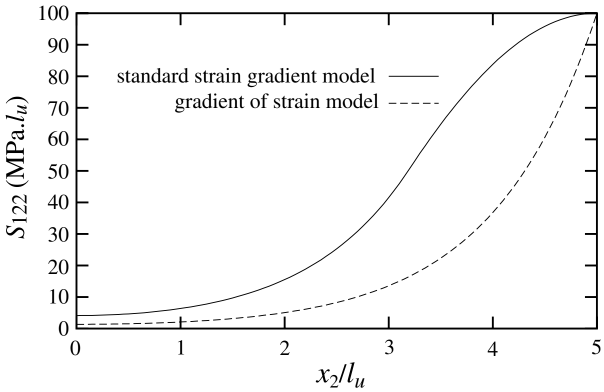

Fig. 5. Simple glide test for a strain gradient material : comparison of the S_{122} hyperstress component for a standard strain gradient material and the modified strain gradient model including the plastic strain as additional degree of freedom. A hyperstress component $S_{122} = 100$ MPa.l_u^2 is prescribed at the top $h = 5l_u$. The material parameters are: E = 70000 MPa, $\nu = 0.3$, $a_{345} = \mu/l_u^2 = 26923$ MPa.l_u^2, $R_0 = 100$ MPa, $l_p = 2l_u$. l_u is a length unit

Two characteristic lengths appear in each model. The elastic length scale $1/\omega_e$ (see Eq. (D.40)) is shared by both models. The yield condition (D.41) of the standard strain gradient model involves a characteristic length l_p that dictates the stress profiles in the plastic zone. The yield condition (D.49) of the model with gradient of plastic strain involves also a characteristic length $\sqrt{c/R_0}$ that dictates the plastic strain distribution, which is found to be parabolic. In contrast, the stress profiles are described in this second model by the elastic length $1/\omega_e$. As a result, there is a discontinuity, at $x_2 = \alpha$, of the stress and hyperstress gradients in the standard model but not for the gradient of strain model. The profile of the non–vanishing hyperstress component for both models is given in Fig. 5.

The value of the position of the limit between the elastic and plastic zones is about the same for both models : $\alpha = 3.19l_u$ for the standard strain gradient model, $\alpha = 3.27l_u$ for the model with gradient of plastic strain, for the load level given in the caption of Fig. 5.

3.4 Gradient of Cosserat deformation

It is has been noticed in Sect. 4 concerning the Cosserat continuum that the curvature tensor can be expressed unambiguously in terms of the gradient of the Cosserat deformation tensor. This has led us to derive a constitutive theory for elastoviscoplastic Cosserat media which is based on a single flow rule (45.1) instead of two (see (52)). However, the question arises why, in such a theory, the only retained variable is $\underset{\approx}{\boldsymbol{\aleph}} : (\underset{\sim}{\mathbf{e}} \otimes \boldsymbol{\nabla})$ and not the full information $\underset{\sim}{\mathbf{e}} \otimes \boldsymbol{\nabla}$. Looking at (33), this amounts to acknowledging the fact that the second grade of the displacement and the Cosserat curvature may have the same magnitude. They should therefore play equal roles in the material response. The simultaneous introduction of the second gradient of the displacement and of the curvature of independent directors has been felt necessary in [39] and [43] (the latter in the case of laminate microstructures or composites). A first setting of the theory with the gradient of the elastic Cosserat deformation as state variable has been presented in [44]. Let us now reformulate this gradient of Cosserat deformation theory in the same spirit as in the previous sections.

The set of degrees of freedom and modelling quantities for the Cosserat continuum also sensitive to the second gradient of displacement with plastic deformation as additional degree of freedom is the largest encountered until now:

$$\mathscr{V} = \{\dot{\underline{\mathbf{u}}}, \dot{\underline{\mathbf{u}}} \otimes \boldsymbol{\nabla}, \dot{\underline{\mathbf{u}}} \otimes \boldsymbol{\nabla} \otimes \boldsymbol{\nabla}, \dot{\underset{\sim}{\boldsymbol{\phi}}}, \dot{\underset{\sim}{\boldsymbol{\phi}}} \otimes \boldsymbol{\nabla}, \dot{\underset{\sim}{\mathbf{e}}}^p, \dot{\underset{\sim}{\mathbf{e}}}^p \otimes \boldsymbol{\nabla}\}, \quad \mathscr{V}^c = \{\dot{\underline{\mathbf{u}}}, \dot{\underset{\sim}{\boldsymbol{\phi}}}, D_n \dot{\underline{\mathbf{u}}}, \dot{\underset{\sim}{\mathbf{e}}}^p\} \tag{62}$$

or equivalently

$$\mathscr{V} = \{\underline{\dot{\mathbf{u}}}, \underline{\dot{\mathbf{\phi}}}, \dot{\underset{\sim}{\mathbf{e}}}, \dot{\underset{\sim}{\mathbf{e}}} \otimes \mathbf{\nabla}, \dot{\underset{\sim}{\mathbf{e}}}^p, \dot{\underset{\sim}{\mathbf{e}}}^p \otimes \mathbf{\nabla}\}, \quad \mathscr{V}^c = \{\underline{\dot{\mathbf{u}}}, \underline{\dot{\mathbf{\phi}}}, D_n\underline{\dot{\mathbf{u}}}, \dot{\underset{\sim}{\mathbf{e}}}^p\}. \tag{63}$$

The expressions of the power of internal and contact forces are:

$$p^{(i)} = \underset{\sim}{\mathbf{\sigma}} : \dot{\underset{\sim}{\mathbf{e}}} + \underset{\approx}{\mathbf{S}} \vdots (\dot{\underset{\sim}{\mathbf{e}}} \otimes \mathbf{\nabla}) + \underset{\sim}{\mathbf{A}} : \dot{\underset{\sim}{\mathbf{e}}}^p + (\underset{\approx}{\mathbf{B}} - \underset{\approx}{\mathbf{S}}) \vdots (\dot{\underset{\sim}{\mathbf{e}}}^p \otimes \mathbf{\nabla}), \tag{64}$$

$$p^{(c)} = \underline{\mathbf{t}}.\underline{\dot{\mathbf{u}}} + \underline{\mathbf{m}}.\dot{\underset{\sim}{\mathbf{\phi}}} + \underline{\mathbf{M}}.D_n\underline{\dot{\mathbf{u}}} + \underset{\sim}{\mathbf{A}}^c : \dot{\underset{\sim}{\mathbf{e}}}^p. \tag{65}$$

The balance equation of momentum, the balance of moment of momentum and the balance of remaining generalized stresses then read:

$$(\underset{\sim}{\mathbf{\sigma}} - \underset{\approx}{\mathbf{S}}.\mathbf{\nabla}).\mathbf{\nabla} = 0, \quad \underset{\approx}{\mathbf{\epsilon}} : (\underset{\sim}{\mathbf{\sigma}} - \underset{\approx}{\mathbf{S}}.\mathbf{\nabla}) = 0, \quad \underset{\sim}{\mathbf{A}} = (\underset{\approx}{\mathbf{B}} - \underset{\approx}{\mathbf{S}}).\mathbf{\nabla}. \tag{66}$$

In addition to the boundary conditions (B.5) found for the classical second-grade continuum, the following ones must hold:

$$\underline{\mathbf{m}} = (\underset{\approx}{\mathbf{\epsilon}} : \underset{\approx}{\mathbf{S}}).\underline{\mathbf{n}}, \quad \underset{\sim}{\mathbf{A}}^c = (\underset{\approx}{\mathbf{B}} - \underset{\approx}{\mathbf{S}}).\underline{\mathbf{n}}. \tag{67}$$

The free energy is a function $\Psi(\underset{\sim}{\mathbf{e}}^e, \underset{\sim}{\mathbf{e}}^e \otimes \mathbf{\nabla}, \underset{\sim}{\mathbf{e}}^p \otimes \mathbf{\nabla}, q)$. The state laws take the form :

$$\underset{\sim}{\mathbf{\sigma}} = \rho \frac{\partial \Psi}{\partial \underset{\sim}{\mathbf{e}}^e}, \quad \underset{\approx}{\mathbf{S}} = \rho \frac{\partial \Psi}{\partial \underset{\sim}{\mathbf{e}}^e \otimes \mathbf{\nabla}}, \quad \underset{\approx}{\mathbf{B}} = \rho \frac{\partial \Psi}{\partial \underset{\sim}{\mathbf{e}}^p \otimes \mathbf{\nabla}}, \quad R = \rho \frac{\partial \Psi}{\partial q}. \tag{68}$$

The intrinsic dissipation involves an effective stress $\underset{\sim}{\mathbf{\tau}}^{eff}$:

$$D = \underset{\sim}{\mathbf{\tau}}^{eff} : \dot{\underset{\sim}{\mathbf{e}}}^p - R\dot{q}, \quad \text{with} \quad \underset{\sim}{\mathbf{\tau}}^{eff} \hat{=} \underset{\sim}{\mathbf{\sigma}} + (\underset{\approx}{\mathbf{B}} - \underset{\approx}{\mathbf{S}}).\mathbf{\nabla} \tag{69}$$

so that a quite general form of the constitutive equations is:

$$\dot{\underset{\sim}{\mathbf{e}}}^p = \frac{\partial \Omega}{\partial \underset{\sim}{\mathbf{\tau}}^{eff}}, \quad \dot{q} = -\frac{\partial \Omega}{\partial R}, \tag{70}$$

where $\Omega(\underset{\sim}{\mathbf{\tau}}^{eff}, R)$ is a convex potential function.

4 Link to the gradient of internal variable approach

Classical nonlinear constitutive equations can be enhanced by incorporating not only time derivatives but also spatial gradients of internal variables, in order to account for observed size effects or deformation patterning in materials [9], [45]. This idea has already been encountered in previous sections dealing with higher-order and higher-grade continua, since the gradient of plastic strain arises naturally in the developments of Sects. 2.4 and 3.2. A thermodynamical formulation of such models for internal variables is presented in the first section, that completes the schemes presented in [46], [44], [47] and [16]. In the second section, it is compared to the alternative formulation proposed in [10] based on the notion of extra-entropy flux.

4.1 Enriched power of internal forces

In such a model, one of the available internal variables q is selected as a good candidate for the introduction of a nonlocal effect. This variable is denoted by γ without reference to its possible tensorial nature and appears explicitly in the set of modelling quantities and degrees of freedom:

$$\mathscr{V} = \{\underline{\dot{\mathbf{u}}},\ \underline{\dot{\mathbf{u}}} \otimes \mathbf{\nabla},\ \dot{\gamma},\ \mathbf{\nabla}\dot{\gamma}\}, \quad \mathscr{V}^c = \{\underline{\dot{\mathbf{u}}},\ \dot{\gamma}\}. \tag{71}$$

The power densities of internal and contact forces are general linear forms with respect to the previous sets:

$$p^{(i)} = \underset{\sim}{\boldsymbol{\sigma}} : \underset{\sim}{\dot{\boldsymbol{\varepsilon}}} + A\dot{\gamma} + \underline{\mathbf{B}}.\mathbf{\nabla}\dot{\gamma}, \quad p^{(c)} = \underline{\mathbf{t}}.\underline{\dot{\mathbf{u}}} + A^c\dot{\gamma}. \tag{72.1,2}$$

The application of the principle of virtual power leads to the classical balance equations for $\underset{\sim}{\boldsymbol{\sigma}}$ and $\underline{\mathbf{t}}$, and two additional ones:

$$\underset{\sim}{\boldsymbol{\sigma}}.\mathbf{\nabla} = 0, \quad A = \underline{\mathbf{B}}.\mathbf{\nabla}, \quad \underline{\mathbf{t}} = \underset{\sim}{\boldsymbol{\sigma}}.\underline{\mathbf{n}}, \quad A^c = \underline{\mathbf{B}}.\underline{\mathbf{n}}. \tag{73}$$

The term $A\dot{\gamma} + \underline{\mathbf{B}}.(\mathbf{\nabla}\dot{\gamma}) = (\dot{\gamma}\underline{\mathbf{B}}).\mathbf{\nabla}$ in (72.1) represents a nonlocal power of internal forces of first grade. The Helmholtz free energy is *a priori* a function $\Psi(\underset{\sim}{\boldsymbol{\varepsilon}}^e, \mathbf{\nabla}\gamma, q)$($\gamma$ can still appear in q). The state laws are:

$$\underset{\sim}{\boldsymbol{\sigma}} = \rho\frac{\partial\Psi}{\partial\underset{\sim}{\boldsymbol{\varepsilon}}^e}, \quad \underline{\mathbf{B}} = \rho\frac{\partial\Psi}{\partial\mathbf{\nabla}\gamma}, \quad R = \rho\frac{\partial\Psi}{\partial q}. \tag{74}$$

The residual intrinsic dissipation then reduces to:

$$D = \underset{\sim}{\boldsymbol{\sigma}} : \underset{\sim}{\dot{\boldsymbol{\varepsilon}}}^p + A\dot{\gamma} - R\dot{q}. \tag{75}$$

A pseudo-potential of dissipation $\Omega(\underset{\sim}{\boldsymbol{\sigma}}, A, R)$ can be chosen to derive the evolution equations:

$$\underset{\sim}{\dot{\boldsymbol{\varepsilon}}}^p = \frac{\partial\Omega}{\partial\underset{\sim}{\boldsymbol{\sigma}}}, \quad \dot{\gamma} = \frac{\partial\Omega}{\partial A}, \quad \dot{q} = -\frac{\partial\Omega}{\partial R}. \tag{76.1–3}$$

Let us now investigate two special cases in order to see the links with well-known Aifantis-like models, on the one hand, and with the generalized continua explored in Sects. 2 and 3.

Scalar internal variable and rate-independent case

In several situations, the internal variable γ is a scalar quantity. Furthermore, one often assumes that an equivalent stress σ_{eq} exists such that:

$$\underset{\sim}{\boldsymbol{\sigma}} : \underset{\sim}{\dot{\boldsymbol{\varepsilon}}}^p = \sigma_{eq}\dot{\gamma}. \tag{77}$$

The residual intrinsic dissipation then takes the simple form:

$$D = \tau^{eff}\dot{\gamma} - R\dot{q} \quad \text{with} \quad \tau^{eff} \hat{=} \sigma_{eq} + \underline{\mathbf{B}}.\mathbf{\nabla}. \tag{78}$$

It suggests that the pseudo-potential of dissipation should be a function of τ^{eff} and R, i.e., $\Omega\ (\tau^{eff}, R)$:

$$\dot{\gamma} = \frac{\partial\Omega}{\partial\tau^{eff}}, \quad \underset{\sim}{\dot{\boldsymbol{\varepsilon}}}^p = \dot{\gamma}\frac{\partial\tau^{eff}}{\partial\underset{\sim}{\boldsymbol{\sigma}}} = \dot{\gamma}\frac{\partial\sigma_{eq}}{\partial\underset{\sim}{\boldsymbol{\sigma}}}, \quad \dot{q} = -\frac{\partial\Omega}{\partial R}. \tag{79}$$

One usually takes a quadratic potential in $\mathbf{\nabla}\gamma$, so that, in the cubic or isotropic case:

$$\underline{\mathbf{B}} = c\mathbf{\nabla}\gamma, \quad A = c\ \Delta\gamma, \tag{80}$$

where c is a material parameter with the unit MPa.l_u^2 (l_u is a length unit). The sign of c is dictated by the definite positivity of the quadratic potential. The Laplacian operator is denoted by Δ. In the rate-independent case, the yield criterion takes, with the yield limit R_y, then the form:

$$\sigma_{eq} = R_y - c\Delta\gamma. \tag{81}$$

One recognizes the well-known form of several strain gradient plasticity models. An example of such enhanced yield condition can be found in [18].

Gradient of plastic strain

In Sects. 2.4 and 3.2, the plastic strain $\underset{\sim}{\boldsymbol{\varepsilon}}^p$ (or $\underset{\sim}{\boldsymbol{e}}^p$ for the Cosserat continuum) has been regarded as an additional degree of freedom, and its gradient has been incorporated in the thermo-mechanical modelling. This situation is of course retrieved here by taking explicitly $\underset{\sim}{\boldsymbol{\varepsilon}}^p$ for γ. The stress-like quantities associated with $\underset{\sim}{\dot{\boldsymbol{\varepsilon}}}^p$ and $\underset{\sim}{\dot{\boldsymbol{\varepsilon}}}^p \otimes \boldsymbol{\nabla}$, respectively, are second- and third–rank tensors. The intrinsic dissipation (75) becomes now:

$$D = \underset{\sim}{\boldsymbol{\tau}}^{eff} : \underset{\sim}{\dot{\boldsymbol{\varepsilon}}}^p, \quad \text{with} \quad \underset{\sim}{\boldsymbol{\tau}}^{eff} \hat{=} \underset{\sim}{\boldsymbol{\sigma}} + \underset{\sim}{\mathbf{A}} = \underset{\sim}{\boldsymbol{\sigma}} + \underset{\approx}{\mathbf{B}}.\boldsymbol{\nabla}. \tag{82}$$

Using a viscoplastic potential $\Omega\,(\underset{\sim}{\tau}^{eff}, R)$, the three evolution equations (76) reduce to two:

$$\underset{\sim}{\dot{\boldsymbol{\varepsilon}}}^p = \frac{\partial \Omega}{\partial \underset{\sim}{\boldsymbol{\tau}}^{eff}} = \frac{\partial \Omega}{\partial \underset{\sim}{\boldsymbol{\sigma}}} = \frac{\partial \Omega}{\partial \underset{\sim}{\mathbf{A}}}, \quad \dot{q} = -\frac{\partial \Omega}{\partial R} \quad . \tag{83}$$

4.2 *Alternative formulation : extra energy vs. extra entropy flux*

At this stage, it may be necessary to justify the intricate form of the density power of internal forces chosen for the various extended continua, see Eqs. (39), (55), (64) and (72). The main argument is that the power densities are taken as general linear forms on the modelling quantities [19]. There is no reason a priori for excluding the associated generalized stresses, even if the physical meaning of each additional term is not obvious at all. The objectivity and conservation laws dictate a posteriori which of the terms must vanish *in fine* or which relations they must satisfy. An example of such a relation is given by the expressions of the tensors A in terms of B tensors in the various investigated situations (see Eqs. (41), (57), (66), (73)).

Several authors however prefer to limit the extension of the work of internal forces and keep it as classical as possible. Accordingly they must act at a different level. In [48] for instance, the expression (48) is not accepted for the second grade continuum. Instead they introduce in the energy balance an additional term called interstitial working. This leads to the same state laws and dissipation inequality but leaves the additional boundary conditions needed for the higher order partial differential equation to be solved unclear. In contrast, for the gradient of internal variable approach, it is proposed in [10] to introduce an extra-entropy flux $\underline{\mathbf{k}}$ instead of extending $p^{(i)}$:

$$\underline{\mathbf{J}}_\eta = \frac{\underline{\mathbf{Q}}}{T} + \underline{\mathbf{k}}. \tag{84}$$

The intrinsic dissipation takes then the form:

$$D = p^{(i)} - \rho \dot{\boldsymbol{\Psi}} + (T\underline{\mathbf{k}}).\boldsymbol{\nabla}. \tag{85}$$

Keeping the classical form of the power of internal forces and considering the free energy $\Psi(\underset{\sim}{\varepsilon}^e, \boldsymbol{\nabla}\dot{\gamma}, q)$, this gives

$$D = \underset{\sim}{\boldsymbol{\sigma}} : \underset{\sim}{\dot{\boldsymbol{\varepsilon}}}^p - \underline{\mathbf{B}}.(\boldsymbol{\nabla}\dot{\gamma}) - R\dot{q} = \underset{\sim}{\boldsymbol{\sigma}} : \underset{\sim}{\dot{\boldsymbol{\varepsilon}}}^p + \dot{\gamma}(\underline{\mathbf{B}}.\boldsymbol{\nabla}) - (\dot{\gamma}\underline{\mathbf{B}}).\boldsymbol{\nabla} + (T\underline{\mathbf{k}}).\boldsymbol{\nabla} - R\dot{q}. \tag{86}$$

In order to eliminate the divergence term $(\dot{\gamma}\underline{\mathbf{B}}).\boldsymbol{\nabla}$, one is led to the following expression of the extra-entropy flux vector:

$$\underline{\mathbf{k}} = \frac{\dot{\gamma}}{T}\underline{\mathbf{B}}. \tag{87}$$

The residual intrinsic dissipation then remains as in (75). The flux $\underline{\mathbf{k}}.\underline{\mathbf{n}}$ on a surface of a body with outward normal $\underline{\mathbf{n}}$ gives the additional boundary conditions necessary for the evolution of γ with a partial differential equation as (76.2).

A priori, both approaches, namely extra energy or entropy flux terms can be justified [21], [78]. The consequences of the choice appear however in the heat equation. It is shown in [23] that the heat equations generally have in fact the same form, but in the same amount as a locally adiabatic process is approached, strongly different equations for temperature evolutions appear. This is however not the place here to discuss this, since this presentation is restricted to isothermal situations.

Another advantage of the extension of the notion of work is that numerical implementation for the application to finite bodies directly follows from the observation of the set of modelling quantities and variational formulation of the balance equations in which the expression of $p^{(i)}$ plays the central role.

5 Micromorphic media

The most general situation envisaged in this work is the case of the micromorphic medium endowed with translational and micro-deformation degrees of freedom. Such an elastic model of materials with microstructure has been introduced in [3], [2]. The mechanical setting including kinematics and balance equations of the theory can be found in [49], [6] and more recently in [50], [7]. The extension to elastoplasticity goes back to [51]. The constitutive framework presented in Subsect. 5.1 represents a generalization of Hlavàček's theory to elastoviscoplasticity. The two subsequent Subsections deal with novel features of non-linear micromorphic continua. The original motivations for considering nonlinear micromorphic continua usually refer to the behavior of crystal lattices [52], polycrystalline and granular materials, although the high number of additional degrees of freedom impeded the development of explicit applications in the past. Available numerical tools make the micromorphic continuum very attractive for future applications in the mechanics of heterogeneous materials.

5.1 Standard micromorphic materials

The kinematical description of the micromorphic continuum is completely defined if one knows two fields, the displacement field $\underline{\mathbf{u}}$ and the independent micro-deformation field $\underset{\sim}{\boldsymbol{\chi}}$. In particular, $\underset{\sim}{\boldsymbol{\chi}}$ is not asked to be a compatible field nor to coincide with the deformation of the displacement field:

$$\mathscr{V} = \{\dot{\underline{\mathbf{u}}},\ \dot{\underline{\mathbf{u}}} \otimes \boldsymbol{\nabla}, \dot{\underset{\sim}{\boldsymbol{\chi}}}, \dot{\underset{\sim}{\boldsymbol{\chi}}} \otimes \boldsymbol{\nabla}\}, \quad \mathscr{V}^c = \{\dot{\underline{\mathbf{u}}}, \dot{\underset{\sim}{\boldsymbol{\chi}}}\}. \tag{88}$$

When the micro-deformation $\underset{\sim}{\boldsymbol{\chi}}$ reduces to a rotation, the micromorphic continuum degenerates into a Cosserat medium. In contrast, a micro-strain continuum can be constructed for which $\underset{\sim}{\boldsymbol{\chi}}$ reduces to a symmetric tensor. The general case is presented here. Several sets of generalized strain measures can be defined. The one used in [49] is retained here:

$$\underset{\sim}{\boldsymbol{\varepsilon}} = \frac{1}{2}(\underline{\mathbf{u}} \otimes \boldsymbol{\nabla} + \boldsymbol{\nabla} \otimes \underline{\mathbf{u}}), \quad {}^{\chi}\underset{\sim}{\mathbf{e}} = \underline{\mathbf{u}} \otimes \boldsymbol{\nabla} - \underset{\sim}{\boldsymbol{\chi}}, \quad {}^{\chi}\underset{\approx}{\mathbf{K}} = \underset{\sim}{\boldsymbol{\chi}} \otimes \boldsymbol{\nabla}, \tag{89}$$

i.e. the strain, relative deformation and micro-deformation gradient tensors. Three generalized stress tensors must therefore be introduced in the virtual power of internal forces:

$$p^{(i)} = \underset{\sim}{\boldsymbol{\sigma}} : \underset{\sim}{\dot{\boldsymbol{\varepsilon}}} + \underset{\sim}{\mathbf{s}} : {}^{\chi}\underset{\sim}{\dot{\mathbf{e}}} + {}^{\chi}\underset{\approx}{\underline{\mathbf{S}}} \,\vdots\, {}^{\chi}\underset{\approx}{\underline{\dot{\mathbf{K}}}}, \quad p^{(c)} = \underline{\mathbf{t}}.\underline{\dot{\mathbf{u}}} + \underset{\sim}{\mathbf{M}} : \underset{\sim}{\dot{\boldsymbol{\chi}}}, \tag{90}$$

where the Cauchy stress tensor $\underset{\sim}{\boldsymbol{\sigma}}$ is symmetric. The balance of momentum and balance of moment of momentum equations read:

$$(\underset{\sim}{\boldsymbol{\sigma}} + \underset{\sim}{\mathbf{s}}).\boldsymbol{\nabla} = 0, \quad {}^{\chi}\underset{\approx}{\underline{\mathbf{S}}}.\boldsymbol{\nabla} + \underset{\sim}{\mathbf{s}} = 0. \tag{91}$$

They are coupled thanks to the micro-stress tensor $\underset{\sim}{\mathbf{s}}$. Equilibrium at the boundary means:

$$\underline{\mathbf{t}} = (\underset{\sim}{\boldsymbol{\sigma}} + \underset{\sim}{\mathbf{s}}).\underline{\mathbf{n}}, \quad \underset{\sim}{\mathbf{M}} = {}^{\chi}\underset{\approx}{\underline{\mathbf{S}}}.\underline{\mathbf{n}}. \tag{92}$$

In the standard model, all strain tensors are decomposed independently into elastic and plastic contributions:

$$\underset{\sim}{\boldsymbol{\varepsilon}} = \underset{\sim}{\boldsymbol{\varepsilon}}^{e} + \underset{\sim}{\boldsymbol{\varepsilon}}^{p}, \quad {}^{\chi}\underset{\sim}{\mathbf{e}} = {}^{\chi}\underset{\sim}{\mathbf{e}}^{e} + {}^{\chi}\underset{\sim}{\mathbf{e}}^{p}, \quad {}^{\chi}\underset{\approx}{\underline{\mathbf{K}}} = {}^{\chi}\underset{\approx}{\underline{\mathbf{K}}}^{e} + {}^{\chi}\underset{\approx}{\underline{\mathbf{K}}}^{p}. \tag{93}$$

The free energy is then a function of all elastic parts and additional internal variables: $\Psi(\underset{\sim}{\boldsymbol{\varepsilon}}^{e}, {}^{\chi}\underset{\sim}{\mathbf{e}}^{e}, {}^{\chi}\underset{\approx}{\underline{\mathbf{K}}}^{e}, q)$. The state laws read:

$$\underset{\sim}{\boldsymbol{\sigma}} = \rho\frac{\partial\Psi}{\partial\underset{\sim}{\boldsymbol{\varepsilon}}^{e}}, \quad \underset{\sim}{\mathbf{s}} = \rho\frac{\partial\Psi}{\partial{}^{\chi}\underset{\sim}{\mathbf{e}}^{e}}, \quad {}^{\chi}\underset{\approx}{\underline{\mathbf{S}}} = \rho\frac{\partial\Psi}{\partial{}^{\chi}\underset{\approx}{\underline{\mathbf{K}}}^{e}}, \quad R = \rho\frac{\partial\Psi}{\partial q}. \tag{94}$$

The residual intrinsic dissipation follows:

$$D = \underset{\sim}{\boldsymbol{\sigma}} : \underset{\sim}{\dot{\boldsymbol{\varepsilon}}}^{p} + \underset{\sim}{\mathbf{s}} : {}^{\chi}\underset{\sim}{\dot{\mathbf{e}}}^{p} + {}^{\chi}\underset{\approx}{\underline{\mathbf{S}}} \,\vdots\, {}^{\chi}\underset{\approx}{\underline{\dot{\mathbf{K}}}}^{p} - R\dot{q}. \tag{95}$$

The evolution equations can be derived from a viscoplastic potential $\Omega(\underset{\sim}{\boldsymbol{\sigma}}, \underset{\sim}{\mathbf{s}}, {}^{\chi}\underset{\approx}{\underline{\mathbf{S}}}, R)$:

$$\underset{\sim}{\dot{\boldsymbol{\varepsilon}}}^{p} = \frac{\partial\Omega}{\partial\underset{\sim}{\boldsymbol{\sigma}}}, \quad {}^{\chi}\underset{\sim}{\dot{\mathbf{e}}}^{p} = \frac{\partial\Omega}{\partial\underset{\sim}{\mathbf{s}}}, \quad {}^{\chi}\underset{\approx}{\underline{\dot{\mathbf{K}}}}^{p} = \frac{\partial\Omega}{\partial{}^{\chi}\underset{\approx}{\underline{\mathbf{S}}}}, \quad \dot{q} = -\frac{\partial\Omega}{\partial R}. \tag{96}$$

The choice of a convex potential then ensures the positivity of intrinsic dissipation.

Again, the viscoplastic potential can involve a single yield function containing a combination of all stress tensors, or, in contrast, be the sum of several mechanisms involving yield functions in which the individual stress tensors intervene separately. These two possibilities have been explored for the Cosserat continuum in Sect. 2.2 and can be extended to the micromorphic continuum in a straightforward manner. Examples of single and multi-mechanism plasticity models for micromorphic media are provided in [36].

5.2 Micromorphic theory with a special dependence on plastic strain and relative deformation as additional degrees of freedom

It must be noted that the gradient of micro-deformation tensor can be expressed in terms of the gradient of the two other strain measures. To show that, let us recall Toupin's relation between gradient of strain and second gradient of displacement, and define the six-rank linear operator $\underset{\approx}{\aleph}$ such that:

$$\underline{\mathbf{u}} \otimes \boldsymbol{\nabla} \otimes \boldsymbol{\nabla} = \underset{\sim}{\boldsymbol{\varepsilon}} \otimes \boldsymbol{\nabla} + \underset{\approx}{\mathbf{1}}_{T} : (\boldsymbol{\nabla} \otimes \underset{\sim}{\boldsymbol{\varepsilon}}) - \boldsymbol{\nabla} \otimes \underset{\sim}{\boldsymbol{\varepsilon}} \Longleftrightarrow \quad \underline{\mathbf{u}} \otimes \boldsymbol{\nabla} \otimes \boldsymbol{\nabla} = \underset{\approx}{\aleph} \,\vdots\, (\underset{\sim}{\boldsymbol{\varepsilon}} \otimes \boldsymbol{\nabla}). \tag{97}$$

The equation on the left-hand side is an expanded version of already mentioned Toupin's relation (34). The four-rank operator $\underset{\approx}{1}_T$ is defined in Appendix A (Eq. (A.3)). The index notation corresponding to the previous expression is given in Appendix A, Eq. (A.15). The application of the gradient operator to the definition (89.2) of relative deformation tensor ${}^{\chi}\underset{\sim}{\mathbf{e}}$ then leads to the following compatibility relation:

$$ {}^{\chi}\underset{\sim}{\underline{\mathbf{K}}} = \underset{\approx}{\aleph}\,\vdots(\underset{\sim}{\boldsymbol{\varepsilon}} \otimes \boldsymbol{\nabla}) - {}^{\chi}\underset{\sim}{\mathbf{e}} \otimes \boldsymbol{\nabla}. \tag{98} $$

As already mentioned for the non-linear Cosserat and second grade continua (Sects. 2.4 and 3.2), the following identification of the elastic and plastic parts of ${}^{\chi}\underset{\sim}{\underline{\mathbf{K}}}$ is natural:

$$ {}^{\chi}\underset{\sim}{\underline{\mathbf{K}}}^{e} \hat{=} \underset{\approx}{\aleph}\,\vdots(\underset{\sim}{\boldsymbol{\varepsilon}}^{e} \otimes \boldsymbol{\nabla}) - {}^{\chi}\underset{\sim}{\mathbf{e}}^{e} \otimes \boldsymbol{\nabla}, \quad {}^{\chi}\underset{\sim}{\underline{\mathbf{K}}}^{p} \hat{=} \underset{\approx}{\aleph}\,\vdots(\underset{\sim}{\boldsymbol{\varepsilon}}^{p} \otimes \boldsymbol{\nabla}) - {}^{\chi}\underset{\sim}{\mathbf{e}}^{p} \otimes \boldsymbol{\nabla}. \tag{99} $$

This gives rise to an alternative nonlinear micromorphic continuum model where plastic strain and plastic relative deformation must be regarded as additional degrees of freedom:

$$ \mathscr{V} = \left\{ \underline{\dot{\mathbf{u}}}, \underline{\dot{\mathbf{u}}} \otimes \boldsymbol{\nabla}, \underset{\sim}{\dot{\boldsymbol{\chi}}}, \underset{\sim}{\dot{\boldsymbol{\chi}}} \otimes \boldsymbol{\nabla}, \underset{\sim}{\dot{\boldsymbol{\varepsilon}}}^{p}, {}^{\chi}\underset{\sim}{\dot{\mathbf{e}}}^{p}, \underset{\approx}{\aleph}\,\vdots(\underset{\sim}{\dot{\boldsymbol{\varepsilon}}}^{p} \otimes \boldsymbol{\nabla}) - {}^{\chi}\underset{\sim}{\dot{\mathbf{e}}}^{p} \otimes \boldsymbol{\nabla} \right\}, \tag{100} $$

$$ \mathscr{V}^{c} = \left\{ \underline{\dot{\mathbf{u}}}, \underset{\sim}{\dot{\boldsymbol{\chi}}}, \underset{\sim}{\dot{\boldsymbol{\varepsilon}}}^{p}, {}^{\chi}\underset{\sim}{\dot{\mathbf{e}}}^{p} \right\}, \tag{101} $$

$$ p^{(i)} = \underset{\sim}{\boldsymbol{\sigma}} : \underset{\sim}{\dot{\boldsymbol{\varepsilon}}} + \underset{\sim}{\mathbf{s}} : {}^{\chi}\underset{\sim}{\dot{\mathbf{e}}} + {}^{\chi}\underset{\sim}{\underline{\mathbf{S}}} \,\vdots\, {}^{\chi}\underset{\sim}{\underline{\dot{\mathbf{K}}}} + \underset{\sim}{\mathbf{A}} : \underset{\sim}{\dot{\boldsymbol{\varepsilon}}}^{p} + {}^{\chi}\underset{\sim}{\mathbf{A}} : {}^{\chi}\underset{\sim}{\dot{\mathbf{e}}}^{p} + ({}^{\chi}\underset{\sim}{\underline{\mathbf{B}}} - {}^{\chi}\underset{\sim}{\underline{\mathbf{S}}}) \,\vdots\, {}^{\chi}\underset{\sim}{\underline{\dot{\mathbf{K}}}}^{p}, \tag{102} $$

$$ p^{(c)} = \underline{\mathbf{t}}.\underline{\dot{\mathbf{u}}} + \underset{\sim}{\mathbf{M}} : \underset{\sim}{\dot{\boldsymbol{\chi}}} + \underset{\sim}{\mathbf{A}}^{c} : \underset{\sim}{\dot{\boldsymbol{\varepsilon}}}^{p} + {}^{\chi}\underset{\sim}{\mathbf{A}}^{c} : {}^{\chi}\underset{\sim}{\dot{\mathbf{e}}}^{p}. \tag{103} $$

The balance and boundary conditions of the classical micromorphic medium are now supplemented by additional balance laws:

$$ \underset{\sim}{\mathbf{A}} = \left(({}^{\chi}\underset{\sim}{\underline{\mathbf{B}}} - {}^{\chi}\underset{\sim}{\underline{\mathbf{S}}}) \,\vdots\, \underset{\approx}{\aleph} \right).\boldsymbol{\nabla}, \quad {}^{\chi}\underset{\sim}{\mathbf{A}} = -({}^{\chi}\underset{\sim}{\underline{\mathbf{B}}} - {}^{\chi}\underset{\sim}{\underline{\mathbf{S}}}).\boldsymbol{\nabla}, \tag{104} $$

$$ \underset{\sim}{\mathbf{A}}^{c} = \left(({}^{\chi}\underset{\sim}{\underline{\mathbf{B}}} - {}^{\chi}\underset{\sim}{\underline{\mathbf{S}}}) \,\vdots\, \underset{\approx}{\aleph} \right).\underline{\mathbf{n}}, \quad {}^{\chi}\underset{\sim}{\mathbf{A}}^{c} = -({}^{\chi}\underset{\sim}{\underline{\mathbf{B}}} - {}^{\chi}\underset{\sim}{\underline{\mathbf{S}}}).\underline{\mathbf{n}}. \tag{105} $$

The free energy depends on an enriched set of variables: $\Psi(\underset{\sim}{\boldsymbol{\varepsilon}}^{e}, {}^{\chi}\underset{\sim}{\mathbf{e}}^{e}, {}^{\chi}\underset{\sim}{\underline{\mathbf{K}}}^{e}, {}^{\chi}\underset{\sim}{\underline{\mathbf{K}}}^{p}, q)$, and:

$$ \underset{\sim}{\boldsymbol{\sigma}} = \rho \frac{\partial \Psi}{\partial \underset{\sim}{\boldsymbol{\varepsilon}}^{e}}, \quad \underset{\sim}{\mathbf{s}} = \rho \frac{\partial \Psi}{\partial {}^{\chi}\underset{\sim}{\mathbf{e}}^{e}}, \quad {}^{\chi}\underset{\sim}{\underline{\mathbf{S}}} = \rho \frac{\partial \Psi}{\partial {}^{\chi}\underset{\sim}{\underline{\mathbf{K}}}^{e}}, \quad {}^{\chi}\underset{\sim}{\underline{\mathbf{B}}} = \rho \frac{\partial \Psi}{\partial {}^{\chi}\underset{\sim}{\underline{\mathbf{K}}}^{p}}, \quad R = \rho \frac{\partial \Psi}{\partial q}. \tag{106} $$

The intrinsic dissipation differs from (95). Taking the previous state laws into account, it reduces to:

$$ D = (\underset{\sim}{\boldsymbol{\sigma}} + \underset{\sim}{\mathbf{A}}) : \underset{\sim}{\dot{\boldsymbol{\varepsilon}}}^{p} + (\underset{\sim}{\mathbf{s}} + {}^{\chi}\underset{\sim}{\mathbf{A}}) : {}^{\chi}\underset{\sim}{\dot{\mathbf{e}}}^{p} - R\dot{q}. \tag{107} $$

Once again, effective stresses appear on which the viscoplastic potential should depend: $\Omega\,(\underset{\sim}{\boldsymbol{\tau}}^{eff} \hat{=} \underset{\sim}{\boldsymbol{\sigma}} + \underset{\sim}{\mathbf{A}}, {}^{\chi}\underset{\sim}{\boldsymbol{\tau}}^{eff} \hat{=} \underset{\sim}{\mathbf{s}} + {}^{\chi}\underset{\sim}{\mathbf{A}}, R)$,

$$ \underset{\sim}{\dot{\boldsymbol{\varepsilon}}}^{p} = \frac{\partial \Omega}{\partial \underset{\sim}{\boldsymbol{\tau}}^{eff}}, \quad {}^{\chi}\underset{\sim}{\dot{\mathbf{e}}}^{p} = \frac{\partial \Omega}{\partial {}^{\chi}\underset{\sim}{\boldsymbol{\tau}}^{eff}}, \quad \dot{q} = -\frac{\partial \Omega}{\partial R}. \tag{108} $$

Thus, an additional flow rule for the plastic part ${}^{\chi}\underset{\sim}{\underline{\mathbf{K}}}^{p}$ of the micro-deformation gradient is not required in this theory with ${}^{\chi}\underset{\sim}{\underline{\mathbf{K}}}^{p}$ represented by the gradient of plastic deformation according to Eq. (99).

5.3 Gradient of strain and relative deformation

The micromorphic theory favours the influence of the micro-deformation gradient on material behavior, and neglects that of the second gradient of displacement. This assumption may be sometimes difficult to justify, especially when generalized continua are derived from homogenization procedures [35], [53]. Indeed, strain gradient and gradient of micro-deformation may well have the same order of magnitude in many situations. This suggests that the second gradient of displacement should be added to the set of modelling quantities (88):

$$\mathscr{V} = \left\{ \dot{\underline{\mathbf{u}}}\ ,\ \dot{\underline{\mathbf{u}}} \otimes \boldsymbol{\nabla},\ \dot{\underline{\mathbf{u}}} \otimes \boldsymbol{\nabla} \otimes \boldsymbol{\nabla},\ \dot{\underset{\sim}{\boldsymbol{\chi}}},\ \dot{\underset{\sim}{\boldsymbol{\chi}}} \otimes \boldsymbol{\nabla} \right\}, \quad \mathscr{V}^c = \left\{ \dot{\underline{\mathbf{u}}},\ D_n \dot{\underline{\mathbf{u}}},\ \dot{\underset{\sim}{\boldsymbol{\chi}}} \right\}, \tag{109}$$

or, equivalently :

$$\mathscr{V} = \left\{ \dot{\underline{\mathbf{u}}}, \dot{\underline{\mathbf{u}}} \otimes \boldsymbol{\nabla}, \dot{\underline{\mathbf{u}}} \otimes \boldsymbol{\nabla} \otimes \boldsymbol{\nabla},\ {}^{\chi}\dot{\underset{\sim}{\mathbf{e}}},\ {}^{\chi}\dot{\underset{\sim}{\mathbf{e}}} \otimes \boldsymbol{\nabla} \right\}, \quad \mathscr{V}^c = \left\{ \dot{\underline{\mathbf{u}}}, D_n \dot{\underline{\mathbf{u}}}, \dot{\underset{\sim}{\boldsymbol{\chi}}} \right\}. \tag{110}$$

This combined strain gradient/micromorphic theory is developed in the same spirit as the gradient of Cosserat deformation model presented in Sect. 3.4. It is sufficient to write the proposed form for the power of internal forces to see what kind of theory is meant:

$$p^{(i)} = \underset{\sim}{\boldsymbol{\sigma}} : \dot{\underset{\sim}{\boldsymbol{\varepsilon}}} + \underset{\approx}{\mathbf{S}} \vdots (\dot{\underset{\sim}{\boldsymbol{\varepsilon}}} \otimes \boldsymbol{\nabla}) + \underset{\sim}{\mathbf{s}} : {}^{\chi}\dot{\underset{\sim}{\mathbf{e}}} + {}^{\chi}\underset{\approx}{\mathbf{S}} \vdots ({}^{\chi}\dot{\underset{\sim}{\mathbf{e}}} \otimes \boldsymbol{\nabla}). \tag{111}$$

The whole theory of this gradient of strain and relative deformation model follows in a straightforward manner and does not need to be explicited. Similarly to the situation of the previous Section, if plastic strain and relative deformation are additional degrees of freedom, then a specific flow rule for the plastic micro-deformation gradient is not required. In this latter case, without the special dependence (99):

$$\mathscr{V} = \left\{ \dot{\underline{\mathbf{u}}}\ ,\ \dot{\underline{\mathbf{u}}} \otimes \boldsymbol{\nabla},\ \dot{\underline{\mathbf{u}}} \otimes \boldsymbol{\nabla} \otimes \boldsymbol{\nabla},\ {}^{\chi}\dot{\underset{\sim}{\mathbf{e}}},\ {}^{\chi}\dot{\underset{\sim}{\mathbf{e}}} \otimes \boldsymbol{\nabla},\ \dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p,\ \dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p \otimes \boldsymbol{\nabla},\ {}^{\chi}\dot{\underset{\sim}{\mathbf{e}}}^p,\ {}^{\chi}\dot{\underset{\sim}{\mathbf{e}}}^p \otimes \boldsymbol{\nabla} \right\}. \tag{112}$$

The power of internal forces is enriched again:

$$\begin{aligned} p^{(i)} = {} & \underset{\sim}{\boldsymbol{\sigma}} : \dot{\underset{\sim}{\boldsymbol{\varepsilon}}} + \underset{\approx}{\mathbf{S}} \vdots (\dot{\underset{\sim}{\boldsymbol{\varepsilon}}} \otimes \boldsymbol{\nabla}) + \underset{\sim}{\mathbf{s}} : {}^{\chi}\dot{\underset{\sim}{\mathbf{e}}} + {}^{\chi}\underset{\approx}{\mathbf{S}} \vdots ({}^{\chi}\dot{\underset{\sim}{\mathbf{e}}} \otimes \boldsymbol{\nabla}) \\ & + \underset{\sim}{\mathbf{A}} : \dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p + {}^{\chi}\underset{\sim}{\mathbf{A}} : {}^{\chi}\dot{\underset{\sim}{\mathbf{e}}}^p + (\underset{\approx}{\mathbf{B}} - \underset{\approx}{\mathbf{S}}) \vdots (\dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p \otimes \boldsymbol{\nabla}) + ({}^{\chi}\underset{\approx}{\mathbf{B}} - {}^{\chi}\underset{\approx}{\mathbf{S}}) \vdots ({}^{\chi}\dot{\underset{\sim}{\mathbf{e}}}^p \otimes \boldsymbol{\nabla}), \end{aligned} \tag{113}$$

which reduces the dissipation power down to the dissipative work done only on the two plastic degrees of freedom $\dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p$ and ${}^{\chi}\dot{\underset{\sim}{\mathbf{e}}}^p$ as in Eq. (107). This refined theory ends up the formulation of proposed constitutive frameworks for generalized continua under the small perturbation hypothesis.

6 Finite deformation formulations

Let $\mathscr{D}$ be a domain of body V in its current state and $\mathscr{D}_0$ the corresponding domain in its reference configuration. The mass density of the material is called ρ in the current configuration and ρ_0 in the reference one. The power of internal forces is defined in both configurations as:

$$\mathscr{P}^{(i)} = \int_{\mathscr{D}} p^{(i)} dV = \int_{\mathscr{D}_0} p^{(i)} \frac{\rho_0}{\rho} dV_0. \tag{114}$$

The deformation of the displacement field is defined by:

$$\underset{\sim}{\mathbf{F}} = \underset{\sim}{\mathbf{1}} + \underline{\mathbf{u}} \otimes \boldsymbol{\nabla}. \tag{115}$$

The energy principle (6) and Clausius–Duhem inequality (9) are still valid in the current configuration, provided that the derivation operator $\boldsymbol{\nabla}$ with respect to Lagrangian coordinates is replaced by the Eulerian operator $\boldsymbol{\nabla}^c$ related to $\boldsymbol{\nabla}$ by Eq. (A.9). We simply write here the Lagrangian version of these principles:

$$\rho_0 \dot{\epsilon} = \frac{\rho_0}{\rho} p^{(i)} - \frac{\rho_0}{\rho} (\underline{\mathbf{Q}} \otimes \boldsymbol{\nabla}) : \underset{\sim}{\mathbf{F}}^{-T}, \tag{116}$$

$$-\rho_0 (\dot{\boldsymbol{\Psi}} + \eta \dot{T}) + \frac{\rho_0}{\rho} p^{(i)} - \frac{\rho_0}{\rho} \frac{1}{T} (\underline{\mathbf{Q}} \otimes (\boldsymbol{\nabla} T)) : \underset{\sim}{\mathbf{F}}^{-T} \geq 0. \tag{117}$$

We will consider the intrinsic dissipation with respect to the reference configuration:

$$D_0 = \frac{\rho_0}{\rho} p^{(i)} - \rho_0 \dot{\boldsymbol{\Psi}}. \tag{118}$$

In the following, most of the previous models are extended to the finite deformation framework. One (or several) choice(s) of invariant generalized strain measures are provided for each model. It appears in particular that a triad of directors (anisotropy directions) allows an unambiguous decomposition of the deformation gradient, in the spirit of [54]. In the case of higher order continua, they coincide with the Cosserat directors. Their physical meaning must be specified for each application, as motivated for polycrystals in [55] and [56]. The dissipation inequality is used to derive the state and evolution laws. Results are given mostly without detailed proof for the sake of conciseness.

6.1 Cosserat continuum

The finite deformation framework of the Cosserat continuum has been settled by [57] for elasticity and independently by [58] and [59] for elastoplasticity (see also [13]).

The degrees of freedom are the displacement $\underline{\mathbf{u}}$ and rotation $\underset{\sim}{\mathbf{R}} = \exp(-\underset{\approx}{\boldsymbol{\epsilon}} . \underline{\boldsymbol{\phi}})$. The invariant deformation and curvature measures are:

$$^\sharp\underset{\sim}{\mathbf{F}} \hat{=} \underset{\sim}{\mathbf{R}}^T . \underset{\sim}{\mathbf{F}}, \quad ^\sharp\underset{\sim}{\boldsymbol{\Gamma}} \hat{=} -\frac{1}{2} \underset{\approx}{\boldsymbol{\epsilon}} : (\underset{\sim}{\mathbf{R}}^T . (\underset{\sim}{\mathbf{R}} \otimes \boldsymbol{\nabla})). \tag{119.1,2}$$

The power density of internal forces then takes the form:

$$p^{(i)} = {}^\sharp\underset{\sim}{\boldsymbol{\sigma}} : ({}^\sharp\dot{\underset{\sim}{\mathbf{F}}} . {}^\sharp\underset{\sim}{\mathbf{F}}^{-1}) + {}^\sharp\underset{\sim}{\boldsymbol{\mu}} : ({}^\sharp\dot{\underset{\sim}{\boldsymbol{\Gamma}}} . {}^\sharp\underset{\sim}{\mathbf{F}}^{-1}), \tag{120}$$

where the invariant stress tensors are related to the Cauchy stress and couple–stress tensors defined on the current configuration:

$$^\sharp\underset{\sim}{\boldsymbol{\sigma}} = \underset{\sim}{\mathbf{R}}^T . \underset{\sim}{\boldsymbol{\sigma}} . \underset{\sim}{\mathbf{R}}, \quad ^\sharp\underset{\sim}{\boldsymbol{\mu}} = \underset{\sim}{\mathbf{R}}^T . \underset{\sim}{\boldsymbol{\mu}} . \underset{\sim}{\mathbf{R}}. \tag{121}$$

The balance equations (momentum and moment of momentum) then are the same as (B.2) provided that $\boldsymbol{\nabla}$ is replaced by $\boldsymbol{\nabla}^c$.

A multiplicative decomposition of the Cosserat deformation and a quasi-additive decomposition of the total curvature are adopted:

$$^\sharp\underset{\sim}{\mathbf{F}} = {}^\sharp\underset{\sim}{\mathbf{F}}^e . {}^\sharp\underset{\sim}{\mathbf{F}}^p, \quad ^\sharp\underset{\sim}{\boldsymbol{\Gamma}} = {}^\sharp\underset{\sim}{\boldsymbol{\Gamma}}^e . {}^\sharp\underset{\sim}{\mathbf{F}}^p + {}^\sharp\underset{\sim}{\boldsymbol{\Gamma}}^p. \tag{122.1,2}$$

Arguments for such a decomposition can be found in [44]. An alternative decomposition is proposed in [60]:

$$^\sharp\underset{\sim}{\mathbf{\Gamma}} = {}^\sharp\underset{\sim}{\mathbf{\Gamma}}^e + {}^\sharp\underset{\sim}{\mathbf{\Gamma}}^p. \tag{123}$$

The Helmholtz free energy is a function $\Psi({}^\sharp\underset{\sim}{\mathbf{F}}^e, {}^\sharp\underset{\sim}{\mathbf{\Gamma}}^e, q)$. The state laws, including the hyperelasticity relations, follow:

$$^\sharp\underset{\sim}{\boldsymbol{\sigma}} = \rho\frac{\partial\Psi}{\partial^\sharp\underset{\sim}{\mathbf{F}}^e}.{}^\sharp\underset{\sim}{\mathbf{F}}^{eT}, \quad {}^\sharp\underset{\sim}{\boldsymbol{\mu}} = \rho\frac{\partial\Psi}{\partial^\sharp\underset{\sim}{\mathbf{\Gamma}}^e}.{}^\sharp\underset{\sim}{\mathbf{F}}^{eT} \quad \text{or} \quad {}^\sharp\underset{\sim}{\boldsymbol{\mu}} = \rho\frac{\partial\Psi}{\partial^\sharp\underset{\sim}{\mathbf{\Gamma}}^e}.{}^\sharp\underset{\sim}{\mathbf{F}}^{T}, \quad R = \rho\frac{\partial\Psi}{\partial q}, \tag{124}$$

whether (122.1) or (123) is adopted. The residual dissipation then is:

$$\frac{\rho}{\rho_0}D_0 = {}^\sharp\underset{\sim}{\boldsymbol{\sigma}} : ({}^\sharp\underset{\sim}{\mathbf{F}}^e.{}^\sharp\underset{\sim}{\dot{\mathbf{F}}}^p.{}^\sharp\underset{\sim}{\mathbf{F}}^{p-1}.{}^\sharp\underset{\sim}{\mathbf{F}}^{e-1}) + {}^\sharp\underset{\sim}{\boldsymbol{\mu}} : ({}^\sharp\underset{\sim}{\dot{\mathbf{\Gamma}}}^p.{}^\sharp\underset{\sim}{\mathbf{F}}^{-1}) + {}^\sharp\underset{\sim}{\boldsymbol{\mu}} : ({}^\sharp\underset{\sim}{\mathbf{\Gamma}}^e.{}^\sharp\underset{\sim}{\dot{\mathbf{F}}}^p.{}^\sharp\underset{\sim}{\mathbf{F}}^{p-1}.{}^\sharp\underset{\sim}{\mathbf{F}}^{e-1}) - R\dot{q}. \tag{125}$$

The second term with ${}^\sharp\underset{\sim}{\boldsymbol{\mu}}$ disappears if (123) is adopted. The positivity of dissipation is therefore ensured if a convex potential $\Omega(\underset{\sim}{\mathbf{\Sigma}}, \underset{\sim}{\mathbf{M}}, R)$ is chosen such that:

$${}^\sharp\underset{\sim}{\dot{\mathbf{F}}}^p.{}^\sharp\underset{\sim}{\mathbf{F}}^{p-1} = \frac{\partial\Omega}{\partial\underset{\sim}{\mathbf{\Sigma}}}, \quad {}^\sharp\underset{\sim}{\dot{\mathbf{\Gamma}}}^p = \frac{\partial\Omega}{\partial\underset{\sim}{\mathbf{M}}}, \quad \dot{q} = -\frac{\partial\Omega}{\partial R}, \tag{126}$$

$$\underset{\sim}{\mathbf{M}} = {}^\sharp\underset{\sim}{\boldsymbol{\mu}}.{}^\sharp\underset{\sim}{\mathbf{F}}^{-T}, \quad \underset{\sim}{\mathbf{\Sigma}} = {}^\sharp\underset{\sim}{\mathbf{F}}^{eT}.{}^\sharp\underset{\sim}{\boldsymbol{\sigma}}.{}^\sharp\underset{\sim}{\mathbf{F}}^{e-T} + {}^\sharp\underset{\sim}{\mathbf{\Gamma}}^{eT}.{}^\sharp\underset{\sim}{\boldsymbol{\mu}}.{}^\sharp\underset{\sim}{\mathbf{F}}^{e-T}. \tag{127}$$

The last term in the definition of $\underset{\sim}{\mathbf{\Sigma}}$ can be dropped if (123) is adopted.

6.2 Strain gradient models

The theoretical basis for the second grade medium has been led directly within the large strain framework [1], [37] and even for elastoplastic solids [39]. The theory is summarized here and linked to the standard material framework. The power density of internal forces takes the form:

$$p^{(i)} = \underset{\sim}{\boldsymbol{\sigma}} : (\underline{\dot{\mathbf{u}}} \otimes \mathbf{\nabla}^c) + \underset{\approx}{\mathbf{S}} \,\vdots\, (\underline{\dot{\mathbf{u}}} \otimes \mathbf{\nabla}^c \otimes \mathbf{\nabla}^c) = \underset{\sim}{\boldsymbol{\sigma}} : (\underset{\sim}{\dot{\mathbf{F}}}.\underset{\sim}{\mathbf{F}}^{-1}) + \underset{\approx}{\mathbf{S}} \,\vdots\, ((\underset{\sim}{\dot{\mathbf{F}}}.\underset{\sim}{\mathbf{F}}^{-1}) \otimes \mathbf{\nabla}^c), \tag{128}$$

where $\underset{\sim}{\boldsymbol{\sigma}}$ is symmetric. The stress and hyperstress tensors must fulfill the same balance and boundary conditions (B.3) and (B.5) on the current configuration, provided that $\mathbf{\nabla}$ is replaced by $\mathbf{\nabla}^c$. The power of internal forces can also be expressed in terms of the time derivatives of Lagrangian strain and strain gradient measures:

$$p^{(i)} = \underset{\sim}{\boldsymbol{\sigma}} : \left(\underset{\sim}{\mathbf{F}}^{-T}.\frac{1}{2}\underset{\sim}{\dot{\mathbf{C}}}.\underset{\sim}{\mathbf{F}}^{-1}\right) + \underset{\approx}{\mathbf{S}} \,\vdots\, (\underset{\sim}{\mathbf{F}}.\underset{\approx}{\dot{\mathbf{K}}} : (\underset{\sim}{\mathbf{F}}^{-1} \boxtimes \underset{\sim}{\mathbf{F}}^{-1})) \tag{129}$$

with

$$\underset{\sim}{\mathbf{C}}\hat{=} \underset{\sim}{\mathbf{F}}^T.\underset{\sim}{\mathbf{F}}, \quad \underset{\approx}{\mathbf{K}} \hat{=} \underset{\sim}{\mathbf{F}}^{-1}.(\underset{\sim}{\mathbf{F}} \otimes \mathbf{\nabla}). \tag{130}$$

The now classical multiplicative decomposition is adopted for the deformation [61],[33], whereas the equation (123) prompts us to choose an additive decomposition of the invariant strain gradient measure $\underset{\approx}{\mathbf{K}}$:

$$\underset{\sim}{\mathbf{F}} = \underset{\sim}{\mathbf{F}}^e.\underset{\sim}{\mathbf{F}}^p, \quad \underset{\sim}{\mathbf{F}}^e = \underset{\sim}{\mathbf{R}}.\underset{\sim}{\mathbf{U}}^e.\underset{\sim}{\mathbf{F}}^p, \quad \underset{\approx}{\mathbf{K}} = \underset{\approx}{\mathbf{K}}^e + \underset{\approx}{\mathbf{K}}^p, \tag{131}$$

where $\underset{\sim}{\mathbf{U}}^e$ is symmetric. Since $\underset{\sim}{\mathbf{F}}$ is not invariant nor objective, a specific definition of a triad of directors must also be provided for this decomposition to become unambiguous. The free energy then is a function $\Psi(\underset{\sim}{\mathbf{C}}^e \hat{=} \underset{\sim}{\mathbf{F}}^{eT}.\underset{\sim}{\mathbf{F}}^e = \underset{\sim}{\mathbf{U}}^{e2}, \underset{\approx}{\mathbf{K}}^e, q)$. The state laws, including hyperelastic relations, read:

$$\underset{\sim}{\sigma} = 2\underset{\sim}{F}^e.\rho\frac{\partial\Psi}{\partial\underset{\sim}{C}^e}.\underset{\sim}{F}^{eT}, \quad \underset{\approx}{S} = \underset{\sim}{F}^{-T}.\rho\frac{\partial\Psi}{\partial\underset{\approx}{K}^e} : (\underset{\sim}{F}^T \boxtimes \underset{\sim}{F}^T), \quad R = \rho\frac{\partial\Psi}{\partial q}. \tag{132}$$

The residual intrinsic dissipation amounts to:

$$\frac{\rho}{\rho_0}D_0 = \underset{\sim}{\sigma} : (\underset{\sim}{F}^e.\dot{\underset{\sim}{F}}^p.\underset{\sim}{F}^{p-1}.\underset{\sim}{F}^{e-1}) + \underset{\approx}{S}_0 \vdots (\underset{\sim}{F}.\dot{\underset{\approx}{K}}^p : (\underset{\sim}{F}^{-1} \boxtimes \underset{\sim}{F}^{-1})) - R\dot{q}. \tag{133}$$

The pseudo-potential of dissipation therefore is a function $\Omega(\underset{\sim}{\Sigma}, \underset{\approx}{S}_0, R)$ such that:

$$\underset{\sim}{\Sigma} = \underset{\sim}{F}^{eT}.\underset{\sim}{\sigma}.\underset{\sim}{F}^{e-T}, \quad \underset{\approx}{S}_0 = \underset{\sim}{F}^T.\underset{\approx}{S} : (\underset{\sim}{F}^{-T} \boxtimes \underset{\sim}{F}^{-T}), \tag{134}$$

$$\dot{\underset{\sim}{F}}^p.\underset{\sim}{F}^{p-1} = \frac{\partial\Omega}{\partial\underset{\sim}{\Sigma}}, \quad \dot{\underset{\approx}{K}}^p = \frac{\partial\Omega}{\partial\underset{\approx}{S}_0}, \quad \dot{q} = -\frac{\partial\Omega}{\partial R}. \tag{135}$$

Note that one may require that $\dot{\underset{\approx}{K}}^p$ should be symmetric with respect to its two last indices, as $\dot{\underset{\approx}{K}}$ does. This implies that $\dot{\underset{\approx}{K}}^e$, $\underset{\approx}{S}$ and $\underset{\approx}{S}_0$ also share this symmetry property.

Alternative decomposition of the strain gradient measure $\underset{\approx}{K}$

For the Cosserat continuum, the advantage of decomposition rule (122) of the curvature tensor with respect to (123) is that the elastic deformation and curvature measures are defined on a common intermediate configuration. It means that, when force stress and couple-stresses are released simultaneously, a single local configuration of the material element does exist. This has a physical meaning in the case of crystal plasticity for instance [13]. One may wish to have a similar property for a second-grade theory. For that purpose, the constitutive equation for the stress $\underset{\approx}{S}$ on the current configuration must have also at large plastic deformation the same form as for pure hyperelastic behavior:

$$\underset{\approx}{S} = \underset{\sim}{F}^{e-T}.\rho\frac{\partial\Psi}{\partial\underset{\approx}{K}^e} : (\underset{\sim}{F}^{eT} \boxtimes \underset{\sim}{F}^{eT}). \tag{136}$$

This is compatible with the following form of the decomposition of $\underset{\approx}{K}$ into elastic and plastic parts:

$$\underset{\approx}{K} = \underset{\sim}{F}^{p-1}.\underset{\approx}{K}^e : (\underset{\sim}{F}^p \boxtimes \underset{\sim}{F}^p) + \underset{\approx}{K}^p. \tag{137}$$

If such a decomposition is preferred, the intrinsic dissipation must be evaluated again and the pseudo-potential of dissipation redefined properly.

Gradient of strain model

As in the small strain case (Sect. 3.2), an alternative theory can be proposed, that explicitly introduces the influence of the gradient of elastic strain $\underset{\sim}{C}^e$ into the free energy. For that purpose, one starts from a different choice of strain gradient measure than (130):

$$\underset{\approx}{K} = \underset{\sim}{C} \otimes \nabla = \underset{\approx}{K}^e + \underset{\approx}{K}^p. \tag{138}$$

Taking the multiplicative decomposition of $\underset{\sim}{F}$ in elastic and plastic parts into account, one can define:

$$\underset{\approx}{K}^e \hat{=} \underset{\sim}{F}^{pT}.\overset{\downarrow}{\underset{\sim}{C}^e}.\underset{\sim}{F}^p \otimes \nabla, \quad \underset{\approx}{K}^p \hat{=} \underset{\sim}{F}^{pT}.\underset{\sim}{C}^e.(\underset{\sim}{F}^p \otimes \nabla) + \overset{\downarrow}{\underset{\sim}{F}^{pT}}.\underset{\sim}{C}^e.\underset{\sim}{F}^p \otimes \nabla. \tag{139}$$

The free energy is still a function $\Psi(\underset{\sim}{\mathbf{C}}^e, \underset{\approx}{\underline{\mathbf{K}}}^e, q)$. As a result, no additional flow rule for $\underset{\approx}{\underline{\mathbf{K}}}^p$ is needed any more, as soon as the field $\underset{\sim}{\mathbf{F}}^p$ is known. For details of elaboration of such a theory of inelasticity see [62].

6.3 Gradient of Cosserat deformation

In Sect. 3.4 and in [44], it has been proposed to reconsider the Cosserat theory in terms of the following modelling quantities:

$$\mathscr{V} = \{\underset{\sim}{\mathbf{F}}, \underset{\sim}{\mathbf{F}} \otimes \nabla, \underset{\sim}{\mathbf{R}}, \underset{\sim}{\mathbf{R}} \otimes \nabla\}, \tag{140}$$

$\underline{\mathbf{u}}, \underset{\sim}{\mathbf{R}}$ being the usual Cosserat degrees of freedom. It is in fact sufficient to consider the reduced set:

$$\mathscr{V} = \{\underset{\sim}{\mathbf{R}}, {}^\sharp\underset{\sim}{\mathbf{F}}, {}^\sharp\underset{\sim}{\mathbf{F}} \otimes \nabla\}. \tag{141}$$

To see this, one notices first that the gradient of the Cosserat rotation can be written as a function of $\underset{\sim}{\mathbf{F}}, \underset{\sim}{\mathbf{R}}, \underset{\sim}{\mathbf{F}} \otimes \nabla$, ${}^\sharp\underset{\sim}{\mathbf{F}} \otimes \nabla$:

$$\underset{\sim}{\mathbf{R}}^T \otimes \nabla = {}^\sharp\overset{\downarrow}{\underset{\sim}{\mathbf{F}}}.\underset{\sim}{\mathbf{F}}^{-1} \otimes \nabla - \underset{\sim}{\mathbf{R}}^T.\overset{\downarrow}{\underset{\sim}{\mathbf{F}}}.\underset{\sim}{\mathbf{F}}^{-1} \otimes \nabla. \tag{142}$$

Furthermore, the second gradient of the displacement field $\underset{\sim}{\mathbf{F}} \otimes \nabla$ is related to $\underset{\sim}{\mathbf{C}} \otimes \nabla$ by Toupin's relation (97) written for large deformation (see [1] for the proof):

$$\underset{\sim}{\mathbf{F}} \otimes \nabla = \frac{1}{2}\ \underset{\sim}{\mathbf{F}}^{-T}.(\underset{\sim}{\mathbf{C}} \otimes \nabla + \underset{\approx}{1}_T : (\nabla \otimes \underset{\sim}{\mathbf{C}}) - \nabla \otimes \underset{\sim}{\mathbf{C}}). \tag{143}$$

Finally, $\underset{\sim}{\mathbf{C}} \otimes \nabla$ is also a function of ${}^\sharp\underset{\sim}{\mathbf{F}}$ and ${}^\sharp\underset{\sim}{\mathbf{F}} \otimes \nabla$, since $\underset{\sim}{\mathbf{C}} = {}^\sharp\underset{\sim}{\mathbf{F}}^T.{}^\sharp\underset{\sim}{\mathbf{F}}$. Accordingly, all quantities of set (140) are functions of the elements of set (141) and vice versa.

The chosen generalized strain measures can now be split into elastic and plastic parts:

$${}^\sharp\underset{\sim}{\mathbf{F}} = {}^\sharp\underset{\sim}{\mathbf{F}}^e.{}^\sharp\underset{\sim}{\mathbf{F}}^p, \quad \underset{\approx}{\underline{\mathbf{\Gamma}}} \hat{=} {}^\sharp\underset{\sim}{\mathbf{F}} \otimes \nabla = \underset{\approx}{\underline{\mathbf{\Gamma}}}^e + \underset{\approx}{\underline{\mathbf{\Gamma}}}^p. \tag{144}$$

The free energy then is a function $\Psi({}^\sharp\underset{\sim}{\mathbf{F}}^e, \underset{\approx}{\underline{\mathbf{\Gamma}}}^e, q)$. A standard viscoplastic framework can be developed based on the free energy and an appropriate pseudo-potential of dissipation. Again, according to such a theory there is no reason why $\underset{\approx}{\underline{\mathbf{\Gamma}}}^e$ should coincide with ${}^\sharp\overset{\downarrow}{\underset{\sim}{\mathbf{F}}^e}.{}^\sharp\underset{\sim}{\mathbf{F}}^p \otimes \nabla$. This indicates the possibility of an alternative theory for which no additional flow rule is needed for $\dot{\mathbf{\Gamma}}^p$ and:

$$\underset{\approx}{\underline{\mathbf{\Gamma}}}^e \hat{=} {}^\sharp\overset{\downarrow}{\underset{\sim}{\mathbf{F}}^e}.{}^\sharp\underset{\sim}{\mathbf{F}}^p \otimes \nabla, \quad \underset{\approx}{\underline{\mathbf{\Gamma}}}^p \hat{=} {}^\sharp\underset{\sim}{\mathbf{F}}^e.({}^\sharp\underset{\sim}{\mathbf{F}}^p \otimes \nabla). \tag{145}$$

6.4 Micromorphic continuum

The finite deformation formulation of the micromorphic theory has been settled in [49], [3] and [6] for hyperelasticity. Extensions to elastoviscoplasticity have been proposed in [63]. A tentative unifying framework with elastoplastic decompositions of strain measures is sketched here. The degrees of freedom are the displacement and micro-deformation fields $\{\underline{\mathbf{u}}, \underset{\sim}{\boldsymbol{\chi}}\}$. The power density of internal forces is defined using stress tensors on the current configuration:

$$p^{(i)} = \underset{\sim}{\boldsymbol{\sigma}} : (\dot{\underset{\sim}{\mathbf{F}}}.\underset{\sim}{\mathbf{F}}^{-1}) + \underset{\sim}{\mathbf{s}} : (\dot{\underset{\sim}{\mathbf{F}}}.\underset{\sim}{\mathbf{F}}^{-1} - \dot{\underset{\sim}{\boldsymbol{\chi}}}.\underset{\sim}{\boldsymbol{\chi}}^{-1}) + {}^{\chi}\underset{\approx}{\mathbf{S}} \vdots ((\dot{\underset{\sim}{\boldsymbol{\chi}}}.\underset{\sim}{\boldsymbol{\chi}}^{-1}) \otimes \boldsymbol{\nabla}^c)$$

$$= \underset{\sim}{\boldsymbol{\sigma}} : (\dot{\underset{\sim}{\mathbf{F}}}.\underset{\sim}{\mathbf{F}}^{-1}) + \underset{\sim}{\mathbf{s}} : (\underset{\sim}{\boldsymbol{\chi}}.(\underset{\sim}{\boldsymbol{\chi}}^{-1}.\underset{\sim}{\mathbf{F}})^{\cdot}.\underset{\sim}{\mathbf{F}}^{-1}) + {}^{\chi}\underset{\approx}{\mathbf{S}} \vdots \left(\underset{\sim}{\boldsymbol{\chi}}.(\underset{\sim}{\boldsymbol{\chi}}^{-1}.(\underset{\sim}{\boldsymbol{\chi}} \otimes \boldsymbol{\nabla}))^{\cdot} : (\underset{\sim}{\boldsymbol{\chi}}^{-1} \boxtimes \underset{\sim}{\mathbf{F}}^{-1}) \right), \tag{146}$$

$\underset{\sim}{\boldsymbol{\sigma}}$ being the symmetric Cauchy stress tensor. The balance equations and boundary conditions on the current configuration have the same form as (91) and (92), provided that the operator $\boldsymbol{\nabla}$ is replaced by $\boldsymbol{\nabla}^c$. Equation (146) indicates the following choice of invariant independent strain measures:

$$\left\{ \underset{\sim}{\mathbf{C}}, \underset{\sim}{\boldsymbol{\chi}}^{-1}.\underset{\sim}{\mathbf{F}}, {}^{\chi}\underset{\approx}{\mathbf{K}} \hat{=} \underset{\sim}{\boldsymbol{\chi}}^{-1}.(\underset{\sim}{\boldsymbol{\chi}} \otimes \boldsymbol{\nabla}) \right\}, \tag{147}$$

i.e., the right Cauchy–Green tensor, the relative deformation and an invariant micro–deformation gradient strain measure, respectively. These tensors correspond to the strain measures used in [49]. A multiplicative decomposition of micro-deformation is then postulated:

$$\underset{\sim}{\boldsymbol{\chi}} = \underset{\sim}{\mathbf{R}}.{}^{\chi}\underset{\sim}{\mathbf{U}}^e.\underset{\sim}{\boldsymbol{\chi}}^p. \tag{148}$$

Such a decomposition is unique once a choice of materials directors is made, the rotation of which with respect to a reference orientation is described by the Cosserat rotation $\underset{\sim}{\mathbf{R}}$. Note that, for finite elastoplastic micro-deformation, the Cosserat rotation is different from the rotation part of the polar decomposition of $\underset{\sim}{\boldsymbol{\chi}}$. The next step is the multiplicative decomposition of the relative deformation:

$$^{\sharp}\underset{\sim}{\mathbf{F}} = \underset{\sim}{\mathbf{R}}^T.\underset{\sim}{\mathbf{F}} = {}^{\sharp}\underset{\sim}{\mathbf{F}}^e.{}^{\sharp}\underset{\sim}{\mathbf{F}}^p. \tag{149}$$

Finally, the micro-deformation gradient strain measure is additively decomposed into elastic and plastic parts:

$$^{\chi}\underset{\approx}{\mathbf{K}} = {}^{\chi}\underset{\approx}{\mathbf{K}}^e + {}^{\chi}\underset{\approx}{\mathbf{K}}^p. \tag{150}$$

Now, the relative deformation tensor takes the form:

$$\underset{\sim}{\boldsymbol{\chi}}^{-1}.\underset{\sim}{\mathbf{F}} = \underset{\sim}{\boldsymbol{\chi}}^{p-1}.\underset{\sim}{\boldsymbol{\Upsilon}}^e.{}^{\sharp}\underset{\sim}{\mathbf{F}}^p, \quad \text{with} \quad \underset{\sim}{\boldsymbol{\Upsilon}}^e = {}^{\chi}\underset{\sim}{\mathbf{U}}^{e-1}.{}^{\sharp}\underset{\sim}{\mathbf{F}}^e. \tag{151}$$

The Helmholtz free energy is therefore a function $\Psi(\underset{\sim}{\mathbf{C}}^e = {}^{\sharp}\underset{\sim}{\mathbf{F}}^{eT}.{}^{\sharp}\underset{\sim}{\mathbf{F}}^e, \underset{\sim}{\boldsymbol{\Upsilon}}^e, {}^{\chi}\underset{\approx}{\mathbf{K}}^e, q)$. The hyperelastic state laws are:

$$^{\sharp}\underset{\sim}{\boldsymbol{\sigma}} = 2{}^{\sharp}\underset{\sim}{\mathbf{F}}^e.\rho \frac{\partial \Psi}{\partial \underset{\sim}{\mathbf{C}}^e}.{}^{\sharp}\underset{\sim}{\mathbf{F}}^{eT}, \quad {}^{\sharp}\underset{\sim}{\mathbf{s}} = {}^{\chi}\underset{\sim}{\mathbf{U}}^{e-1}.\rho \frac{\partial \Psi}{\partial \underset{\sim}{\boldsymbol{\Upsilon}}^e}.{}^{\sharp}\underset{\sim}{\mathbf{F}}^{eT} \tag{152}$$

$$^{\chi}\underset{\approx}{\mathbf{S}} = \underset{\sim}{\boldsymbol{\chi}}^{-T}.\rho \frac{\partial \Psi}{\partial {}^{\chi}\underset{\approx}{\mathbf{K}}^e} : (\underset{\sim}{\boldsymbol{\chi}}^T \boxtimes \underset{\sim}{\mathbf{F}}^T). \tag{153}$$

The intrinsic dissipation then reduces to:

$$\frac{\rho}{\rho_0} D_0 = ({}^{\sharp}\underset{\sim}{\mathbf{F}}^{eT}.({}^{\sharp}\underset{\sim}{\boldsymbol{\sigma}} + {}^{\sharp}\underset{\sim}{\mathbf{s}}).{}^{\sharp}\underset{\sim}{\mathbf{F}}^{e-T}) : ({}^{\sharp}\dot{\underset{\sim}{\mathbf{F}}}^p.{}^{\sharp}\underset{\sim}{\mathbf{F}}^{p-1})$$

$$- ({}^{\chi}\underset{\sim}{\mathbf{U}}^e.{}^{\sharp}\underset{\sim}{\mathbf{s}}.{}^{\chi}\underset{\sim}{\mathbf{U}}^{e-1}) : (\dot{\underset{\sim}{\boldsymbol{\chi}}}^p.\underset{\sim}{\boldsymbol{\chi}}^{p-1}) + (\underset{\sim}{\boldsymbol{\chi}}^T.{}^{\chi}\underset{\approx}{\mathbf{S}} : (\underset{\sim}{\boldsymbol{\chi}}^{-T} \boxtimes \underset{\sim}{\mathbf{F}}^{-T})) \vdots {}^{\chi}\dot{\underset{\approx}{\mathbf{K}}}^p - R\dot{q}. \tag{154}$$

Positivity of dissipation is ensured by the choice of a convex potential $\Omega(\underset{\sim}{\boldsymbol{\Sigma}}, \underset{\sim}{\mathscr{S}}, {}^{\chi}\underset{\approx}{\mathbf{S}}_{\mathbf{0}}, R)$ such that the flow and hardening rules read:

$$ {}^\sharp\dot{\mathbf{F}}^p . {}^\sharp\mathbf{F}^{p-1} = \frac{\partial\Omega}{\partial {}^\chi\boldsymbol{\Sigma}}, \quad \dot{\boldsymbol{\chi}}^p . \boldsymbol{\chi}^{p-1} = \frac{\partial\Omega}{\partial \mathscr{S}}, \quad {}^\chi\dot{\mathbf{K}}^p = \frac{\partial\Omega}{\partial {}^\chi\mathbf{S}_0}, \quad \dot{q} = -\frac{\partial\Omega}{\partial R} \tag{155} $$

with

$$ {}^\chi\boldsymbol{\Sigma} = {}^\sharp\mathbf{F}^{eT} . ({}^\sharp\boldsymbol{\sigma} + {}^\sharp\mathbf{s}) . {}^\sharp\mathbf{F}^{e-T}, \quad \mathscr{S} = -{}^\chi\mathbf{U}^e . {}^\sharp\mathbf{s} . {}^\chi\mathbf{U}^{e-1} \tag{156} $$

$$ {}^\chi\mathbf{S}_0 = \boldsymbol{\chi}^T . {}^\chi\mathbf{S} : (\boldsymbol{\chi}^{-T} \boxtimes \mathbf{F}^{-T}). \tag{157} $$

Alternative decomposition of the micro–deformation gradient strain measure ${}^\chi\mathbf{K}$

The decomposition (150) is far from being the only possible one. One may for instance prefer a decomposition such that the elastic generalized strain measures are defined on a common intermediate configuration. It has been discussed in the case of the Cosserat and second grade continua, see Eqs. (122.2) and (137), respectively. For that purpose the constitutive equation for ${}^\chi\mathbf{S}$ on the current configuration must have also at large plastic deformations the same form as for pure hyperelastic behavior:

$$ {}^\chi\mathbf{S} = \boldsymbol{\chi}^{e-T} . \rho \frac{\partial\Psi}{\partial {}^\chi\mathbf{K}^e} : (\boldsymbol{\chi}^{eT} \boxtimes {}^\sharp\mathbf{F}^{eT}) \quad \text{with} \quad \boldsymbol{\chi}^e = \mathbf{R} . {}^\chi\mathbf{U}^e. \tag{158} $$

This is compatible with the following form of the decomposition of $\mathbf{K}$ into elastic and plastic parts:

$$ {}^\chi\mathbf{K} = \boldsymbol{\chi}^{p-1} . {}^\chi\mathbf{K}^e : (\boldsymbol{\chi}^p \boxtimes {}^\sharp\mathbf{F}^p) + {}^\chi\mathbf{K}^p. \tag{159} $$

If such a decomposition is preferred, the intrinsic dissipation must be evaluated again and the pseudo-potential of dissipation redefined properly. Note that if the micro-deformation reduces to a pure rotation $\mathbf{R}$ (meaning that the micromorphic medium degenerates into a Cosserat one), then $\boldsymbol{\chi}^p = \mathbf{1}$, and the previous decomposition becomes

$$ {}^\chi\mathbf{K} = {}^\chi\mathbf{K}^e . {}^\sharp\mathbf{F}^p + {}^\chi\mathbf{K}^p \tag{160} $$

and that for the Cosserat strain measure (119)

$$ {}^\sharp\boldsymbol{\Gamma} \hat{=} -\frac{1}{2}\,\boldsymbol{\epsilon} : (\mathbf{R}^T . (\mathbf{R} \otimes \nabla)) = -\frac{1}{2}\,\boldsymbol{\epsilon} : {}^\chi\mathbf{K} = -\left(\frac{1}{2}\,\boldsymbol{\epsilon} : {}^\chi\mathbf{K}^e . {}^\sharp\mathbf{F}^p + \frac{1}{2}\,\boldsymbol{\epsilon} : {}^\chi\mathbf{K}^p\right) \tag{161} $$

which has the same form as (122.2) with the identifications

$$ {}^\sharp\boldsymbol{\Gamma}^e = -\frac{1}{2}\,\boldsymbol{\epsilon} : {}^\chi\mathbf{K}^e, \quad {}^\sharp\boldsymbol{\Gamma}^p = -\frac{1}{2}\,\boldsymbol{\epsilon} : {}^\chi\mathbf{K}^p. \tag{162} $$

Gradient of macro- and micro-deformation

A combination of second grade and micromorphic theories has been proposed at small strain in Sect. 5.3. The set of invariant generalized strain measures at finite deformation is:

$$ \left\{\mathbf{C},\ \mathbf{K} \hat{=} \mathbf{F}^{-1} . (\mathbf{F} \otimes \nabla),\ \boldsymbol{\chi}^{-1} . \mathbf{F},\ {}^\chi\mathbf{K} \hat{=} \boldsymbol{\chi}^{-1} . (\boldsymbol{\chi} \otimes \nabla)\right\}. \tag{163} $$

Possible decompositions into elastic and plastic parts are then:

$$\underset{\sim}{\boldsymbol{\chi}} = \underset{\sim}{\mathbf{R}}.{}^{\chi}\underset{\sim}{\mathbf{U}}^{e}.\underset{\sim}{\boldsymbol{\chi}}^{p}, \quad {}^{\sharp}\underset{\sim}{\mathbf{F}} = \underset{\sim}{\mathbf{R}}^{T}.\underset{\sim}{\mathbf{F}} = {}^{\sharp}\underset{\sim}{\mathbf{F}}^{e}.{}^{\sharp}\underset{\sim}{\mathbf{F}}^{p}, \tag{164}$$

$$\underset{\approx}{\mathbf{K}} = \underset{\approx}{\mathbf{K}}^{e} + \underset{\approx}{\mathbf{K}}^{p}, \quad {}^{\chi}\underset{\approx}{\mathbf{K}} = {}^{\chi}\underset{\approx}{\mathbf{K}}^{e} + {}^{\chi}\underset{\approx}{\mathbf{K}}^{p}. \tag{165}$$

7 Conclusions

The present work provides a unifying thermomechanical constitutive framework for generalized continua including additional degrees of freedom or/and the second gradient of displacement. For each continuum, appropriate generalized strain measures have been chosen and split into elastic and viscoplastic parts. Based on the analysis of the dissipation, state laws and evolution equations, including flow rules, have been proposed. A constitutive model is therefore completely characterized by two functions: Helmholtz free energy Ψ depending on elastic strains and hardening variables and a pseudo-potential of dissipation depending on the conjugate effective stresses. The gradient of internal variable approach has been somehow reconciled with higher-order and higher-grade theories in the sense that the nonlocal internal variable has been treated as an actual degree of freedom, thus requiring a contribution in the power of internal forces. Variants of the higher order and higher grade theories are also proposed based on the explicit introduction of plastic strain as additional degree of freedom. This corresponds to the class of models called *gradient of strain* in this work, in contrast to the more classical strain gradient theories. This amounts to recognizing that, in a second grade theory for instance, the plastic part of the strain gradient can be identified with the gradient of plastic strain.

Simple examples have been given dealing with the shearing or bending of an elastoplastic Cosserat or strain gradient material. In both cases, deformation is not homogeneous, and a plastic zone develops in which the stress profiles are significantly different, for the same continuum, depending on the choice of a single coupled yield criterion or of a multi-mechanism model. Furthermore, in the same situation, the standard formulation of second grade elastoplasticity has been compared to the "gradient of strain" version. Adequate experiments with local strain field measurements could be useful to decide which of the models provides the most realistic answer.

The complete gradient of Cosserat deformation model and the gradient of strain and micro-deformation model presented in this work are new extensions of the classical Cosserat and micromorphic theories, respectively, that acknowledge the fact that the gradient of strain and of the additional rotation or deformation degrees of freedom can have an equal influence on material behavior. Second gradient of strain [64] or more general grade-n theories, involving higher orders than the second gradient of displacement ([65], also with respect to dissipative behavior at fully developed turbulent flow [66]) or including the second gradient of the Cosserat rotation (for dissipative fluids [67]) do exist but they have not been considered here.

The class of standard materials considered in this work is far from being the most general constitutive framework, but represents a starting point for more refined models. In particular, the wanted evolution equations may not admit such a potential, or the potential may be different for the plastic flow rules and for the hardening variables [68]. Lastly, non–associative plasticity for which the yield criterion is different from the flow potential are necessary for instance in soil mechanics. Such extensions have been proposed in [36].

The finite deformation formulations of the models have been sketched in the last Section, focusing on the choice of adequate generalized strain measures and proposing some decomposition rules for them into elastic and plastic parts. Pseudo–potentials of dissipation can also be proposed depending on adequate stress measures, thus ensuring the positivity of dissipation. However, other decompositions of the deformation measures are possible depending on the specific microstructural meaning of the additional degrees of freedom. The decomposition (122.2) is for instance adequate for crystal plasticity for which the Cosserat directors are lattice vectors [13].

Two important questions remain to be raised once a generalized elastoviscoplastic model has been retained for a specific material and a specific range of loading conditions : the identification of the numerous material parameters, and its numerical implementation for structural computations. The latter issue is not a strong obstacle to the use of such models any longer. The finite element formulation of the problem for a specific generalized continuum is straightforward once the proper generalized version of the principle of virtual power is provided. This can be deduced from the expressions $p^{(i)}$ given for each model in this work. Efficient implicit algorithms are available for the integration of the flow and evolution equations [31]. The question of the number and identification method of material parameters is far more challenging. Two recent methodologies may well be able to make the 40–year old dream become reality. The first one is the identification of the parameters from experimental results including not only overall response curves (of, e.g. indentation or crack growth [69]) but also local strain fields measurements on inhomogeneously strained specimens [70]. The second method is based on generalized homogenization methods, since nonlocal models are sometimes used to describe heterogeneous materials made of classical constituents with a high contrast of properties, and subjected to strong gradients of macroscopic loading conditions. Generalized homogenization methods have been designed to construct a homogeneous equivalent higher-order or higher-grade medium, replacing the heterogeneous material. They can be used to identify the parameters of the macroscopic model directly from the responses of a unit cell of heterogeneous material subjected to specific boundary conditions [71], [72], [53]. This has been undertaken for linear elastic constituents, as well as in the nonlinear case [35], [73]. When such a homogenization procedure is possible, the number of material parameters of the macroscopic model does not really matter any longer.

Appendix A

Notations

In this work, $\underline{\mathbf{A}}$ denotes a vector of the Euclidean space $\mathbf{E}$, $\underset{\sim}{\mathbf{A}}$ a second-rank Euclidean tensor, and $\underset{\sim}{\underline{\mathbf{A}}}$ (resp. $\underset{\simeq}{\mathbf{A}}$) a third-rank tensor when operating on a vector (resp. a second-rank tensor). The same third-rank tensor is denoted $\underline{\underline{\underline{\mathbf{A}}}}$ when regarded as a 3-linear form. In a positively oriented orthonormal basis $(\underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2, \underline{\mathbf{e}}_3)$ of oriented $\mathbf{E}$ with dimension 3, these quantities are represented by their components:

$$\underline{\mathbf{A}} = A_i \underline{\mathbf{e}}_i, \quad \underset{\sim}{\mathbf{A}} = A_{ij} \underline{\mathbf{e}}_i \otimes \underline{\mathbf{e}}_j, \quad \underset{\sim}{\underline{\mathbf{A}}} = \underset{\simeq}{\mathbf{A}} = \underline{\underline{\underline{\mathbf{A}}}} = A_{ijk} \underline{\mathbf{e}}_i \otimes \underline{\mathbf{e}}_j \otimes \underline{\mathbf{e}}_k \tag{A.1}$$

with summation over all repeated indices. The transpose of second rank tensors is denoted by subscript ^{T}:

$$\underset{\sim}{\mathbf{A}}^T = A_{ji}\underline{\mathbf{e}}_i \otimes \underline{\mathbf{e}}_j. \tag{A.2}$$

The four–rank transposition operator $\underset{\approx}{\mathbf{1}}_T$ is then defined as:

$$\underset{\approx}{\mathbf{1}}_T : \underset{\sim}{\mathbf{A}} = \underset{\sim}{\mathbf{A}}^T. \tag{A.3}$$

We denote $\underset{\sim}{\mathbf{1}}$ the second rank unity tensor and $\underset{\approx}{\boldsymbol{\epsilon}}$ the Levi-Civita tensor

$$\underset{\sim}{\mathbf{1}} = \delta_{ij}\underline{\mathbf{e}}_i \otimes \underline{\mathbf{e}}_j, \quad \underline{\underline{\underline{\boldsymbol{\epsilon}}}} = \mathrm{Det}(\underline{\mathbf{e}}_j, \underline{\mathbf{e}}_k, \underline{\mathbf{e}}_l)\underline{\mathbf{e}}_j \otimes \underline{\mathbf{e}}_k \otimes \underline{\mathbf{e}}_l. \tag{A.4}$$

The tensor product $\otimes$ applies to vectors and higher-order tensors:

$$\underset{\sim}{\mathbf{A}} \otimes \underset{\sim}{\mathbf{B}} = A_{ij}B_{kl}\underline{\mathbf{e}}_i \otimes \underline{\mathbf{e}}_j \otimes \underline{\mathbf{e}}_k \otimes \underline{\mathbf{e}}_l. \tag{A.5}$$

A modified tensor product linking second-rank tensors is also used:

$$\underset{\sim}{\mathbf{A}} \boxtimes \underset{\sim}{\mathbf{B}} = A_{ik}B_{jl}\underline{\mathbf{e}}_i \otimes \underline{\mathbf{e}}_j \otimes \underline{\mathbf{e}}_k \otimes \underline{\mathbf{e}}_l. \tag{A.6}$$

The simple, double and triple contractions of tensors read:

$$\underline{\mathbf{A}}.\underline{\mathbf{B}} = A_iB_i, \quad \underset{\sim}{\mathbf{A}} : \underset{\sim}{\mathbf{B}} = A_{ij}B_{ij}, \quad \underset{\simeq}{\mathbf{A}} \vdots \underset{\simeq}{\mathbf{B}} = A_{ijk}B_{ijk}. \tag{A.7}$$

The nabla operator $\boldsymbol{\nabla}$ is used extensively to compute the gradient or divergence of tensors:

$$\boldsymbol{\nabla} = {}_{,i}\ \underline{\mathbf{e}}_i, \tag{A.8}$$

where the comma denotes differentiation with respect to material (Lagrangian) coordinates $\underline{\mathbf{X}}$. An alternative nabla operator $\boldsymbol{\nabla}^c$ is used also when differentiation is to be taken with respect to current (Eulerian) coordinates $\underline{\mathbf{x}}$:

$$\boldsymbol{\nabla}^c = \boldsymbol{\nabla}.\underset{\sim}{\mathbf{F}}^{-1}, \quad \underset{\sim}{\mathbf{F}} = \frac{\partial\underline{\mathbf{x}}}{\partial\underline{\mathbf{X}}}, \tag{A.9}$$

where $\underset{\sim}{\mathbf{F}}$ is the classical deformation gradient tensor. The gradient of scalars, vectors and tensors takes the form:

$$\boldsymbol{\nabla}T = T_{,i}\underline{\mathbf{e}}_i, \quad \underline{\mathbf{u}} \otimes \boldsymbol{\nabla} = u_{i,j}\underline{\mathbf{e}}_i \otimes \underline{\mathbf{e}}_j, \quad \boldsymbol{\nabla} \otimes \underline{\mathbf{u}} = u_{j,i}\underline{\mathbf{e}}_i \otimes \underline{\mathbf{e}}_j, \tag{A.10}$$

$$\underset{\sim}{\mathbf{F}} \otimes \boldsymbol{\nabla} = F_{ij,k}\underline{\mathbf{e}}_i \otimes \underline{\mathbf{e}}_j \otimes \underline{\mathbf{e}}_k, \quad \boldsymbol{\nabla} \otimes \underset{\sim}{\mathbf{F}} = F_{jk,i}\underline{\mathbf{e}}_i \otimes \underline{\mathbf{e}}_j \otimes \underline{\mathbf{e}}_k. \tag{A.11}$$

The divergence operator is written as follows:

$$\underline{\mathbf{Q}}.\boldsymbol{\nabla} = \boldsymbol{\nabla}.\underline{\mathbf{Q}} = Q_{i,i}, \quad \underset{\sim}{\boldsymbol{\sigma}}.\boldsymbol{\nabla} = \sigma_{ij,j}\underline{\mathbf{e}}_i, \quad \boldsymbol{\nabla}.\underset{\sim}{\boldsymbol{\sigma}} = \sigma_{ij,i}\underline{\mathbf{e}}_j. \tag{A.12}$$

To give examples showing how the product of tensors of different orders is meant, the formulae (119.2), (35) and (97) are written below in index form:

$$^\sharp\boldsymbol{\Gamma}_{ij} = -\frac{1}{2}\epsilon_{ipq}R^T_{pr}\,R_{rq,j}, \tag{A.13}$$

$$\kappa_{lk} = \frac{1}{2}\epsilon_{lij}(e_{ij,k} + (e_{jk} + e_{kj})_{,i}), \tag{A.14}$$

$$u_{i,jk} = \varepsilon_{ij,k} + \varepsilon_{ki,j} - \varepsilon_{jk,i}. \tag{A.15}$$

When the gradient operator acts on an intermediate tensor, the application on that is indicated by an arrow $\downarrow$, as in Eq. (139) for instance, which reads in components:

$$\underset{\sim}{\mathbf{F}}^{pT}.\overset{\downarrow}{\underset{\sim}{\mathbf{C}}^e}.\underset{\sim}{\mathbf{F}}^p \otimes \boldsymbol{\nabla} = F^{pT}_{ip}C^e_{pq,k}F^p_{qj}\underline{\mathbf{e}}_i \otimes \underline{\mathbf{e}}_j \otimes \underline{\mathbf{e}}_k. \tag{A.16}$$

More about vector and tensor analysis in relation to classical and generalized continuum mechanics can be found in [74].

New quantities are defined by the sign $\hat{=}$.

Appendix B

Balance equations for generalized continua

Balance equations and associated boundary conditions are briefly recalled here without proof, for the Cosserat and second grade continua. The corresponding ones for the micromorphic medium are given in Sect. 5.1. The derivation can be found in [20] and [6]. For the sake of simplicity, body forces, couples and double forces are excluded throughout this work.

B.1 Cosserat continuum

The main characteristics and applications of the Cosserat continuum are recalled in [75]. The degrees of freedom, deformation measures and corresponding density of power of internal forces are given by Eqs. (10), (11) and (12). These equations are complemented by the power of contact forces which takes the form:

$$p^{(c)} = \underline{\mathbf{t}}.\underline{\dot{\mathbf{u}}} + \underline{\mathbf{m}}.\underline{\dot{\boldsymbol{\phi}}}, \quad \text{with} \quad \underline{\mathbf{t}} = \underset{\sim}{\boldsymbol{\sigma}}.\underline{\mathbf{n}}, \quad \underline{\mathbf{m}} = \underset{\sim}{\boldsymbol{\mu}}.\underline{\mathbf{n}}, \tag{B.1.1-3}$$

where $\underline{\mathbf{t}}$ and $\underline{\mathbf{m}}$ are the traction and surface couple vectors acting on a surface element of normal $\underline{\mathbf{n}}$. In the bulk of the material, the stress tensors must fulfill the equations of balance of momentum and of moment of momentum:

$$\underset{\sim}{\boldsymbol{\sigma}}.\boldsymbol{\nabla} = 0, \quad \underset{\sim}{\boldsymbol{\mu}}.\boldsymbol{\nabla} - \underset{\approx}{\underline{\boldsymbol{\epsilon}}} : \underset{\sim}{\boldsymbol{\sigma}} = 0. \tag{B.2}$$

B.2 Second-grade model

Balance equations for a theory incorporating the first and second gradients of the displacement have been derived in [4] and in [20]. They involve the force stress tensor $\underset{\sim}{\boldsymbol{\sigma}}$ and the hyperstress tensor $\underset{\approx}{\underline{\mathbf{S}}}$ and reduce to a single set of equations:

$$\underset{\sim}{\boldsymbol{\tau}}.\boldsymbol{\nabla} = 0, \quad \text{with} \quad \underset{\sim}{\boldsymbol{\tau}} = \underset{\sim}{\boldsymbol{\sigma}} - \underset{\approx}{\underline{\mathbf{S}}}.\boldsymbol{\nabla}. \tag{B.3}$$

The corresponding power of contact forces takes the form:

$$p^{(c)} = \underline{\mathbf{t}}.\underline{\dot{\mathbf{u}}} + \underline{\mathbf{M}}.D_n\underline{\dot{\mathbf{u}}}, \tag{B.4}$$

$$\underline{\mathbf{t}} = \underset{\sim}{\boldsymbol{\tau}}.\underline{\mathbf{n}} + 2R\underset{\simeq}{\mathbf{S}} : (\underline{\mathbf{n}} \otimes \underline{\mathbf{n}}) - D_t(\underset{\approx}{\underline{\mathbf{S}}}.\underline{\mathbf{n}}), \quad \underline{\mathbf{M}} = \underset{\simeq}{\mathbf{S}} : (\underline{\mathbf{n}} \otimes \underline{\mathbf{n}}), \tag{B.5}$$

where $\underline{\mathbf{n}}$ and R are the normal vector and the mean curvature of the surface. The surface traction and double traction vectors are $\underline{\mathbf{t}}$ and $\underline{\mathbf{M}}$. The previous conditions are valid for a smooth surface. Special conditions at edges and corners can be found in [20].

Appendix C

Simple glide and bending in Cosserat elastoplasticity

C.1. Simple glide test

A two–dimensional layer of Cosserat material with infinite extension in direction 1 and height h is considered on Fig. 1. The unknowns of the problem are $\underline{\mathbf{u}} = [u(x_2), 0, 0]^T$ and $\underline{\boldsymbol{\phi}} = [0, 0, \phi(x_2)]^T$. Various types of boundary conditions are possible. For example, we consider:

$$u(0) = 0, \phi(0) = 0, \ \underline{\mathbf{t}} = \sigma_{12}\underline{\mathbf{e}}_1 = 0, \ \underline{\mathbf{m}} = \mu_{32}\underline{\mathbf{e}}_3 = \mu^0_{32}\underline{\mathbf{e}}_3. \tag{C.6}$$

Note that the solution of this problem for the classical Cauchy continuum would be a vanishing u. The material exhibits an elastoplastic behavior with a generalized von Mises yield function (22) or (28). Let us recall the elasticity relations in the isotropic case:

$$\underset{\sim}{\boldsymbol{\sigma}} = \lambda(\text{trace}\, \underset{\sim}{\mathbf{e}}^e)\underset{\sim}{\mathbf{1}} + 2\mu^{\{}\underset{\sim}{\mathbf{e}}^{e\}} + 2\mu_c^{\}}\underset{\sim}{\mathbf{e}}^{e\{}, \quad \underset{\sim}{\boldsymbol{\mu}} = \alpha(\text{trace}\, \underset{\sim}{\boldsymbol{\kappa}}^e)\underset{\sim}{\mathbf{1}} + 2\beta^{\{}\underset{\sim}{\boldsymbol{\kappa}}^{e\}} + 2\gamma^{\}}\underset{\sim}{\boldsymbol{\kappa}}^{e\{}, \tag{C.7}$$

where λ, μ are the Lamé constants and $\mu_c, \alpha, \ \beta, \gamma$ are additional moduli. The brackets {} (resp.}{) denote the symmetric (resp. skew-symmetric) part of the tensor. One usually takes $\beta = \gamma$ at least in the two-dimensional case [31]. An elastic Cosserat characteristic length $l_e = \sqrt{\beta/\mu}$ can be defined. Under the prescribed boundary conditions, a plastic zone develops starting from the top.

Elastic zone, $0 \leq x_2 \leq \alpha$

The evaluation of elasticity law and balance equations leads to the following equations:

$$\sigma_{12} = (\mu + \mu_c)u_{,2} + 2\mu_c\phi, \quad \sigma_{21} = (\mu - \mu_c)u_{,2} - 2\mu_c\phi, \quad \mu_{32} = 2\beta\phi_{,2}, \tag{C.8}$$

$$\sigma_{12,2} = 0, \quad \mu_{32,2} + \sigma_{21} - \sigma_{12} = 0, \tag{C.9}$$

from which two differential equations are deduced:

$$\phi_{,22} = \omega_e^2\phi, \quad u_{,2} = -\frac{2\mu_c}{\mu + \mu_c}\phi, \quad \omega_e = \sqrt{\frac{2\mu\mu_c}{\beta(\mu + \mu_c)}}. \tag{C.10}$$

Taking the boundary conditions at the bottom into account, the solutions follow, including an integration constant B to be determined:

$$\phi(x_2) = B\sinh(\omega_e x_2), \quad u(x) = \frac{2\mu_c B}{\omega_e(\mu + \mu_c)}(1 - \cosh(\omega_e x_2)), \tag{C.11}$$

$$\mu_{32} = 2B\beta\omega_e\cosh(\omega_e x_2), \quad \sigma_{21} = -\frac{4\mu\mu_c}{\mu + \mu_c}B\sinh(\omega_e x_2). \tag{C.12}$$

Plastic zone, $\alpha \leq x_2 \leq h$

In the generalized von Mises criteria (22) or (28), the simplifying assumption $a_1 = a, a_2 = 0, b_1 = b, b_2 = 0$ is adopted, together with a constant threshold $R = R_0$. The yield criterion (22) requires:

$$a\sigma_{21}^2 + b\mu_{32}^2 = R_0^2. \tag{C.13}$$

Combining this condition with balance equations (C.9), the solution takes the following form including integration constants C and D:

$$\mu_{32} = C\cos(\omega_p x_2) + D\sin(\omega_p x_2), \quad \sigma_{21} = \omega_p(C\sin(\omega_p x_2) - D\cos(\omega_p x_2)), \tag{C.14}$$

$$\omega_p = \frac{1}{l_p} = \sqrt{\frac{b}{a}}, \tag{C.15}$$

where a characteristic length l_p comes into play. The constants C and D are solutions of the following system of equations:

$$C^2 + D^2 = \frac{R_0^2}{b}, \quad C\,\cos(\omega_p h) + D\sin(\omega_p h) = \mu_{32}^0. \tag{C.16}$$

The continuity of surface couple vector and yield condition at $x_2 = \alpha$ provides the system of equations for the unknowns B and α:

$$2\beta\omega_e B\cosh(\omega_e\alpha) = C\,\cos(\omega_p\alpha) + D\sin(\omega_p\alpha), \tag{C.17}$$

$$16a\left(\frac{\mu\mu_c}{\mu+\mu_c}\right)^2 B^2\sinh^2(\omega_e\alpha) + 4b\beta^2\omega_e^2 B^2\cosh^2(\omega_e\alpha) = R_0^2. \tag{C.18}$$

The numerical resolution of both systems of equations leads to a semi–analytical solution of the simple glide test, that can be used as test for the implementation of Cosserat elastoplasticity in a Finite Element code. This has been checked for the simulation presented on Fig. 3.

In contrast, the use of two separate criteria (28) without hardening and assuming plastic loading for both deformation and curvature leads to a plastic zone with no extension with constant force and couple-stress σ_{21} and μ_{32} at the upper boundary.

C.2 Simple bending

Simple bending is a well-suited test to investigate the effect of curvature on the overall response of the material. The bending of metal sheets has been studied experimentally in the elastic regime [76] and in the plastic regime [77]: size effects have been observed only in the latter case. The solution of the simple bending problem is given here for the elastic and elastoplastic cases.

Elastic solution

The beam of thickness h and width W of Fig. 2 is considered for simple bending under plane stress conditions and for an elastic isotropic Cosserat material. Two types of boundary conditions are possible: imposed couple M on the beam, or rotation of left and right sides of the beam. In the latter case, one can prescribe a micro-rotation equal to the rotation of the section, but it does not matter in the sense of Saint-Venant. The solution takes the form:

$$u_1 = Ax_1x_2, \quad u_2 = -\frac{A}{2}x_1^2 + \frac{D}{2}\left(x_2^2 - x_3^2\right), \quad u_3 = Dx_2x_3, \tag{C.19}$$

$$\phi_1 = Dx_3, \quad \phi_2 = 0, \quad \phi_3 = -Ax_1 \tag{C.20}$$

in the coordinate frame defined in Fig. 2. Under these conditions the non-vanishing components of the deformation and curvature tensors are:

$$e_{11} = Ax_2, \quad e_{22} = e_{33} = Dx_2, \quad \kappa_{31} = -A, \quad \kappa_{13} = D, \tag{C.21}$$

which shows that the solution is in principle fully three-dimensional. The fact that the deformation tensor is found to be symmetric means that there is no relative rotation between material lines and the Cosserat directors. The plane stress condition implies that the constants A and D are related by $D = -\nu A$. The non-vanishing stress components are then:

$$\sigma_{11} = EAx_2, \quad \mu_{13} = -A(\beta(1+\nu) - \gamma(1-\nu)), \tag{C.22}$$

$$\mu_{31} = -A(\beta(1+\nu) + \gamma(1-\nu)) = -\beta^\star A. \tag{C.23}$$

The couple stress component μ_{13} is an out-of-plane component that should vanish under plane couple stress condition. This can be regarded as a reaction stress that will not be considered here in order to keep the simple form of the solution. Note also that it vanishes for the choice $\gamma = (1+\nu)/(1-\nu)$. The resulting moment M with respect to axis 3 is computed as

$$M = \int (\sigma_{11}x_2 + \mu_{31})\mathrm{d}x_2\mathrm{d}x_3 = WA\left(\frac{Eh^3}{12} + \beta^\star h\right) \tag{C.24}$$

which gives A for a given couple M. The additional resistance due to the Cosserat character of the material can be readily seen in the term $\beta^\star$. Formula (C.24) reduces to the classical solution when the Cosserat characteristic length goes to zero or when l_e is much smaller than h.

Plastic case

Some elements of the solution of the bending problem for Cosserat elastoplasticity with a single coupled plastic potential (22) are provided here, that can be compared to Finite Element simulations. We still look for a solution of the form:

$$u_1 = Ax_1x_2, \quad \phi_3 = -Ax_1,$$

where A is the loading parameter. The non-vanishing stress components are σ_{11} and μ_{31}. The component μ_{13} may exist but is not taken into account in the present two-dimensional solution. The yield condition reads:

$$\frac{2}{3}a\sigma_{11}^2 + b\mu_{31}^2 = R_0^2. \tag{C.25}$$

As in the classical solution, a plastic zone starts from the top and the bottom up to $x_2 = \pm\alpha$. In the plastic zone, the plastic deformation and curvature are deduced from the flow rules (23):

$$e_{11}^p = p\frac{2}{3}\frac{a}{R_0}\sigma_{11}, \quad e_{22}^p = e_{33}^p = -p\frac{1}{3}\frac{a}{R_0}\sigma_{11}, \quad \kappa_{31}^p = p\frac{b}{R_0}\mu_{31}. \tag{C.26}$$

Combining the elastic and plastic parts of deformation and curvature, we get:

$$e_{11} = Ax_2 = e_{11}^e + e_{11}^p = \left(\frac{1}{E} + \frac{2}{3}\frac{a}{R_0}p\right)\sigma_{11}, \tag{C.27}$$

$$\kappa_{31} = -A = \kappa_{31}^e + \kappa_{31}^p = \left(\frac{1}{2\beta} + \frac{b}{R_0}p\right)\mu_{31} \tag{C.28}$$

Eliminating p from (C.27) using (C.28), we get:

$$\frac{x_2}{\sigma_{11}} + \frac{2}{3}\frac{a}{b}\frac{1}{\mu_{31}} = \frac{1}{A}\left(\frac{1}{E} - \frac{1}{3\beta}\frac{a}{b}\right). \tag{C.29}$$

This equation combined with (C.25) leads to a system of two equations in σ_{11} and μ_{31}. It can be shown that σ_{11} then is a root of an algebraic equation of degree 3, the coefficients depending on the material parameters and on x_2. A simple solution can be given however if the right-hand side of (C.29) is neglected, which is possible for sufficiently large values of prescribed A and of $\beta b/a$ which is proportional to l_e^2/l_p^2. In this case, we get, for a fully plastic beam ($\alpha = 0$):

$$\mu_{31} = \frac{R_0 l_p}{\sqrt{a + \frac{3}{2} b x_2^2}}, \tag{C.30}$$

and the profile of σ_{11} is deduced from the yield condition (C.25). The role played by the plastic characteristic length l_p defined by (C.15) appears clearly. When l_p tends towards zero, the classical solution $\sigma_{11} = \pm R_0\sqrt{3/2a}$ is retrieved. This dependence on l_p is illustrated in Fig. 4. The approximate solution is found to be a good one when compared to a finite element simulation.

In contrast, the use of two potentials f and f_c according to Eq. (28) leads to a linear profile of σ_{11} in the elastic zone and a constant value $\sigma_{11} = \pm R_0/\sqrt{a}$ in the plastic zone of the beam. The component μ_{31} remains constant in the beam with the value $R_{c0}/\sqrt{b}$.

Appendix D

Simple glide in strain gradient elastoplasticity

The two-dimensional layer of Fig. 1 is considered as an elastoplastic strain gradient material. The unknown of the problem is $\underline{\mathbf{u}} = [u(x_2), 0, 0]^T$. The constitutive frameworks presented in Sects. 3.1 and 3.2 are applied successively to this situation. The non-vanishing components of the strain and strain gradient tensors are:

$$\varepsilon_{12} = u_{,2}/2, \quad K_{122} = K_{212} = u_{,22}/2. \tag{D.31}$$

D.1 Standard strain gradient model

Various types of boundary conditions are possible. For example, we consider:

$$u(0) = 0, \quad (D_n\underline{\mathbf{u}})(0) = \frac{1}{2}u_{,2}(0)\underline{\mathbf{e}}_1 = 0, \tag{D.32}$$

$$\underline{\mathbf{t}}(h) = (\sigma_{12} - S_{122,2})\underline{\mathbf{e}}_1 = 0, \quad \underline{\mathbf{M}}(h) = S_{122}\underline{\mathbf{e}}_1 = S_0\underline{\mathbf{e}}_1, \tag{D.33}$$

which are similar to the conditions (C.6). The solution corresponding to this problem with the classical Cauchy continuum is a vanishing displacement u. In contrast, the strain gradient medium deforms due to the application of the double force $S_{122} = S_0$ at the upper boundary. An elastic and an elastoplastic zone coexist with interface $x_2 = \alpha$.

Elastic zone, $0 \le x_2 \le \alpha$

The isotropic elasticity laws for a second-grade medium can be found in [4]. The elastic part of the free energy reads:

$$\Psi^e(\underset{\sim}{\boldsymbol{\varepsilon}}^e, \underset{\approx}{\mathbf{K}}^e) = \frac{1}{2}\lambda\varepsilon^e_{ii}\varepsilon^e_{jj} + \mu\varepsilon^e_{ij}\varepsilon^e_{ij} + a_1K^e_{kii}K^e_{jjk} + a_2K^e_{jji}K^e_{kki} + a_3K^e_{kii}K^e_{kjj} + a_4K^e_{jki}K^e_{jki} + a_5K^e_{jki}K^e_{jik}. \tag{D.34}$$

The components of the hyperstress tensor are deduced as:

$$S_{pqr} = a_1\left(K^e_{iip}\delta_{qr} + K^e_{rii}\delta_{pq}\right) + 2a_2K^e_{iir}\delta_{pq} + 2a_3K^e_{pii}\delta_{qr} + 2a_4K^e_{pqr} + 2a_5K^e_{rpq}. \tag{D.35}$$

In the present example, the non-vanishing components of the hyperstress tensor are

$$S_{111} = (a_1 + 2a_3)K^e_{122}, \quad S_{221} = (a_1 + 2a_5)K^e_{122}, \quad S_{122} = S_{212} = 2a_{345}K^e_{122} \tag{D.36}$$

with $a_{345} = a_3 + a_4 + a_5$. In the sequel, we assume that the components S_{111} and S_{221} vanish, by a proper choice of the elastic constants a_3 and a_5. This choice is compatible with the conditions of definite positivity of the elasticity tensor, established in [4, Eq. (5.25)]. This makes possible a simple semi-analytical solution of the elastoplastic problem, similar to the situation solved with the Cosserat continuum in Sect. C.1.

The balance of momentum equation reduces to

$$\sigma_{12,2} - S_{122,22} = 0, \tag{D.37}$$

which means that $\sigma_{12} - S_{122,2}$ is constant and equal to zero due to the boundary condition at the upper part. Accordingly, in the elastic zone, the solution takes the form:

$$u(x_2) = A(\cosh(\omega_e x_2) - 1), \quad \sigma_{12} = \mu A\omega_e \sinh(\omega_e x_2), \quad S_{122} = \mu A \cosh(\omega_e x_2), \tag{D.38}$$

where A is a constant to be determined and

$$\omega_e = \sqrt{\frac{\mu}{a_{345}}}. \tag{D.39}$$

Plastic zone, $\alpha \le x_2 \le h$

The generalized J_2-theory of plasticity proposed in [15] is used. The extension of the von Mises equivalent stress involves the deviatoric parts of the stress tensors $\underset{\sim}{\boldsymbol{\sigma}}$ and $\underset{\approx}{\mathbf{S}}$. In the simple example considered here, the yield condition reduces to:

$$3\sigma^2_{12} + \frac{4}{l_p^2}S^2_{122} = R_0^2, \tag{D.40}$$

where l_p is an effective characteristic length, that can be expressed in terms of the three lengths introduced in [15]. The stress and hyperstress components are therefore solutions of the system of equations (D.37) and (D.40). One finds:

$$\sigma_{12} = -C\omega_p \cos(\omega_p x_2) + D\omega_p \sin(\omega_p x_2), \quad S_{122} = C \cos(\omega_p x_2) + D \sin(\omega_p x_2), \tag{D.41}$$

$$\omega_p = \frac{2}{\sqrt{3}l_p}. \tag{D.42}$$

The integration constants C and D are determined from the yield condition (D.40) and from the boundary condition:

$$C \cos(\omega_p h) + D \sin(\omega_p h) = S_0. \tag{D.43}$$

A condition for this system of equations to have a solution is $S_0^2 \le R_0^2/3\omega_p^2$. The constant A appearing in (D.38) and position α are deduced from the continuity of S_{122} and yield condition at α:

$$A\mu\cosh(\omega_e\alpha) = C \cos(\omega_p\alpha) + D \sin(\omega_p\alpha), \tag{D.44}$$

$$A^2\mu^2\left(\omega_e^2\sinh^2(\omega_e\alpha) + \omega_p^2\cosh^2(\omega_e\alpha)\right) = R_0^2/3. \tag{D.45}$$

The profile of hyperstress component S_{122}, found numerically, is given on Fig. 5.

D.2 Strain gradient plasticity with gradient of plastic strain

Let us now apply to the same boundary value problem the alternative constitutive framework for strain gradient plasticity presented in Sect. 3.2. The boundary conditions must however be complemented as follows, in accordance to the specific choice of modelling quantities (54) and associated forces (56):

$$\underset{\sim}{\mathbf{A}}^c(h) = 0. \tag{D.46}$$

The elastic solution still is the same as in the standard case. Balance equation (D.38) still holds. The model involves another hyperstress tensor $\underset{\approx}{\mathbf{B}}$ in addition to $\underset{\approx}{\mathbf{S}}$. A simple linear relationship between $\underset{\approx}{\mathbf{B}}$ and the gradient of plastic strain is assumed, including material parameter c with the dimension of pressure multiplied by the square of a length:

$$B_{122} = c\varepsilon^p_{12,2}. \tag{D.47}$$

The effective stress driving plastic flow is $\underset{\sim}{\tau}^{eff}$:

$$\tau^{eff}_{12} = \sigma_{12} + B_{122,2} - S_{122,2} = \sigma_{12} + c\varepsilon^p_{12,22} - S_{122,2}. \tag{D.48}$$

The yield criterion is chosen as:

$$J_2(\underset{\sim}{\tau}^{eff}) - R_0 = \sqrt{3}|\sigma_{12} + c\varepsilon^p_{12,22} - S_{122,2}| - R_0 = 0. \tag{D.49}$$

The final equation is given by the elasticity law:

$$S_{122} = 2a_{345}\varepsilon^e_{12,2} = \frac{a_{345}}{\mu}\sigma_{12,2} = \sigma_{12,2}/\omega_e^2. \tag{D.50}$$

It follows that the stress components take the form:

$$\sigma_{12} = C\omega_e\cosh(\omega_e x_2) + D\omega_e\sinh(\omega_e x_2), \quad S_{122} = C\cosh(\omega_e x_2) + D\sinh(\omega_e x_2). \tag{D.51}$$

Combining Eq. (D.49) with balance equation (D.37), the plastic deformation obeys to:

$$\Delta\varepsilon^p_{12} = \frac{R_0}{\sqrt{3}c} \quad \Longrightarrow \quad \varepsilon^p_{12} = \frac{R_0}{2c\sqrt{3}}(x_2 - \alpha)^2 + E(x_2 - \alpha) + F. \tag{D.52}$$

Since plastic and elastic strain are now known, the total displacement field in the plastic zone can be computed:

$$u_{,2} = \frac{\omega_e}{\mu}(C\cosh(\omega_e x_2) + D\sinh(\omega_e x_2)) + \frac{R_0}{c\sqrt{3}}(x_2 - \alpha)^2 + 2E(x_2 - \alpha) + 2F. \tag{D.53}$$

The displacement field in the elastic zone is still given by (D.38). To determine the integration constants C, D, E, F, α and A (A characterizes the elastic zone, see Eq. (D.38)), the following requirements must be taken into account:

(i) Continuity of ε^p_{12} at $x_2 = \alpha$. This implies $F = 0$.
(ii) Continuity of $\underset{\sim}{\mathbf{A}}^c = (c\varepsilon^p_{12,2} - S_{122})(\underline{\mathbf{e}}_1 \otimes \underline{\mathbf{e}}_2 + \underline{\mathbf{e}}_2 \otimes \underline{\mathbf{e}}_1)$ at $x_2 = \alpha$. This condition together with the continuity of S_{122} implies the continuity of $\varepsilon_{12,2}$. This means $E = 0$.
(iii) Continuity of $u_{,2}$:

$$A\omega_e\sinh(\omega_e\alpha) = \frac{\omega_e}{\mu}(C\sinh(\omega_e\alpha) + D\cosh(\omega_e\alpha)) \tag{D.54}$$

(iv) Continuity of S_{122}:

$$\mu A\cosh(\omega_e \alpha) = C\cosh(\omega_e \alpha) + D\sinh(\omega_e \alpha) \tag{D.55}$$

(v) Boundary condition $S_{122}(h) = S_0$
(vi) Additional boundary condition (D.46):

$$\underset{\sim}{\mathbf{A}}^c = (\underset{\approx}{\mathbf{B}} - \underset{\approx}{\mathbf{S}}).\underline{\mathbf{n}} = 0, \quad \text{i.e.,} \quad \varepsilon^p_{12,2} - S_{122} = 0. \tag{D.56}$$

This provides the value of the position of the interface of the elastic/plastic zone:

$$\alpha = h - \sqrt{3}S_0/R_0. \tag{D.57}$$

This results in:

$$A = \frac{S_0}{\mu\cosh(\omega_e h)}, \quad C = \frac{S_0}{\cosh(\omega_e h)}, \quad D = 0.$$

It turns out that the expression (D.38) of the stress components σ_{12} and S_{122} is valid for both zones. The profile of S_{122} according to this model is given in Fig. 5 and compared to the one found with the standard model.

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