Micromorphic Approach for Gradient Elasticity, Viscoplasticity, and Damage

Samuel Forest[1]

Abstract: A unifying thermomechanical framework is presented that reconciles several classes of gradient elastoviscoplasticity and damage models proposed in the literature during the last 40 years. It is based on the introduction of the micromorphic counterpart ${}^{\chi}\phi$ of a selected state or internal variable ϕ in a standard constitutive model. In addition to the classical balance of momentum equation, a balance of micromorphic momentum is derived that involves generalized stress tensors. The corresponding additional boundary conditions are also deduced from the procedure. The power of generalized forces is assumed to contribute to the energy balance equation. The free energy density function is then chosen to depend on a relative generalized strain, typically ϕ-${}^{\chi}\phi$, and the microstrain gradient $\nabla^{\chi}\phi$. When applied to the deformation gradient itself, $\phi \equiv \underset{\sim}{\mathbf{F}}$, the method yields the micromorphic theory of Eringen and Mindlin together with its extension to finite deformation elastoviscoplasticity by Forest and Sievert. If the selected variable is the cumulative plastic strain, the theory reduces to the so-called "nonlocal implicit gradient-enhanced elastoplasticity model" by Engelen, Geers, and Peerlings, provided that simplified linear relationships are adopted between generalized stresses and strains. The same holds if the micromorphic variable coincides with a microdamage variable. If the internal constraint is introduced that the micromorphic variable ${}^{\chi}\phi$ remains as close as possible to the macroscopic variable ϕ, the micromorphic model reduces to the second gradient or gradient of internal variable approach as defined by Maugin. If the selected variable is the cumulative plastic strain or the full plastic strain tensor, the constrained micromorphic theory delivers Aifantis-like strain gradient plasticity models. The advantage of the micromorphic approach is that it provides the generalized balance equation under nonisothermal conditions and offers the setting for anisotropic nonlinear constitutive relations between generalized stress and strains in contrast to most existing models. In rate-independent plasticity, it is shown that there is generally no need for a variational formulation of the yield condition.

DOI: 10.1061/(ASCE)0733-9399(2009)135:3(117)

CE Database subject headings: Elasticity; Damage; Viscoplasticity.

Introduction

In the continuum modeling of materials with microstructure as developed in the early 1960s, microscopic variables are introduced in the classical continuum setting, that fulfill additional balance equations complementing the balance of momentum equations. The paradigm of such a continuum model accounting for microstructure effect is the micromorphic medium introduced by Eringen and Suhubi (1964a,b) and Mindlin (1964), which introduces a full (generally noncompatible) microdeformation field in addition to the classical displacement field. In view of the tremendous computational effort associated with the introduction of additional degrees-of-freedom, it was shown in the 1970s that, in many cases, the generalized balance equations for microvariables can be reduced to differential evolution equations (Sidoroff 1975). In this process, the *internal degrees-of-freedom* become *internal variables* (Maugin 1999). This approach led to considerable successes in the modeling of the elastoviscoplastic behavior of materials.

Due to the current endeavor to model size effects in the constitutive and fracture behavior of materials, there is a need to restore the status of internal degrees-of-freedom to some variables involved in nowadays classical elastoviscoplastic constitutive models. This gives rise to a large variety of generalized continuum models based on well-established continua such as the Cosserat, second gradient, and micromorphic theories (Forest and Sievert 2003), as well as on more recent plastic strain gradient approaches (Aifantis 1987; Dorgan and Voyiadjis 2003), and the so-called implicit theory of plasticity and damage (Peerlings et al. 2001). In the latter approaches, the additional partial differential equations to be solved are very often postulated without providing the most general possible boundary conditions, nor the thermodynamic framework required for the extensibility of the models to nonisothermal and nonlinear (geometrical and material) behavior. The abundant literature conveys the image of a plethoric zoology of generalized continuum models, which makes difficult the choice of the most appropriate model for a given material. There is a real need for unifying and classifying these approaches.

The objective of the present work is to show that most models in the strain gradient literature can be related to a systematic thermomechanical method of construction of higher order media, that we call the *micromorphic approach*. The name *micromorphic* is used here in a broader sense than in the original expression coined by Eringen and Suhubi (1964a,b). The class of models derived in this work share two aspects with the original one, called here the full micromorphic medium: an internal degree-of-freedom related to strain quantities and a dependence of the free

[1]Mines Paristech, Centre des Matériaux, CNRS UMR 7633, BP 87, 91003 Evry Cedex, France. E-mail: samuel.forest@ensmp.fr

Note. Associate Editor: George Z. Voyiadjis. Discussion open until August 1, 2009. Separate discussions must be submitted for individual papers. The manuscript for this paper was submitted for review and possible publication on July 18, 2007; approved on November 7, 2007. This paper is part of the ***Journal of Engineering Mechanics***, Vol. 135, No. 3, March 1, 2009. ©ASCE, ISSN 0733-9399/2009/3-117–131/$25.00.

energy on the difference between the macro and microvariables. According to the proposed approach, an internal variable ϕ, of any tensorial nature, is selected in a classical elastoviscoplastic model. The status of internal degree-of-freedom is restored for the associated micromorphic variable ${}^{\chi}\phi$, appearing explicitly in the balance equations (Svendsen 1999). For isotropic materials, introducing the quadratic contributions $(\phi-{}^{\chi}\phi)^2$ and $\nabla\, {}^{\chi}\phi . \nabla\, {}^{\chi}\phi$ into the free energy will turn out to be the necessary hypothesis to recover most of the existing formulations.

The systematic formal procedure called *micromorphic approach* is explained in the next section. It is shown in which cases the standard Cosserat, second gradient, and micromorphic theories are recovered. The procedure is then applied subsequently to strain gradient plasticity. In particular, the additional partial differential equations to be solved are derived to include the effect of gradient of isotropic and kinematic hardening variables. The consistency conditions in plasticity and the extensions to anisothermal conditions are explored. Microdamage is considered in the next section. Specific difficulties arising for multimechanisms in anisotropic plasticity and damage, exemplified by crystal plasticity and cleavage, are investigated following that. In the last section, we consider the internal constraint forcing the micromorphic variable ${}^{\chi}\phi$ to remain as close as possible to the macrovariable ϕ. In this way, the well-known Aifantis model and its recent variants are retrieved.

In this work, zeroth, first, second, and third order tensors are denoted by $a, \underline{a}, \underset{\sim}{a}, \underset{\approx}{a}$, respectively. The simple, double, and triple contractions are written ., :, and $\vdots$, respectively. In index form with respect to an orthonormal Cartesian basis, these notations correspond to

$$\underline{\mathbf{a}} \, . \, \underline{\mathbf{b}} = a_i b_i \quad \underset{\sim}{\mathbf{a}} : \underset{\sim}{\mathbf{b}} = a_{ij} b_{ij} \quad \underset{\approx}{\mathbf{a}} \vdots \underset{\approx}{\mathbf{b}} = a_{ijk} b_{ijk} \tag{1}$$

where repeated indices are summed up. The tensor product is denoted by $\otimes$. The nabla operator with respect to the reference configuration is denoted by ∇. For example, the component ijk of $\nabla \underset{\sim}{\mathbf{F}}$ is $F_{ij,k}$. The sign $:=$ defines the quantity on the left-hand side.

General Procedure for Introducing Micromorphic Variables

Micromorphic Approach

We start from an elastoviscoplasticity model formulation within the framework of the classical Cauchy continuum and classical continuum thermodynamics according to Germain et al. (1983) and Maugin (1999). The material behavior is characterized by the reference sets of degrees-of-freedom and state variables

$$\text{DOF0} = \{\underline{\mathbf{u}}\} \quad \text{STATE } 0 = \{\underset{\sim}{\mathbf{F}},\, T,\, \alpha\} \tag{2}$$

on which the free energy density function ψ may depend. The displacement vector is $\underline{\mathbf{u}}$. The deformation gradient is denoted by $\underset{\sim}{\mathbf{F}}$ whereas α represents the whole set of internal variables of arbitrary tensorial order accounting for nonlinear processes at work inside the material volume element, like isotropic and kinematic hardening variables. The absolute temperature is T.

The proposed systematic method for the enhancement of the previous continuum and constitutive theory to incorporate generalized strain gradient effects proceeds as follows:

1. Select a variable ϕ from the set of state variables, which is supposed to carry the targeted gradient effects

$$\phi \in \{\underset{\sim}{\mathbf{F}},\, T,\, \alpha\} \tag{3}$$

   It can be a tensor variable of arbitrary rank. For the illustration, it is treated as a scalar quantity in this section.
2. Introduce the micromorphic variable ${}^{\chi}\phi$ associated with ϕ. It has the same tensor rank and same physical dimension as ϕ. Regard it as an additional internal degree-of-freedom

$$\text{DOF} = \{\underline{\mathbf{u}},\, {}^{\chi}\phi\} \tag{4}$$

3. Extend the virtual power of internal forces to the power done by the micromorphic variable and its first gradient

$$\mathcal{P}^{(i)}(\underline{\mathbf{v}}^\star, {}^{\chi}\dot{\phi}^\star) = -\int_{\mathcal{D}} p^{(i)}(\underline{\mathbf{v}}^\star, {}^{\chi}\dot{\phi}^\star)dV$$

$$p^{(i)}(\underline{\mathbf{v}}^\star, {}^{\chi}\dot{\phi}^\star) = \underset{\sim}{\boldsymbol{\sigma}} : \nabla \underline{\mathbf{v}}^\star + a\, {}^{\chi}\dot{\phi}^\star + \underline{\mathbf{b}} \, . \, \nabla\, {}^{\chi}\dot{\phi}^\star \tag{5}$$

   where $\mathcal{D}$=subdomain of the current configuration Ω of the body. The Cauchy stress is $\underset{\sim}{\boldsymbol{\sigma}}$ and a and $\underline{\mathbf{b}}$=generalized stresses associated with the micromorphic variable and its first gradient.
4. Extend then the power of contact forces as follows:

$$\mathcal{P}^{(c)}(\underline{\mathbf{v}}^\star, {}^{\chi}\dot{\phi}^\star) = \int_{\mathcal{D}} p^{(c)}(\underline{\mathbf{v}}^\star, {}^{\chi}\dot{\phi}^\star)dV$$

$$p^{(c)}(\underline{\mathbf{v}}^\star, {}^{\chi}\dot{\phi}^\star) = \underline{\mathbf{t}} \, . \, \underline{\mathbf{v}}^\star + a^c\, {}^{\chi}\dot{\phi}^\star \tag{6}$$

   where $\underline{\mathbf{t}}$=traction vector and a^c a generalized traction.
5. Extend the power of forces acting at a distance by introducing, if necessary, generalized body forces

$$\mathcal{P}^{(e)}(\underline{\mathbf{v}}^\star, {}^{\chi}\dot{\phi}^\star) = \int_{\mathcal{D}} p^{(e)}(\underline{\mathbf{v}}^\star, {}^{\chi}\dot{\phi}^\star)dV$$

$$p^{(e)}(\underline{\mathbf{v}}^\star, {}^{\chi}\dot{\phi}^\star) = \rho \underline{\mathbf{f}} \, . \, \underline{\mathbf{v}}^\star + a^e\, {}^{\chi}\dot{\phi}^\star + \underline{\mathbf{b}}^e \, . \, \nabla\, {}^{\chi}\dot{\phi}^\star \tag{7}$$

   where $\rho \underline{\mathbf{f}}, a^e, \underline{\mathbf{b}}^e$ account for given simple and generalized body forces. Following Germain (1973), given body couples and double forces working with the gradient of the velocity field, could also be introduced in the theory.
6. Formulate the generalized principle of virtual power with respect to the velocity and micromorphic variable fields, presented here in the static case only

$$\mathcal{P}^{(i)}(\underline{\mathbf{v}}^\star, {}^{\chi}\dot{\phi}) + \mathcal{P}^{(e)}(\underline{\mathbf{v}}^\star, {}^{\chi}\dot{\phi}^\star) + \mathcal{P}^{(c)}(\underline{\mathbf{v}}^\star, {}^{\chi}\dot{\phi}) = 0$$

$$\forall\, \mathcal{D} \subset \Omega \quad \forall\, \underline{\mathbf{v}}^\star \quad \forall\, {}^{\chi}\dot{\phi} \tag{8}$$

   The method of virtual power according to Maugin (1980) is used then to
7. Derive the standard local balance of momentum equation

$$\operatorname{div} \underset{\sim}{\boldsymbol{\sigma}} + \rho \underline{\mathbf{f}} = 0 \quad \forall\, \underline{\mathbf{x}} \in \Omega \tag{9}$$

   and the generalized balance of micromorphic momentum equation

$$\operatorname{div}(\underline{\mathbf{b}} - \underline{\mathbf{b}}^e) - a + a^e = 0 \quad \forall\, \underline{\mathbf{x}} \in \Omega \tag{10}$$

8. Derive the associated boundary conditions for the simple and generalized tractions

$$\underline{\mathbf{t}} = \underset{\sim}{\boldsymbol{\sigma}} \, . \, \underline{\mathbf{n}} \quad \forall\, \underline{\mathbf{x}} \in \partial\mathcal{D} \tag{11}$$

$$a^c = (\underline{\mathbf{b}} - \underline{\mathbf{b}}^e) \, . \, \underline{\mathbf{n}} \quad \forall\, \underline{\mathbf{x}} \in \partial\mathcal{D} \tag{12}$$

9. Enhance the local balance of energy by the generalized micromorphic power already included in the power of internal forces [Eq. (5)]

$$\rho\dot{\epsilon} = p^{(i)} - \mathrm{div}\, \underline{\mathbf{q}} + \rho r \tag{13}$$

where ϵ=specific internal energy, $\underline{\mathbf{q}}$=heat flux vector; and r=external heat sources.

10. Enlarge the state space to include the micromorphic variable and its first gradient

$$\mathrm{STATE} = \{\underset{\sim}{\mathbf{F}},\ T,\ \alpha,\ {}^{\chi}\phi,\ \boldsymbol{\nabla}\, {}^{\chi}\phi\} \tag{14}$$

11. Formulate the entropy principle in its local form

$$-\rho(\dot{\psi} + \eta\dot{T}) + p^{(i)} - \frac{\underline{\mathbf{q}}}{T} \cdot \boldsymbol{\nabla} T \geqslant 0 \tag{15}$$

where it is assumed that the entropy production vector is still equal to the heat vector divided by temperature, as in classical thermomechanics according to Coleman and Noll (1963). Again, the enhancement of the theory goes through the enriched power density of internal forces [Eq. (5)].

12. Exploit the entropy principle according to classical continuum thermodynamics to derive the state laws. For that purpose, the following constitutive functions are introduced (more general dependencies, especially dissipative micromechanisms are considered in the section Dissipative Micromechanisms)

$$\psi = \hat{\psi}(\underset{\sim}{\mathbf{F}}^{e}, T, \alpha, {}^{\chi}\phi, \boldsymbol{\nabla}{}^{\chi}\phi) \tag{16}$$

$$\underset{\sim}{\boldsymbol{\sigma}} = \hat{\underset{\sim}{\boldsymbol{\sigma}}}(\underset{\sim}{\mathbf{F}}^{e}, T, \alpha, {}^{\chi}\phi, \boldsymbol{\nabla}{}^{\chi}\phi) \tag{17}$$

$$a = \hat{a}(\underset{\sim}{\mathbf{F}}^{e}, T, \alpha, {}^{\chi}\phi, \boldsymbol{\nabla}{}^{\chi}\phi) \tag{18}$$

$$\eta = \hat{\eta}(\underset{\sim}{\mathbf{F}}^{e}, T, \alpha, {}^{\chi}\phi, \boldsymbol{\nabla}{}^{\chi}\phi) \tag{19}$$

$$\underline{\mathbf{b}} = \hat{\underline{\mathbf{b}}}(\underset{\sim}{\mathbf{F}}^{e}, T, \alpha, {}^{\chi}\phi, \boldsymbol{\nabla}{}^{\chi}\phi) \tag{20}$$

where $\underset{\sim}{\mathbf{F}}^{e}$ represents the elastic part of total deformation. Its precise definition depends, however, on the retained decomposition of total deformation into elastic and plastic contributions. The usual multiplicative decomposition is adopted below for the illustration. The state laws follow:

$$\underset{\sim}{\boldsymbol{\sigma}} = \rho\frac{\partial\hat{\psi}}{\partial\underset{\sim}{\mathbf{F}}^{e}} \cdot \underset{\sim}{\mathbf{F}}^{eT} \tag{21}$$

$$\eta = -\frac{\partial\hat{\psi}}{\partial T} \tag{22}$$

$$X = \rho\frac{\partial\hat{\psi}}{\partial\alpha} \tag{23}$$

$$a = \rho\frac{\partial\hat{\psi}}{\partial{}^{\chi}\phi} \tag{24}$$

$$\underline{\mathbf{b}} = \rho\frac{\partial\hat{\psi}}{\partial\boldsymbol{\nabla}\,{}^{\chi}\phi} \tag{25}$$

and the residual dissipation is

$$D^{\mathrm{res}} = W^{p} - X\dot{\alpha} - \frac{\underline{\mathbf{q}}}{T} \cdot \boldsymbol{\nabla} T \geqslant 0 \tag{26}$$

where W^{p} represents the (visco-)plastic power, and X=thermodynamic force associated with the internal variable α. The existence of a convex dissipation potential depending on the thermodynamic forces can then be assumed from which the evolution rules for internal variables are derived, that identically fulfill the entropy inequality, as usually done in classical continuum thermomechanics (Germain et al. 1983).

At this first stage, there will be no need for considering generalized external body forces so that $a^{e}=0$, $\underline{\mathbf{b}}^{e}=0$. In the following, this mehodology will be applied to existing theories of plasticity and damage.

After presenting the general approach, we readily give the most simple example, which provides a direct connection to several existing generalized continuum models. We consider first cases where ϕ and ${}^{\chi}\phi$ are observer invariant quantities. The free energy density function ψ is chosen as a function of the generalized relative strain variable e defined as

$$e = \phi - {}^{\chi}\phi \tag{27}$$

thus, introducing a coupling between macro and micromorphic variables. Assuming isotropic material behavior for brevity, the additional contributions to the free energy can be taken as quadratic functions of e and $\boldsymbol{\nabla}\,{}^{\chi}\phi$

$$\psi(\underset{\sim}{\mathbf{F}}^{e}, T, \alpha, {}^{\chi}\phi, \boldsymbol{\nabla}{}^{\chi}\phi) = \psi^{(1)}(\underset{\sim}{\mathbf{F}}^{e}, T, \alpha) + \psi^{(2)}(e = \phi - {}^{\chi}\phi, \boldsymbol{\nabla}{}^{\chi}\phi, T) \tag{28}$$

with

$$\rho\psi^{(2)} = \frac{1}{2}H_{\chi}(\phi - {}^{\chi}\phi)^{2} + \frac{1}{2}A\,\boldsymbol{\nabla}\,{}^{\chi}\phi \cdot \boldsymbol{\nabla}{}^{\chi}\phi \tag{29}$$

After inserting the state laws [Eqs. (24) and (25)]

$$a = \rho\frac{\partial\psi}{\partial{}^{\chi}\phi} = -H_{\chi}(\phi - {}^{\chi}\phi) \tag{30}$$

$$\underline{\mathbf{b}} = \rho\frac{\partial\psi}{\partial\boldsymbol{\nabla}\,{}^{\chi}\phi} = A\,\boldsymbol{\nabla}\,{}^{\chi}\phi \tag{31}$$

into the additional balance Eq. (10)

$$a = \mathrm{div}\,\underline{\mathbf{b}} \tag{32}$$

the following partial differential equation is obtained, at least for a homogeneous material under isothermal conditions:

$$\phi = {}^{\chi}\phi - \frac{A}{H_{\chi}}\Delta{}^{\chi}\phi \tag{33}$$

where Δ=Laplace operator. This type of equation is encountered at several places in the mechanics of generalized continua especially in the linear micromorphic theory (Mindlin 1964; Eringen 1999; Dillard et al. 2006) and in the so-called implicit gradient theory of plasticity and damage (Peerlings et al. 2001; Engelen et al. 2003; Peerlings et al. 2004). Note, however, that this equation corresponds to a special quadratic potential and represents the simplest micromorphic extension of the classical theory. It involves a characteristic length scale defined by

$$l_c^2 = \frac{A}{H_\chi} \quad (34)$$

This length is real for positive values of the ratio A/H_χ. The additional material parameters H_χ and A are assumed to be positive in this work. This does not exclude a softening material behavior that can be induced by the proper evolution of the internal variables (including $\phi=\alpha$ itself).

Full Micromorphic and Microstrain Theories

The micromorphic theory proposed in Eringen and Suhubi (1964a,b) and Mindlin (1964) is retrieved by choosing

$$\phi \equiv \underset{\sim}{\mathbf{F}} \quad (35)$$

i.e., the selected variable ϕ is the full deformation gradient itself. The associated micromorphic variable is

$$^\chi\phi \equiv \underset{\sim}{\boldsymbol{\chi}} \quad (36)$$

where $\underset{\sim}{\boldsymbol{\chi}}(\underline{\mathbf{x}})$ = generally nonsymmetric and noncompatible field of microdeformation introduced by these authors. Following the approach sketched in the previous section, the power of internal forces is extended by the micromorphic power, written here in the small deformation framework for the sake of brevity

$$p^{(i)} = \underset{\sim}{\boldsymbol{\sigma}}:\nabla\dot{\underline{\mathbf{u}}} + \underset{\sim}{\mathbf{a}}:\dot{\underset{\sim}{\boldsymbol{\chi}}} + \underset{\approx}{\mathbf{B}} \,\vdots\, \nabla\dot{\underset{\sim}{\boldsymbol{\chi}}} \quad (37)$$

An (infinitesimal) change of observer of rate $\underset{\sim}{\mathbf{w}}$ changes the gradient of the velocity field into $\nabla\dot{\underline{\mathbf{u}}}+\underset{\sim}{\mathbf{w}}$ and the microdeformation rate into $\dot{\underset{\sim}{\boldsymbol{\chi}}}+\underset{\sim}{\mathbf{w}}$. The principle of (infinitesimal) material frame indifference requires the invariance of $p^{(i)}$ with respect to (infinitesimal) Euclidean changes of observers (Gurtin 2003). As a result, the sum $\underset{\sim}{\boldsymbol{\sigma}}+\underset{\sim}{\mathbf{a}}$ must be a symmetric second-rank tensor. The power density of internal forces can, therefore, be rewritten in the following form:

$$p^{(i)} = \underset{\sim}{\boldsymbol{\sigma}}:\dot{\underset{\sim}{\boldsymbol{\varepsilon}}} + \underset{\sim}{\mathbf{s}}:(\nabla\dot{\underline{\mathbf{u}}} - \dot{\underset{\sim}{\boldsymbol{\chi}}}) + \underset{\approx}{\mathbf{S}} \,\vdots\, \nabla\dot{\underset{\sim}{\boldsymbol{\chi}}} \quad (38)$$

where $\underset{\sim}{\boldsymbol{\sigma}}$ is symmetric; $\underset{\sim}{\mathbf{s}}$ = generally nonsymmetric relative stress tensor, and $\underset{\approx}{\mathbf{S}}=\underset{\approx}{\mathbf{B}}$ = higher order stress tensor introduced in the formulation of Eringen (1999). The infinitesimal strain tensor is $\underset{\sim}{\boldsymbol{\varepsilon}}$. The generalized strain rates $\nabla\dot{\underline{\mathbf{u}}}-\dot{\underset{\sim}{\boldsymbol{\chi}}}$ and $\nabla\dot{\underset{\sim}{\boldsymbol{\chi}}}$ are invariant with respect to (infinitesimal) changes of observers [the reasoning holds true at finite deformation as done in Mindlin (1964) and Eringen and Suhubi (1964a,b)]. The balance equations of momentum and of generalized moment of momentum take the form

$$\mathrm{div}(\underset{\sim}{\boldsymbol{\sigma}} + \underset{\sim}{\mathbf{s}}) + \rho\underline{\mathbf{f}} = 0 \quad \mathrm{div}\,\underset{\approx}{\mathbf{S}} + \underset{\sim}{\mathbf{s}} = 0 \quad (39)$$

which shows an explicit coupling between both balance equations via the relative stress tensor $\underset{\sim}{\mathbf{s}}$. Such a coupling was not explicit in the general formulation [see Eqs. (9) and (10)], but it finally becomes evident through the constitutive coupling in Eq. (30).

The microstrain theory proposed in Forest and Sievert (2006) is an application of the micromorphic approach when taking

$$\phi \equiv \underset{\sim}{\mathbf{C}} = \underset{\sim}{\mathbf{F}}^T\,.\,\underset{\sim}{\mathbf{F}} \quad {}^\chi\phi \equiv {}^\chi\underset{\sim}{\mathbf{C}} \quad (40)$$

or

$$\phi \equiv \underset{\sim}{\boldsymbol{\varepsilon}} \quad {}^\chi\phi \equiv {}^\chi\underset{\sim}{\boldsymbol{\varepsilon}} \quad (41)$$

within the small strain approximation. Because of the symmetry of the microstrain tensor, it is not necessary to introduce a relative stress tensor in the enriched power of internal forces. Instead, the standard form [Eq. (5)] is adopted

$$p^{(i)} = \underset{\sim}{\boldsymbol{\sigma}}:\dot{\underset{\sim}{\boldsymbol{\varepsilon}}} + \underset{\sim}{\mathbf{a}}:{}^\chi\dot{\underset{\sim}{\boldsymbol{\varepsilon}}} + \underset{\approx}{\mathbf{B}}:\nabla{}^\chi\dot{\underset{\sim}{\boldsymbol{\varepsilon}}} \quad (42)$$

The coupling between macro and microstrain arises at the constitutive level since the free energy density was proposed in Forest and Sievert (2006) to be a function

$$\psi(\underset{\sim}{\boldsymbol{\varepsilon}}^e,\ T,\ \alpha,\ \underset{\sim}{\mathbf{e}} := \underset{\sim}{\boldsymbol{\varepsilon}} - {}^\chi\underset{\sim}{\boldsymbol{\varepsilon}},\ \underset{\approx}{\mathbf{K}} := \nabla{}^\chi\underset{\sim}{\boldsymbol{\varepsilon}}) \quad (43)$$

The additional arguments may be limited to their nondissipative parts $\underset{\sim}{\mathbf{e}}^e, \underset{\approx}{\mathbf{K}}^e$ according to Forest and Sievert (2006). The existence of dissipative micromechanisms is considered in a later section. When the following simplified constitutive equations are adopted:

$$\underset{\sim}{\mathbf{a}} = H_\chi\underset{\sim}{\mathbf{e}} \quad \underset{\approx}{\mathbf{b}} = A\,\nabla\,{}^\chi\underset{\sim}{\boldsymbol{\varepsilon}} \quad (44)$$

the extra balance equation takes the following simple form:

$$^\chi\underset{\sim}{\boldsymbol{\varepsilon}} - l_c^2\Delta{}^\chi\underset{\sim}{\boldsymbol{\varepsilon}} = \underset{\sim}{\boldsymbol{\varepsilon}} \quad \text{with } l_c^2 = \frac{A}{H_\chi} \quad (45)$$

as shown in Dillard et al. (2006). This represents an extension of the scalar partial differential [Eq. (33)] to a tensor valued micromorphic variable. The Laplace operator applies here to each individual tensor component, within a Cartesian orthonormal coordinate system.

The full micromorphic continuum can be regarded as the combination of the microstrain and Cosserat continua (Forest and Sievert 2006). The Cosserat continuum itself can be interpreted in terms of the proposed methodology. It corresponds to the choice

$$\phi = \underset{\sim}{\mathbf{R}} \quad {}^\chi\phi = {}^\chi\underset{\sim}{\mathbf{R}} \quad (46)$$

where $\underset{\sim}{\mathbf{R}}$ = material rotation in the polar decomposition of $\underset{\sim}{\mathbf{F}}$. The associated quantity ${}^\chi\underset{\sim}{\mathbf{R}}$ is nothing but the micropolar rotation representing the rotation of a triad of directors attributed to each material point. The generalized stress $\underset{\sim}{\mathbf{a}}$ in the enriched power of internal forces is then related to the skew-symmetric part of the Cosserat stress tensor. Cosserat models are known to be able to account for some size effects in the softening plastic behavior of granular materials (Mühlhaus and Vardoulakis 1987) and in the hardening behavior of metals (Forest et al. 2000a).

Microstrain Gradient Plasticity

The proposed methodology is now applied to the simplest models available for the isothermal elastic-plastic behavior of materials. Different generalized models are worked out and compared to the existing models in the literature. It turns out that several existing strain gradient plasticity models are recovered using the proposed systematic procedure. Some differences are evidenced due to the precise thermodynamical background of the approach, which is not always present in the earlier approaches. In particular, the approach can be used to tackle coupled problems, thus, providing coupled strain gradient equations not present in the literature. A simple example of this coupling is given to account for a possible dependence of material parameters with temperature. The framework is applicable to plasticity and viscoplasticity.

Scalar Microstrain Gradient Plasticity

Balance and Constitutive Equations

The reference state space corresponding to a classical elastoplasticity model retained in this subsection is

$$\mathrm{DOF0} = \{\underline{\mathbf{u}}\} \quad \mathrm{STATE}\ 0 = \{\underset{\sim}{\boldsymbol{\varepsilon}}^e,\ p,\ \alpha\} \tag{47}$$

where $\underset{\sim}{\boldsymbol{\varepsilon}}^e$=(infinitesimal) elastic strain tensor; p=cumulated plastic strain variable; and α denotes another possible internal variable of any tensorial rank. The selected variable for the micromorphic approach is

$$\phi \equiv p \tag{48}$$

which the micromorphic variable ${}^{\chi}p$ is associated to. The classical powers of internal and contact forces are extended in the following way:

$$p^{(i)} = \underset{\sim}{\boldsymbol{\sigma}}:\dot{\underset{\sim}{\boldsymbol{\varepsilon}}} + a^{\chi}\dot{p} + \underline{\mathbf{b}} \cdot \boldsymbol{\nabla}^{\chi}\dot{p} \quad p^{(c)} = \underline{\mathbf{t}} \cdot \dot{\underline{\mathbf{u}}} + a^{c\chi}\dot{p} \tag{49}$$

in which generalized stresses a and $\underline{\mathbf{b}}$ have been introduced. The application of the method of virtual power leads to the following additional local balance equation and boundary conditions in addition to the classical local balance of momentum and traction condition at the outer boundary

$$\mathrm{div}\,\underline{\mathbf{b}} - a = 0 \quad \forall\ \underline{\mathbf{x}} \in \Omega \quad a^c = \underline{\mathbf{b}} \cdot \underline{\mathbf{n}} \quad \forall\ \underline{\mathbf{x}} \in \partial\Omega \tag{50}$$

Generalized body forces a^e and $\underline{\mathbf{b}}^e$ could be introduced in the case of a necessity in the balance equations. The total strain is split into its elastic and plastic parts

$$\underset{\sim}{\boldsymbol{\varepsilon}} = \underset{\sim}{\boldsymbol{\varepsilon}}^e + \underset{\sim}{\boldsymbol{\varepsilon}}^p \tag{51}$$

The extended state space on which constitutive functions may depend is

$$\mathrm{STATE} = \{\underset{\sim}{\boldsymbol{\varepsilon}}^e,\ p,\ \alpha,\ {}^{\chi}p,\ \boldsymbol{\nabla}\,{}^{\chi}p\} \tag{52}$$

The free energy density function ψ is assumed to be a function of the previous set STATE. The Clausius-Duhem inequality then takes the form

$$\left(\underset{\sim}{\boldsymbol{\sigma}} - \rho\frac{\partial\psi}{\partial\underset{\sim}{\boldsymbol{\varepsilon}}^e}\right):\dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^e + \left(a - \rho\frac{\partial\psi}{\partial{}^{\chi}p}\right){}^{\chi}\dot{p} + \left(\underline{\mathbf{b}} - \rho\frac{\partial\psi}{\partial\boldsymbol{\nabla}\,{}^{\chi}p}\right)\cdot\boldsymbol{\nabla}^{\chi}\dot{p}$$
$$+ \underset{\sim}{\boldsymbol{\sigma}}:\dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p - \rho\frac{\partial\psi}{\partial p}\dot{p} - \rho\frac{\partial\psi}{\partial\alpha}\dot{\alpha} \geqslant 0 \tag{53}$$

from which the following state laws and residual dissipation are derived:

$$\underset{\sim}{\boldsymbol{\sigma}} = \rho\frac{\partial\psi}{\partial\underset{\sim}{\boldsymbol{\varepsilon}}^e} \quad a = \rho\frac{\partial\psi}{\partial{}^{\chi}p} \quad \underline{\mathbf{b}} = \rho\frac{\partial\psi}{\partial\boldsymbol{\nabla}\,{}^{\chi}p} \quad R = \rho\frac{\partial\psi}{\partial p} \quad X = \rho\frac{\partial\psi}{\partial\alpha} \tag{54}$$

$$D^{\mathrm{res}} = \underset{\sim}{\boldsymbol{\sigma}}:\dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p - R\dot{p} - X\dot{\alpha} \geqslant 0 \tag{55}$$

The plastic behavior is characterized by the yield function $f(\underset{\sim}{\boldsymbol{\sigma}},R,X)$. In the micromorphic model, the yield function can still be treated as the dissipation potential providing the flow and evolution rules for the internal variables. This corresponds to the hypothesis of maximal dissipation or normality rule

$$\dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p = \dot{\lambda}\frac{\partial f}{\partial\underset{\sim}{\boldsymbol{\sigma}}} \quad \dot{p} = -\dot{\lambda}\frac{\partial f}{\partial R} \quad \dot{\alpha} = -\dot{\lambda}\frac{\partial f}{\partial X} \tag{56}$$

where $\dot{\lambda}$=plastic multiplier. At this stage, a coupling between the macroscopic and microscopic variables must be introduced, for instance via the relative cumulative plastic strain $p - {}^{\chi}p$. An example of such a possible coupling is given in the next paragraph.

Example

A quadratic form is proposed to model the free energy density function, with respect to elastic strain, cumulative plastic strain, relative plastic strain, and micromorphic plastic strain gradient

$$\rho\psi(\underset{\sim}{\boldsymbol{\varepsilon}}^e,p,{}^{\chi}p,\boldsymbol{\nabla}^{\chi}p) = \frac{1}{2}\underset{\sim}{\boldsymbol{\varepsilon}}^e:\underset{\approx}{\boldsymbol{\Lambda}}:\underset{\sim}{\boldsymbol{\varepsilon}}^e + \frac{1}{2}Hp^2 + \frac{1}{2}H_{\chi}(p - {}^{\chi}p)^2$$
$$+ \frac{1}{2}\boldsymbol{\nabla}\,{}^{\chi}p \cdot \underset{\sim}{\mathbf{A}} \cdot \boldsymbol{\nabla}^{\chi}p \tag{57}$$

The corresponding classical model describes an elastoplastic material behavior with linear elasticity characterized by the tensor of elastic moduli $\underset{\approx}{\boldsymbol{\Lambda}}$ and the linear hardening modulus H. Two additional material parameters are introduced in the micromorphic extension of this classical model, namely, the coupling modulus H_{χ} (unit MPa) and the micromorphic "stiffness" $\underset{\sim}{\mathbf{A}}$ (unit MPa m^2). The thermodynamic forces associated with the state variables are given by the relations [Eq. (54)]

$$\underset{\sim}{\boldsymbol{\sigma}} = \underset{\approx}{\boldsymbol{\Lambda}}:\underset{\sim}{\boldsymbol{\varepsilon}}^e \tag{58a}$$

$$a = -H_{\chi}(p - {}^{\chi}p) \tag{58b}$$

$$\underline{\mathbf{b}} = \underset{\sim}{\mathbf{A}} \cdot \boldsymbol{\nabla}^{\chi}p \tag{58c}$$

$$R = (H + H_{\chi})p - H_{\chi}{}^{\chi}p \tag{58d}$$

Note that when the relative plastic strain $e = p - {}^{\chi}p$ is close to zero, the linear hardening rule retrieves its classical form and the generalized stress a vanishes. Only the strain gradient effect $\boldsymbol{\nabla}p$ remains in the enriched work of internal forces [Eq. (49)]. This is the situation encountered in the strain gradient plasticity models developed in Fleck and Hutchinson (2001). When inserted in the additional balance Eq. (50), the previous state laws lead to the following partial differential equation:

$${}^{\chi}p - \frac{1}{H_{\chi}}\,\mathrm{div}(\underset{\sim}{\mathbf{A}} \cdot \boldsymbol{\nabla}^{\chi}p) = p \tag{59}$$

Let us specialize this equation to the case of isotropic materials, for which the second order tensor of micromorphic stiffness reduces to

$$\underset{\sim}{\mathbf{A}} = A\underset{\sim}{\mathbf{1}} \tag{60}$$

which involves a single additional material parameter. Eq. (59) then becomes

$${}^{\chi}p - \frac{A}{H_{\chi}}\Delta^{\chi}p = p \tag{61}$$

which is identical to the additional partial differential equation used in the so-called implicit gradient-enhanced elastoplasticity in Engelen et al. (2003). The microstrain ${}^{\chi}p$ is called there the "nonlocal strain measure" $\bar{p}$. Note, however, that the latter model involves only one additional material parameter, namely, $l_c^2 = A/H_{\chi}$ instead of two in the micromorphic approach. It will turn out to be a special case of the micromorphic model for a specific value of the coupling modulus H_{χ}. No thermodynamical framework was proposed for the elastoplasticity model in the original contribution (Engelen et al. 2003). Such a framework has been sketched in the reference (Peerlings et al. 2004) where a quadratic potential similar to Eq. (57) is introduced, which involves in particular the same coupling term. In contrast to the micromorphic approach, however, no additional contribution is introduced in the power of internal forces so that the additional partial differential equation is derived as a sufficient condition to identically fulfill the global form of the entropy inequality. In the micromorphic approach, the coupling modulus H_χ plays a central role and makes it possible to have a fully consistent thermomechanical basis for the model. When its value is high enough, it acts as a penalty term forcing the micromorphic plastic strain to follow the macroscopic one as close as possible.

The necessity of an additional boundary condition associated with the nonlocal strain measure is recognized in Engelen et al. (2003). The associated Neumann condition is used in the form

$$\nabla^{\chi} p \,.\, \underline{n} = 0 \quad \text{on } \partial\Omega \tag{62}$$

It coincides with the more general boundary condition derived in the micromorphic approach

$$\underline{b} \,.\, \underline{n} = a^c \quad \text{on } \partial\Omega \tag{63}$$

when $a^c=0$ and when $\underline{b}$ is linear with respect to $\nabla\, {}^{\chi}p$, as it is the case for the quadratic potential [Eq. (57)].

The yield function is now chosen as

$$f(\underset{\sim}{\sigma}, R) = \sigma_{\text{eq}} - \sigma_Y - R \tag{64}$$

where σ_{eq}=equivalent stress measure and σ_Y=initial yield stress. The hardening rule then takes the following form:

$$R = \rho \frac{\partial \psi}{\partial p} = (H + H_\chi)p - H_\chi {}^{\chi}p \tag{65}$$

After substituting the balance [Eq. (59)] into the hardening law, yielding takes place when

$$\sigma_{\text{eq}} = \sigma_Y + H^{\chi}p - A\left(1 + \frac{H}{H_\chi}\right)\Delta^{\chi}p \tag{66}$$

This expression coincides with the enhanced yield criterion originally proposed in Aifantis (1987) and used for strain localization simulations in de Borst et al. (1993) when the micromorphic variable remains as close as possible to the plastic strain: ${}^{\chi}p \simeq p$. In the latter references, the Laplace operator is directly introduced in the yield function as a postulate, whereas its presence is derived here from the combination of the additional balance equation and the linear generalized constitutive equations.

In the reference (Engelen et al. 2003), after introducing the partial differential Eq. (61) in addition to the classical balance and constitutive equations, the authors propose to substitute the classical hardening law $R(p)$ by the same function $R({}^{\chi}p)$ where the argument is replaced by the nonlocal equivalent plastic strain. If such a hardening law is adopted, this model turns out to be a special case of the present microstrain theory for the following specific value of the hardening modulus:

$$H_\chi = -H \tag{67}$$

which follows from the identification $(H+H_\chi)p - H_\chi\, {}^{\chi}p = H\, {}^{\chi}p$ according to Eq. (65). This assumption indeed reduces the number of free additional parameters to one, namely, the choice of A related to the intrinsic length of the material. Such a choice, however, is acceptable only for softening materials for which $H<0$. Otherwise, the additional contribution to the free energy associated with ${}^{\chi}p$ in Eq. (57) will act as a destabilizing term in the material behavior. Furthermore, the type of the partial differential Eq. (61) would be changed. The authors also point out the limitations of the simplistic method consisting of substituting the microstrain ${}^{\chi}p$ in the classical hardening law instead of p, especially regarding the subsequent evolution inside plastic strain localization bands.

Keeping both H_χ and A as free parameters of the theory makes it possible, in principle, to envisage applications to strain localization phenomena in softening materials, as done in Engelen et al. (2003) and Dillard et al. (2006), but also to size effects in hardening plasticity, as done in Hutchinson (2000), Forest et al. (2000a), and Dillard et al. (2006). In the application to metallic foams in Dillard et al. (2006), the microstrain approach was used for both the hardening part of the material and subsequent softening. For a detailed discussion of the pros and the cons of various available strain gradient plasticity models for both types of applications, the reader is referred to Engelen et al. (2006).

Consistency Condition and Extension to Viscoplasticity

The development of the microstrain gradient plasticity model must be complemented by the exploitation of the consistency condition in order to provide the required information for possible numerical implementation of the model. The isotropic hardening variable is assumed to be a function $R(p, {}^{\chi}p)$. The consistency condition states the continuing fulfillment of the yield condition [Eq. (64)] during plastic yielding

$$\dot{f} = \frac{\partial f}{\partial \underset{\sim}{\sigma}} : \dot{\underset{\sim}{\sigma}} + \frac{\partial f}{\partial R}\dot{R} \tag{68}$$

$$\dot{f} = \frac{\partial \sigma_{\text{eq}}}{\partial \underset{\sim}{\sigma}} : \underset{\approx}{\Lambda} : (\dot{\underset{\sim}{\varepsilon}} - \dot{\underset{\sim}{\varepsilon}}^p) - \frac{\partial R}{\partial p}\dot{p} - \frac{\partial R}{\partial {}^{\chi}p} {}^{\chi}\dot{p} = 0 \tag{69}$$

Taking the flow rule [Eq. (56)] into account, we obtain the expression of the plastic multiplier

$$\dot{\lambda} = \dot{p} = \frac{\underset{\sim}{N} : \underset{\approx}{\Lambda} : \dot{\underset{\sim}{\varepsilon}} - \dfrac{\partial R}{\partial {}^{\chi}p} {}^{\chi}\dot{p}}{\underset{\sim}{N} : \underset{\approx}{\Lambda} : \underset{\sim}{N} + \dfrac{\partial R}{\partial p}} \quad \text{with } \underset{\sim}{N} = \frac{\partial \sigma_{\text{eq}}}{\partial \underset{\sim}{\sigma}} \tag{70}$$

The plastic multiplier is expressed as a function of the controlled variables $\dot{\underset{\sim}{\varepsilon}}$ and ${}^{\chi}\dot{p}$, which are prescribed incrementally and independently at the material point. Plastic flow will occur if

$$\underset{\sim}{N} : \underset{\approx}{\Lambda} : \dot{\underset{\sim}{\varepsilon}} - \frac{\partial R}{\partial {}^{\chi}p} {}^{\chi}\dot{p} > 0 \tag{71}$$

provided that $\underset{\sim}{N} : \underset{\approx}{\Lambda} : \underset{\sim}{N} + H + H_\chi$ remains positive (no snap back at the material point). The stabilizing character of H_χ when it is positive appears in this expression even in the case of a softening behavior $H<0$.

Numerical integration of the previous constitutive equations can be implemented according to the standard implicit θ method based on the Newton algorithm (Besson et al. 2001) and its generalization to higher order continua with additional degrees-of-freedom as done in de Borst (1993) and de Borst et al. (1993). According to this approach, there is no need for a variational formulation of the consistency condition contrary to the strain gradient plasticity theories envisaged in Mühlhaus and Aifantis (1991) and Liebe et al. (2001, 2003). In the latter references, besides the balance of linear momentum, the algorithmic consistency condition has to be solved in weak form. Thereby, the crucial issue is the determination of the active domains exhibiting plastic loading, which is solved by an active set search algorithm borrowed from convex nonlinear programming. In contrast, in the present microstrain plasticity model, there is no need for tracking the frontier of the elastic-plastic domain. The additional boundary conditions [Eq. (12)] are to be prescribed at the physical boundary of the body $\partial\Omega$. Across the interface between the elastic and plastically flowing zones of the body, the displacement $\underline{u}$, microstrain ${}^{\chi}p$, traction vector $\underline{t}$, and generalized traction $\underline{b}.\underline{n}$ are continuous. This represents a remarkable advantage of the micromorphic approach.

The extension of the approach to viscoplasticity is straightforward. It can be based for instance on the introduction of a viscoplastic potential $\Omega(\underset{\sim}{\sigma}, R, X)$

$$\dot{\underset{\sim}{\varepsilon}}^p = \frac{\partial\Omega}{\partial\underset{\sim}{\sigma}} \quad \dot{p} = -\frac{\partial\Omega}{\partial R} \quad \dot{\alpha} = \frac{\partial\Omega}{\partial X} \tag{72}$$

Specific convexity properties of the dissipation potential are sufficient conditions for the entropy inequality to be identically fulfilled. The introduction of additional micromorphic degrees-of-freedom does not modify the structure of the local thermomechanics of continua (Forest and Sievert 2003).

Thermal Effects

The temperature dependence of the free energy has not been considered in the previous presentation of the microstrain gradient plasticity model. The systematic procedure makes it possible to derive the thermal effects associated with the thermomechanical microstrain theory. This is illustrated here briefly, by considering the possible temperature dependence of the material parameters. In this section only, we depart from the isothermal situation. If the moduli $A(T)$, $H(T)$, and $H_\chi(T)$ are regarded as temperature dependent, the additional balance Eq. (61) and the consistency conditions [Eq. (70)] must be amended in the following way. The generalized stresses are related by

$$a = \text{div}\, \underline{\mathbf{b}} = \text{div}(A\, \nabla\, {}^{\chi}p) = A\Delta^{\chi}p + \frac{\partial A}{\partial T}\nabla\, T\,.\,\nabla^{\chi}p \tag{73}$$

Combining this expression with the state law [Eq. (58*b*)], the modified additional partial differential equation is found

$${}^{\chi}p - \frac{A}{H_\chi}\Delta^{\chi}p - \frac{1}{H_\chi}\frac{\partial A}{\partial T}\nabla\, T\,.\,\nabla^{\chi}p = p \tag{74}$$

Similarly, the consistency condition [Eq. (69)] is reconsidered

$$\dot{f} = \frac{\partial f}{\partial\underset{\sim}{\sigma}}:\dot{\underset{\sim}{\sigma}} + \frac{\partial f}{\partial R}\dot{R} \tag{75}$$

$$= \frac{\partial\sigma_{\text{eq}}}{\partial\sigma}:\underset{\approx}{\Lambda}:(\dot{\underset{\sim}{\varepsilon}} - \dot{\underset{\sim}{\varepsilon}}^{\text{th}} - \dot{\underset{\sim}{\varepsilon}}^{p}) - \frac{\partial R}{\partial p}\dot{p} - \frac{\partial R}{\partial^{\chi}p}{}^{\chi}\dot{p} - \frac{\partial R}{\partial T}\dot{T} \tag{76}$$

where the thermal strain $\underset{\sim}{\varepsilon}^{\text{th}}$ has been introduced

$$\dot{p} = \frac{\underset{\sim}{\mathbf{N}}:\underset{\approx}{\Lambda}:(\dot{\underset{\sim}{\varepsilon}} - \dot{\underset{\sim}{\varepsilon}}^{\text{th}}) - \dfrac{\partial R}{\partial^{\chi}p}{}^{\chi}\dot{p} - \dfrac{\partial R}{\partial T}\dot{T}}{\underset{\sim}{\mathbf{N}}:\underset{\approx}{\Lambda}:\underset{\sim}{\mathbf{N}} + \dfrac{\partial R}{\partial p}} \tag{77}$$

The variables $\dot{\underset{\sim}{\varepsilon}}$, ${}^{\chi}\dot{p}$, and $\dot{T}$ are known at each increment and used to evaluate $\dot{p}$. For the sake of brevity, the classical additional thermal contributions to the free energy [Eq. (57)], involving in particular the thermal strain, the associated expression of entropy according to Eq. (19) and the residual dissipation, which takes its classical form, are not explicited here.

Gradient of a Dislocation Density-Like Variable

The cumulative plastic strain p is generally not the appropriate internal variable to describe realistic plastic material behavior in metals (Lemaitre and Chaboche 1994). The choice of an internal variable ϱ, which saturates for increasing plastic strain, thus, crudely mimicking the evolution of dislocation density, which is at the basis of plastic deformation in metals, is more relevant (Teodosiu 1997). One of the most simple evolution equations that complies with this requirement is

$$\dot{\varrho} = \dot{p}(1 - b\varrho) \tag{78}$$

where b=dimensionless material parameter accounting for dynamic recovery (Lemaitre and Chaboche 1994). For b=0, ϱ coincides with the cumulative plastic strain. The isotropic hardening variable is then assumed to be proportional to ϱ

$$R = bQ\varrho \tag{79}$$

For monotonous loading, Eq. (78) can be integrated as

$$\varrho = \frac{1}{b}(1 - \exp(-bp)) \quad R = Q(1 - \exp(-bp)) \tag{80}$$

which coincides with the nonlinear isotropic hardening rule used in Lemaitre and Chaboche (1994). The set of state variables of the initial model is then

$$\text{STATE 0} = \{\underset{\sim}{\varepsilon}^e,\ \varrho\} \tag{81}$$

The selected variable in the micromorphic extension of the model is

$$\phi \equiv \varrho \quad {}^{\chi}\phi \equiv {}^{\chi}\varrho \quad \text{STATE} = \{\underset{\sim}{\varepsilon}^e,\ \varrho,\ {}^{\chi}\varrho,\ \nabla\, {}^{\chi}\varrho\} \tag{82}$$

We now give successively the expressions of the enriched power of internal forces and of the quadratic form for the free energy density

$$p^{(i)} = \underset{\sim}{\sigma}:\dot{\underset{\sim}{\varepsilon}} + a\,{}^{\chi}\dot{\varrho} + \underline{\mathbf{b}}\,.\,\nabla^{\chi}\dot{\varrho} \tag{83}$$

$$\rho\psi(\underset{\sim}{\varepsilon}^e, \varrho, {}^{\chi}\varrho, \nabla^{\chi}\varrho) = \frac{1}{2}\underset{\sim}{\varepsilon}^e:\underset{\approx}{\Lambda}:\underset{\sim}{\varepsilon}^e + \frac{1}{2}bQ\varrho^2 + \frac{1}{2}bQ_\chi(\varrho - {}^{\chi}\varrho)^2 + \frac{1}{2}A\,\nabla\,{}^{\chi}\varrho\,.\,\nabla^{\chi}\varrho \tag{84}$$

in the case of isotropic material behavior. The state laws are

$$R = bQ\varrho + bQ_\chi(\varrho - {}^{\chi}\varrho) \quad a = -bQ_\chi(\varrho - {}^{\chi}\varrho) \quad \underline{\mathbf{b}} = A\,\nabla\,{}^{\chi}\varrho \tag{85}$$

When inserted into the additional balance equation

$$a = \text{div}\,\underline{\mathbf{b}} = A\Delta^{\chi}\varrho = -bQ_\chi(\varrho - {}^{\chi}\varrho) \tag{86}$$

we obtain the following partial differential equation:

$${}^{\chi}\varrho - \frac{A}{bQ_\chi}\Delta^{\chi}\varrho = \varrho \quad \text{or equivalently}$$

$$b^{\chi}\varrho - \frac{A}{Q_\chi}\Delta^{\chi}\varrho = 1 - \exp(-bp) \tag{87}$$

If plasticity can invade the whole structure, the limiting case $p \to \infty$ will be $\varrho = {}^{\chi}\varrho = 1/b$ with vanishing plastic strain gradient effects. Such a model is probably not well suited for the simulation of strain localization phenomena, but rather for size effect in hardening plasticity.

A yield criterion $f(\underset{\sim}{\sigma},R)$ of the form [Eq. (64)] is introduced. Under plastic loading, we have

$$\sigma_{eq}=\sigma_Y+R=\sigma_Y+b(Q+Q_\chi)\varrho-bQ_\chi{}^\chi\varrho \tag{88}$$

$$=\sigma_Y+bQ^\chi\varrho-A\left(1+\frac{Q}{Q_\chi}\right)\Delta^\chi\varrho \tag{89}$$

For extended plasticity, the limit stress level is σ_Y+Q as in the classical case.

The intrinsic dissipation is

$$D^{res}=\underset{\sim}{\sigma}:\dot{\underset{\sim}{\varepsilon}}^p-R\dot{\varrho}=f\dot{p}+(\sigma_Y+Rb\varrho)\dot{p}=(\sigma_Y+Qb^2\varrho^2)\dot{p}\geqslant 0 \tag{90}$$

It remains positive as long as $Q\geqslant-\sigma_Y$.

The formulation is flexible enough to allow straightforward extensions to anisothermal and anisotropic material behavior following the guidelines given in the previous two sections.

The idea of introducing nonlocal hardening variables associated with the classical isotropic and kinematic hardening variables has been pursued in Dorgan and Voyiadjis (2003), Voyiadjis et al. (2004), Voyiadjis and Dorgan (2004), and Voyiadjis and Abu Al-Rub (2005), where the corresponding thermodynamical framework is settled. Following the incentive of Walgraef and Aifantis (1988) to introduce a diffusion-like contribution in the evolution of dislocation densities, these authors postulate a dependence of the free energy on the Laplacian of the hardening variables. Instead, we consider here a first gradient theory for which the free energy can depend on the first gradient of the degrees-of-freedom only. The combination of the first gradient framework with the additional balance equation leads also to the presence of the Laplace operator in the generalized evolution equations.

Full Microstrain Gradient Plasticity

The approach is not restricted to scalar micromorphic variables. As in Eringen's micromorphic model where the selected variable is the full deformation gradient itself, it can be applied to the full plastic strain tensor

$$\phi\equiv\underset{\sim}{\varepsilon}^p\quad{}^\chi\phi\equiv{}^\chi\underset{\sim}{\varepsilon}^p \tag{91}$$

This corresponds to five additional degrees-of-freedom if the micromorphic plastic strain ${}^\chi\underset{\sim}{\varepsilon}^p$ is treated as a deviatoric tensor like in dense metals. The generalized stresses are symmetric second and third order tensors, respectively

$$p^{(i)}=\underset{\sim}{\sigma}:\dot{\underset{\sim}{\varepsilon}}^p+\underset{\sim}{a}:{}^\chi\dot{\underset{\sim}{\varepsilon}}^p+\underset{\sim}{b}\,\vdots\,\nabla^\chi\dot{\underset{\sim}{\varepsilon}}^p \tag{92}$$

The symmetry condition applies only to the first two indices of b_{ijk}. The power of internal forces is indeed invariant with respect to (infinitesimal) changes of observers, due to the invariance of $\underset{\sim}{\varepsilon}^p$ and ${}^\chi\underset{\sim}{\varepsilon}^p$ themselves.

The initial and extended sets of state variables are

$$\text{STATE 0}=\{\underset{\sim}{\varepsilon}^e,\ \underset{\sim}{\varepsilon}^p\}\quad\text{STATE}=\{\underset{\sim}{\varepsilon}^e,\ \underset{\sim}{\varepsilon}^p,\ {}^\chi\underset{\sim}{\varepsilon}^p,\ \nabla\,{}^\chi\underset{\sim}{\varepsilon}^p\} \tag{93}$$

When the micromorphic variable is constrained to remain as close as possible to the macroscopic one, the theories of gradient of plastic strain presented in Forest and Sievert (2003), Gurtin (2003), and Abu Al-Rub et al. (2007) are recovered. In these works, generalized stresses are associated with the plastic strain rate tensor and its first gradient in the extended power of internal forces.

As an illustration, we adopt the following quadratic form for the free energy potential:

$$\rho\psi(\underset{\sim}{\varepsilon}^e,\underset{\sim}{\varepsilon}^p,{}^\chi\underset{\sim}{\varepsilon}^p,\nabla^\chi\underset{\sim}{\varepsilon}^p)=\frac{1}{2}\underset{\sim}{\varepsilon}^e:\underset{\approx}{\Lambda}:\underset{\sim}{\varepsilon}^e+\frac{1}{3}C\underset{\sim}{\varepsilon}^p:\underset{\sim}{\varepsilon}^p$$
$$+\frac{1}{3}(\underset{\sim}{\varepsilon}^p-{}^\chi\underset{\sim}{\varepsilon}^p):\underset{\approx}{C}_\chi:(\underset{\sim}{\varepsilon}^p-{}^\chi\underset{\sim}{\varepsilon}^p)$$
$$+\frac{1}{2}\nabla\,{}^\chi\underset{\sim}{\varepsilon}^p\,\vdots\,\underset{\approx}{A}\,\vdots\,\nabla^\chi\underset{\sim}{\varepsilon}^p \tag{94}$$

from which the state laws are derived

$$\underset{\sim}{\sigma}=\underset{\approx}{\Lambda}:\underset{\sim}{\varepsilon}^e\quad\underset{\sim}{X}=\frac{2}{3}C\underset{\sim}{\varepsilon}^p+\frac{2}{3}\underset{\approx}{C}_\chi:(\underset{\sim}{\varepsilon}^p-{}^\chi\underset{\sim}{\varepsilon}^p)$$
$$\underset{\sim}{a}=-\frac{2}{3}\underset{\approx}{C}_\chi:(\underset{\sim}{\varepsilon}^p-{}^\chi\underset{\sim}{\varepsilon}^p)\quad\underset{\sim}{b}=\underset{\approx}{A}\,\vdots\,\nabla^\chi\underset{\sim}{\varepsilon}^p \tag{95}$$

In the simplified situation for which

$$\underset{\approx}{C}_\chi=C_\chi\underset{\approx}{1}\quad\underset{\approx}{A}=A\underset{\approx}{1} \tag{96}$$

where $\underset{\approx}{1}$ and $\underset{\approx}{1}$ are the fourth rank and sixth rank identity tensors operating, respectively, on symmetric second order tensors and symmetric (with respect to the first two indices) third rank tensors, the combination of the additional balance equation and state laws leads to the following partial differential equation:

$$\underset{\sim}{a}=\operatorname{div}\underset{\sim}{b}=A\Delta^\chi\underset{\sim}{\varepsilon}^p=-\frac{2}{3}C_\chi(\underset{\sim}{\varepsilon}^p-{}^\chi\underset{\sim}{\varepsilon}^p) \tag{97}$$

$${}^\chi\underset{\sim}{\varepsilon}^p-\frac{3A}{2C_\chi}\Delta^\chi\underset{\sim}{\varepsilon}^p=\underset{\sim}{\varepsilon}^p \tag{98}$$

The differential operators act in the following way with respect to a Cartesian frame $(\underline{e}_i)_{i=1,3}$:

$$\operatorname{div}\underset{\sim}{b}=b_{ijk,k}\underline{e}_i\otimes\underline{e}_j\quad\Delta^\chi\underset{\sim}{\varepsilon}^p=(\Delta^\chi\varepsilon^p_{ij})\underline{e}_i\otimes\underline{e}_j \tag{99}$$

The associated boundary conditions on the boundary of the body are given by a set of six equations

$$\underset{\sim}{b}\,.\,\underline{n}=\underset{\sim}{a}^c \tag{100}$$

The internal variable $\underset{\sim}{\alpha}=\underset{\sim}{\varepsilon}^p$=proper state variable for a plasticity theory incorporating linear kinematic hardening, $\underset{\sim}{X}$ being the back-stress tensor. The retained isotropic yield function for extended J_2-plasticity is

$$f(\underset{\sim}{\sigma},\underset{\sim}{X})=J_2(\underset{\sim}{\sigma}-\underset{\sim}{X})-\sigma_Y \tag{101}$$

$$=J_2\left(\underset{\sim}{\sigma}-\frac{2}{3}(C+C_\chi)\underset{\sim}{\varepsilon}^p-\frac{2}{3}C_\chi{}^\chi\underset{\sim}{\varepsilon}^p\right)-\sigma_Y \tag{102}$$

$$=J_2\left(\underset{\sim}{\sigma}-\frac{2}{3}C_\chi{}^\chi\underset{\sim}{\varepsilon}^p+A\left(1+\frac{C}{C_\chi}\right)\Delta^\chi\underset{\sim}{\varepsilon}^p\right)-\sigma_Y \tag{103}$$

where $J_2(\underset{\sim}{\sigma})=\sqrt{3(\underset{\sim}{\sigma}^{dev}:\underset{\sim}{\sigma}^{dev})/2}$=von Mises second invariant for symmetric second rank tensors. The normality rule is adopted

$$\dot{\underset{\sim}{\varepsilon}}^p=\dot{\lambda}\frac{\partial f}{\partial\underset{\sim}{\sigma}}=-\dot{\lambda}\frac{\partial f}{\partial\underset{\sim}{X}}=\dot{p}\underset{\sim}{N} \tag{104}$$

The intrinsic dissipation then takes its classical form

$$D^{res}=\underset{\sim}{\sigma}:\dot{\underset{\sim}{\varepsilon}}^p-\underset{\sim}{X}:\dot{\underset{\sim}{\varepsilon}}^p=f\dot{p}+\sigma_Y\dot{p}\geqslant 0 \tag{105}$$

with energy storage associated with kinematic hardening. The plastic multiplier is deduced from the consistency condition

$$\dot{f} = \underset{\sim}{\mathbf{N}}:\dot{\underset{\sim}{\boldsymbol{\sigma}}} - \underset{\sim}{\mathbf{N}}:\dot{\underset{\sim}{\mathbf{X}}} = \underset{\sim}{\mathbf{N}}:\underset{\approx}{\boldsymbol{\Lambda}}:(\dot{\underset{\sim}{\boldsymbol{\varepsilon}}} - \dot{p}\underset{\sim}{\mathbf{N}}) - \underset{\sim}{\mathbf{N}}:\left(\frac{2}{3}(C + C_\chi)\dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p - \frac{2}{3}C_\chi {}^\chi\dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p\right) = 0 \tag{106}$$

$$\dot{p} = \frac{\underset{\sim}{\mathbf{N}}:\left(\underset{\approx}{\boldsymbol{\Lambda}}:\dot{\underset{\sim}{\boldsymbol{\varepsilon}}} + \frac{2}{3}C_\chi {}^\chi\dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p\right)}{\underset{\sim}{\mathbf{N}}:\underset{\approx}{\boldsymbol{\Lambda}}:\underset{\sim}{\mathbf{N}} + C + C_\chi} \tag{107}$$

where both $\dot{\underset{\sim}{\boldsymbol{\varepsilon}}}$ and ${}^\chi\dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p$=controllable independent variables at the material point.

The fact that the gradient of the plastic strain tensor (or part of it, in models retaining only the rotational part) mainly impacts on the kinematic hardening of the material has been recognized in Steinmann (1996), Forest et al. (2002c), Forest and Sievert (2003), and Gurtin (2003). In these references, the divergence of the higher order generalized stress tensor acts as a back-stress.

Application to Nonlinear Kinematic Hardening

Linear kinematic hardening has a limited range of validity in nonlinear mechanics of materials. The nonlinear extension of kinematic hardening as formulated in Lemaitre and Chaboche (1994) and Maugin (1992) is recalled

$$\underset{\sim}{\mathbf{X}} = \frac{2}{3}C\underset{\sim}{\boldsymbol{\alpha}} \quad \dot{\underset{\sim}{\boldsymbol{\alpha}}} = \dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p - D\dot{p}\underset{\sim}{\boldsymbol{\alpha}} \tag{108}$$

Linear kinematic hardening is retrieved when the material parameter D vanishes. In the latter case, the internal variable $\underset{\sim}{\boldsymbol{\alpha}}$ reduces to the plastic strain tensor itself. The initial state space is

$$\text{STATE } 0 = \{\underset{\sim}{\boldsymbol{\varepsilon}}^p,\ \underset{\sim}{\boldsymbol{\alpha}}\} \tag{109}$$

The variable selected for the application of the micromorphic approach is

$$\phi \equiv \underset{\sim}{\boldsymbol{\alpha}} \quad {}^\chi\phi \equiv {}^\chi\underset{\sim}{\boldsymbol{\alpha}} \tag{110}$$

This corresponds to five additional degrees-of-freedom if the ${}^\chi\underset{\sim}{\boldsymbol{\alpha}}$ is treated as a deviatoric tensor like the macroscopic kinematic hardening variable $\underset{\sim}{\boldsymbol{\alpha}}$. The state space is enlarged as follows:

$$\text{STATE } 0 = \{\underset{\sim}{\boldsymbol{\varepsilon}}^p,\ \underset{\sim}{\boldsymbol{\alpha}},\ {}^\chi\underset{\sim}{\boldsymbol{\alpha}},\ \boldsymbol{\nabla}\ {}^\chi\underset{\sim}{\boldsymbol{\alpha}}\} \tag{111}$$

The quadratic free energy function [Eq. (94)] is changed into

$$\begin{aligned}\rho\psi(\underset{\sim}{\boldsymbol{\varepsilon}}, \underset{\sim}{\boldsymbol{\alpha}},\ {}^\chi\underset{\sim}{\boldsymbol{\alpha}},\ \boldsymbol{\nabla}\,{}^\chi\underset{\sim}{\boldsymbol{\alpha}}) &= \frac{1}{2}\underset{\sim}{\boldsymbol{\varepsilon}}^e:\underset{\approx}{\boldsymbol{\Lambda}}:\underset{\sim}{\boldsymbol{\varepsilon}}^e + \frac{1}{3}C\underset{\sim}{\boldsymbol{\alpha}}:\underset{\sim}{\boldsymbol{\alpha}} \\ &+ \frac{1}{3}(\underset{\sim}{\boldsymbol{\alpha}} - {}^\chi\underset{\sim}{\boldsymbol{\alpha}}):\underset{\approx}{\mathbf{C}}_\chi:(\underset{\sim}{\boldsymbol{\alpha}} - {}^\chi\underset{\sim}{\boldsymbol{\alpha}}) \\ &+ \frac{1}{2}\boldsymbol{\nabla}\,{}^\chi\underset{\sim}{\boldsymbol{\alpha}} \,\vdots\, \underset{\approx}{\mathbf{A}} \,\vdots\, \boldsymbol{\nabla}{}^\chi\underset{\sim}{\boldsymbol{\alpha}}\end{aligned} \tag{112}$$

Compared to Eq. (95), the state laws modify to

$$\underset{\sim}{\boldsymbol{\sigma}} = \underset{\approx}{\boldsymbol{\Lambda}}:\underset{\sim}{\boldsymbol{\varepsilon}}^e \quad \underset{\sim}{\mathbf{X}} = \frac{2}{3}C\underset{\sim}{\boldsymbol{\alpha}} + \frac{2}{3}\underset{\approx}{\mathbf{C}}_\chi:(\underset{\sim}{\boldsymbol{\alpha}} - {}^\chi\underset{\sim}{\boldsymbol{\alpha}})$$

$$\underset{\sim}{\mathbf{a}} = -\frac{2}{3}\underset{\approx}{\mathbf{C}}_\chi:(\underset{\sim}{\boldsymbol{\alpha}} - {}^\chi\underset{\sim}{\boldsymbol{\alpha}}) \quad \underset{\sim}{\mathbf{b}} = \underset{\approx}{\mathbf{A}} \,\vdots\, \boldsymbol{\nabla}{}^\chi\underset{\sim}{\boldsymbol{\alpha}} \tag{113}$$

In the simplified situation [Eq. (96)], the combination of the additional balance equation and of the previous state laws gives

$$\underset{\sim}{\mathbf{a}} = -\frac{2}{3}C_\chi(\underset{\sim}{\boldsymbol{\alpha}} - {}^\chi\underset{\sim}{\boldsymbol{\alpha}}) = \text{div}\ \underset{\sim}{\mathbf{b}} = A\Delta{}^\chi\underset{\sim}{\boldsymbol{\alpha}} \tag{114}$$

The partial differential equation, which the micromorphic kinematic hardening variable obeys, is then

$${}^\chi\underset{\sim}{\boldsymbol{\alpha}} - \frac{3A}{2C_\chi}\Delta{}^\chi\underset{\sim}{\boldsymbol{\alpha}} = \underset{\sim}{\boldsymbol{\alpha}} \tag{115}$$

The idea of introducing a nonlocal kinematic hardening variable associated with the classical one is due to Dorgan and Voyiadjis (2003). The classical evolution law [Eq. (108)] for kinematic hardening can be kept in the micromorphic model

$$\dot{\underset{\sim}{\boldsymbol{\alpha}}} = \dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p - D\dot{p}\underset{\sim}{\boldsymbol{\alpha}} = \dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p - D\dot{p}\left({}^\chi\underset{\sim}{\boldsymbol{\alpha}} - \frac{3A}{2C_\chi}\Delta{}^\chi\underset{\sim}{\boldsymbol{\alpha}}\right) \tag{116}$$

It has, however, a nonlocal evolution in terms of the micromorphic kinematic hardening variable.

We finally give the yield function and exploit the consistency condition

$$f = J_2(\underset{\sim}{\boldsymbol{\sigma}} - \underset{\sim}{\mathbf{X}}) - \sigma_Y = J_2\left(\underset{\sim}{\boldsymbol{\sigma}} - \frac{2}{3}C{}^\chi\underset{\sim}{\boldsymbol{\alpha}} + A\left(1 + \frac{C}{C_\chi}\right)\Delta{}^\chi\underset{\sim}{\boldsymbol{\alpha}}\right) - \sigma_Y = 0 \tag{117}$$

$$\dot{p} = \frac{\underset{\sim}{\mathbf{N}}:\left(\underset{\approx}{\boldsymbol{\Lambda}}:\dot{\underset{\sim}{\boldsymbol{\varepsilon}}} + \frac{2}{3}C_\chi {}^\chi\dot{\underset{\sim}{\boldsymbol{\alpha}}}\right)}{\underset{\sim}{\mathbf{N}}:\underset{\approx}{\boldsymbol{\Lambda}}:\underset{\sim}{\mathbf{N}} + (C + C_\chi)\left(1 - \frac{2}{3}D\underset{\sim}{\mathbf{N}}:\underset{\sim}{\boldsymbol{\alpha}}\right)} \tag{118}$$

The intrinsic dissipation is then evaluated as

$$D^{\text{res}} = \underset{\sim}{\boldsymbol{\sigma}}:\dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p - \underset{\sim}{\mathbf{X}}:\dot{\underset{\sim}{\boldsymbol{\alpha}}} = \underset{\sim}{\boldsymbol{\sigma}}:\dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p - \underset{\sim}{\mathbf{X}}:\dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p + \dot{p}D\underset{\sim}{\mathbf{X}}:\underset{\sim}{\boldsymbol{\alpha}} \tag{119}$$

$$= J_2(\underset{\sim}{\boldsymbol{\sigma}} - \underset{\sim}{\mathbf{X}})\dot{p} + D\dot{p}\underset{\sim}{\mathbf{X}}:\underset{\sim}{\boldsymbol{\alpha}} \tag{120}$$

$$= f\dot{p} + \left(\sigma_Y + \frac{D}{C + C_\chi}J_2^2(\underset{\sim}{\mathbf{X}}) + \frac{3}{2}\frac{DC_\chi}{C + C_\chi}\underset{\sim}{\mathbf{X}}:{}^\chi\underset{\sim}{\boldsymbol{\alpha}}\right)\dot{p} \tag{121}$$

Because of the last term, it is not possible to ensure the positivity of dissipation for any values of the state variables and degrees-of-freedom. If the evolution equation for kinematic hardening is kept in the classical form [Eq. (116)], it will be necessary for any computation based on this model to check the positivity numerically at each material point and at each loading time.

A slight modification of Eq. (116) can be proposed to alleviate this weakness of the previous model, as follows:

$$\dot{\underset{\sim}{\boldsymbol{\alpha}}} = \dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p - D\dot{p}\left(\underset{\sim}{\boldsymbol{\alpha}} + \frac{C_\chi}{C}(\underset{\sim}{\boldsymbol{\alpha}} - {}^\chi\underset{\sim}{\boldsymbol{\alpha}})\right) \tag{122}$$

This evolution law for kinematic hardening still has a local character that makes its numerical implementation straightforward within an implicit integration scheme. It introduces, however, a coupling between the macroscopic and micromorphic kinematic hardening variables. In the two limit cases C_χ=0 (uncoupled evolution of the micromorphic variable) or $\underset{\sim}{\boldsymbol{\alpha}} = {}^\chi\underset{\sim}{\boldsymbol{\alpha}}$ (constrained evolution of the micromorphic variable), the enhanced evolution rule [Eq. (122)] reduces to the classical one [Eq. (108)]. In all situations, it implies

$$D^{\text{res}} = J_2(\sigma - \underset{\sim}{\mathbf{X}})\dot{p} + \frac{D}{C + C_\chi}\dot{p}J_2^2(\underset{\sim}{\mathbf{X}}) \geqslant 0 \tag{123}$$

The combination of micromorphic isotropic and kinematic hardening in a more elaborate elastoplastic model is straightforward based on the developments of the preceding sections. So does the extension to viscoplasticity as discussed earlier.

Dissipative Micromechanisms

In all the previous examples, it has been assumed that the micromorphic variable and its gradient contribute to the free energy density of the material only and not to the intrinsic dissipation. This is a strong hypothesis motivated by the pragmatic argument of simplicity. Indeed the extended elastoplasticity theories then require only two additional material parameters, a coupling modulus and a characteristic length. However, the microdeformation can be associated with dissipative mechanisms and the systematic procedure sketched in an earlier section must be complemented by the following concepts.

The micromorphic variable and its gradient are split into dissipative and nondissipative parts, called elastic and plastic parts, respectively

$$ {}^{\chi}\phi = {}^{\chi}\phi^{e} + {}^{\chi}\phi^{p} \quad \underline{\mathbf{K}} = \nabla {}^{\chi}\phi = \underline{\mathbf{K}}^{e} + \underline{\mathbf{K}}^{p} \tag{124} $$

Note that the decomposition of ${}^{\chi}\phi$ into elastic and plastic parts in an unambiguous way is possible only if it has some invariance properties with respect to change of observers. For instance, such a decomposition has no meaning if $\phi \equiv \underset{\sim}{\mathbf{R}}$, i.e., for a Cosserat theory since the material rotation and the microrotation are not objective quantities. Instead, the relative rotation, written $\phi - {}^{\chi}\phi$ within the small deformation framework, is invariant w.r.t. change of observer and can, therefore, be decomposed unambiguously. Such elastic-plastic decompositions of the additional strain measures have been proposed for the full micromorphic and full microstrain continua in Sansour (1998a,b) and Forest and Sievert (2003, 2006) within the finite deformation framework.

The space of state variables becomes

$$ \mathrm{STATE} = \{\underset{\sim}{\boldsymbol{\varepsilon}}^{e},\ \alpha,\ {}^{\chi}\phi^{e},\ \underline{\mathbf{K}}^{e}\} \tag{125} $$

The Clausius-Duhem inequality now takes the form

$$ -\rho\dot{\psi} + \underset{\sim}{\boldsymbol{\sigma}}:\dot{\underset{\sim}{\boldsymbol{\varepsilon}}} + a\,{}^{\chi}\dot{\phi} + \underline{\mathbf{b}}\,.\,\dot{\underline{\mathbf{K}}} \geqslant 0 \tag{126} $$

$$ \left(\underset{\sim}{\boldsymbol{\sigma}} - \rho\frac{\partial\psi}{\partial\underset{\sim}{\boldsymbol{\varepsilon}}^{e}}\right):\dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^{e} + \left(a - \rho\frac{\partial\psi}{\partial{}^{\chi}\phi^{e}}\right){}^{\chi}\dot{\phi}^{e} + \left(\underline{\mathbf{b}} - \rho\frac{\partial\psi}{\partial\underline{\mathbf{K}}^{e}}\right).\,\dot{\underline{\mathbf{K}}}^{e} + \underset{\sim}{\boldsymbol{\sigma}}:\dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^{p} $$

$$ -\rho\frac{\partial\psi}{\partial\alpha}\dot{\alpha} + a\,{}^{\chi}\dot{\phi}^{p} + \underline{\mathbf{b}}\,.\,\dot{\underline{\mathbf{K}}}^{p} \geqslant 0 \tag{127} $$

from which the state laws are derived

$$ \underset{\sim}{\boldsymbol{\sigma}} = \rho\frac{\partial\psi}{\partial\underset{\sim}{\boldsymbol{\varepsilon}}^{e}} \quad X = \rho\frac{\partial\psi}{\partial\alpha} \quad a = \frac{\partial\psi}{\partial{}^{\chi}\phi^{e}} \quad \underline{\mathbf{b}} = \frac{\partial\psi}{\partial\underline{\mathbf{K}}^{e}} \tag{128} $$

The plastic parts of the micromorphic variables now contribute to the intrinsic dissipation

$$ D^{\mathrm{res}} = \underset{\sim}{\boldsymbol{\sigma}}:\dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^{p} - X\dot{\alpha} + a\,{}^{\chi}\dot{\phi}^{p} + \underline{\mathbf{b}}\,.\,\dot{\underline{\mathbf{K}}}^{p} \geqslant 0 \tag{129} $$

The positivity of dissipation can be identically fulfilled by the proper choice of a dissipation potential $\Omega(\underset{\sim}{\boldsymbol{\sigma}}, X, a, \underline{\mathbf{b}})$ that depends on all thermodynamic forces

$$ \dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^{p} = \frac{\partial\Omega}{\partial\underset{\sim}{\boldsymbol{\sigma}}} \quad \dot{\alpha} = -\frac{\partial\Omega}{\partial X} \quad {}^{\chi}\dot{\phi}^{p} = \frac{\partial\Omega}{\partial a} \quad \dot{\underline{\mathbf{K}}}^{p} = \frac{\partial\Omega}{\partial\underline{\mathbf{b}}} \tag{130} $$

Example

For the illustration, let us consider the extreme situation for which the micromorphic variables contribute to the dissipation only so that all the parts ${}^{\chi}\phi^{e}$ and $\underline{\mathbf{K}}^{e}$ can be neglected. It corresponds to the opposite situation to that analyzed in the micromophic approach. The case of scalar microstrain gradient viscoplasticity $\phi \equiv p$, ${}^{\chi}\phi \equiv {}^{\chi}p$ is handled as an example.

The following enhanced viscoplastic potential is proposed, starting from a template viscoplastic model from Lemaitre and Chaboche (1994):

$$ \Omega(\underset{\sim}{\boldsymbol{\sigma}}, R, a, \underline{\mathbf{b}}) = \frac{K}{n+1}\left\langle\frac{f(\underset{\sim}{\boldsymbol{\sigma}}, R, a)}{K}\right\rangle^{n+1} + \frac{a^{2}}{2K_{\chi}} + \frac{1}{2A}\underline{\mathbf{b}}\,.\,\underline{\mathbf{b}} \tag{131} $$

where $\langle\cdot\rangle$ represents the positive part of the quantity in brackets; and f is the following yield function:

$$ f(\underset{\sim}{\boldsymbol{\sigma}}, R, a, \underline{\mathbf{b}}) = \sigma_{\mathrm{eq}} - \sigma_{Y} - R + a \tag{132} $$

The coupling between macroscopic and micromorphic variables goes through the introduction of the generalized stress a in the yield function f. The additional contributions are quadratic with respect to the generalized stresses. Derivation of the dissipation potential w.r.t. to all thermodynamical forces provide the evolution equations

$$ \dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^{p} = \frac{\partial\Omega}{\partial\underset{\sim}{\boldsymbol{\sigma}}} = \dot{p}\underset{\sim}{\mathbf{N}} \tag{133} $$

$$ \dot{p} = -\frac{\partial\Omega}{\partial R} = \left\langle\frac{f}{K}\right\rangle^{n} \tag{134} $$

$$ {}^{\chi}\dot{p} = \frac{\partial\Omega}{\partial a} = \dot{p} + \frac{a}{K_{\chi}} \tag{135} $$

$$ \dot{\underline{\mathbf{K}}}^{p} = \frac{\partial\Omega}{\partial\underline{\mathbf{b}}} = \frac{1}{A}\underline{\mathbf{b}} \tag{136} $$

The partial differential equation satisfied by the micromorphic variable is derived after combining the additional balance equation relating the generalized stresses and the previous evolution equations

$$ a = \mathrm{div}\,\underline{\mathbf{b}} = A\Delta{}^{\chi}\dot{p} = -K_{\chi}(\dot{p} - {}^{\chi}\dot{p}) \tag{137} $$

$$ {}^{\chi}\dot{p} - \frac{A}{K_{\chi}}\Delta{}^{\chi}\dot{p} = \dot{p} \tag{138} $$

The latter equation turns out to be the time derivative of Eq. (61). Under plastic loading, the current equivalent stress level can be evaluated as

$$ \sigma_{\mathrm{eq}} = \sigma_{Y} + K\dot{p}^{1/n} + K_{\chi}(\dot{p} - {}^{\chi}\dot{p}) \tag{139} $$

which we expand further in the specific simple case $n=1$ of linear viscosity

$$ \sigma_{\mathrm{eq}} = \sigma_{Y} + R + K\,{}^{\chi}\dot{p} - A\left(1 + \frac{K}{K_{\chi}}\right)\Delta{}^{\chi}\dot{p} \tag{140} $$

which is to be compared to Eq. (66). In contrast to the plastic microstrain model proposed in the microstrain gradient plasticity section, the micromorphic variable and its Laplacian solely modify the viscous part of the stress instead of the strain hardening part. This dissipative model can be regarded as the purely viscous counterpart of the elastoplastic microstrain model depicted in that section. Combining both models through simultaneous microstrain energy storage and dissipation will affect both work hardening and viscosity of the material.

Gradient of Microdamage

The application of the micromorphic approach to damage thermomechanics is tackled here, starting from the basic classical damage model based on formulations from Lemaitre and Chaboche (1994) and Besson (2004). Several features of available gradient of damage models are recovered, especially Frémond and Nedjar (1996) and Peerlings et al. (2004).

Brittle Microdamage

The selected variable is first a scalar damage parameter D

$$\phi \equiv D \quad {}^{\chi}\phi = {}^{\chi}D \quad \mathrm{STATE} = \{\underset{\sim}{\boldsymbol{\varepsilon}}, D, {}^{\chi}D, \nabla {}^{\chi}D\} \tag{141}$$

The power of internal forces is extended as follows:

$$p^{(i)} = \underset{\sim}{\boldsymbol{\sigma}} : \dot{\underset{\sim}{\boldsymbol{\varepsilon}}} + a\,{}^{\chi}\dot{D} + \underset{\sim}{\mathbf{b}} \,.\, \nabla {}^{\chi}\dot{D} \tag{142}$$

thus, introducing the generalized stresses a and $\underset{\sim}{\mathbf{b}}$. The application of the method of virtual power gives the balance Eq. (32) and the boundary condition Eq. (63) that these stresses must satisfy.

The idea of treating the damage variable as an actual degree-of-freedom goes back to the work of Frémond and Nedjar (1996). In contrast, this status is attributed here to the microdamage variable ${}^{\chi}D$. The Clausius-Duhem inequality taking the microdamage gradient into account is

$$-\rho\dot{\psi} + \underset{\sim}{\boldsymbol{\sigma}} : \dot{\underset{\sim}{\boldsymbol{\varepsilon}}} + a\,{}^{\chi}\dot{D} + \underset{\sim}{\mathbf{b}} \,.\, \nabla {}^{\chi}\dot{D} \geqslant 0 \tag{143}$$

$$\left(\underset{\sim}{\boldsymbol{\sigma}} - \rho\frac{\partial\psi}{\partial \underset{\sim}{\boldsymbol{\varepsilon}}}\right) : \dot{\underset{\sim}{\boldsymbol{\varepsilon}}} + \left(a - \rho\frac{\partial\psi}{\partial {}^{\chi}D}\right){}^{\chi}\dot{D} + \left(\underset{\sim}{\mathbf{b}} - \rho\frac{\partial\psi}{\partial \nabla {}^{\chi}D}\right) . \nabla {}^{\chi}\dot{D} - \rho\frac{\partial\psi}{\partial D}\dot{D} \geqslant 0 \tag{144}$$

The state laws follow:

$$\underset{\sim}{\boldsymbol{\sigma}} = \rho\frac{\partial\psi}{\partial \underset{\sim}{\boldsymbol{\varepsilon}}} \quad a = \rho\frac{\partial\psi}{\partial {}^{\chi}D} \quad \underset{\sim}{\mathbf{b}} = \rho\frac{\partial\psi}{\partial \nabla {}^{\chi}D} \quad Y = -\rho\frac{\partial\psi}{\partial D} \tag{145}$$

The intrinsic dissipation reduces then to

$$D^{\mathrm{res}} = Y\dot{D} \geqslant 0 \tag{146}$$

At this stage, a convex damage function $f(Y)$ can be chosen as a damage criterion and dissipation potential

$$\dot{D} = \dot{\lambda}\frac{\partial f}{\partial Y} \tag{147}$$

The most simple example is perhaps Marigo's model (Marigo 1981; Besson 2004; Lemaitre and Desmorat 2004) for which the damage evolution is explicit

$$f(Y) = Y - \kappa(D) \quad D = \kappa^{-1}(Y) \tag{148}$$

The corresponding enhanced free energy density can be taken of the form

$$\rho\psi(\underset{\sim}{\boldsymbol{\varepsilon}}, D, {}^{\chi}D, \nabla {}^{\chi}D) = \frac{1}{2}(1-D)\underset{\sim}{\boldsymbol{\varepsilon}} : \underset{\approx}{\boldsymbol{\Lambda}} : \underset{\sim}{\boldsymbol{\varepsilon}} + \frac{1}{2}H_{\chi}(D - {}^{\chi}D)^2 + \frac{1}{2}A\,\nabla {}^{\chi}D \,.\, \nabla {}^{\chi}D \tag{149}$$

Damage then affects the elastic response of the material according to

$$\underset{\sim}{\boldsymbol{\sigma}} = \rho\frac{\partial\psi}{\partial \underset{\sim}{\boldsymbol{\varepsilon}}} = (1-D)\underset{\approx}{\boldsymbol{\Lambda}} : \underset{\sim}{\boldsymbol{\varepsilon}} \tag{150}$$

The damage variable D is constrained to remain lower than 1, complete failure being reached at this limit. We do not introduce here any ingredient for imposing such a constraint to the microdamage ${}^{\chi}D$. The driving force for damage is

$$Y = -\rho\frac{\partial\psi}{\partial D} = \frac{1}{2}\underset{\sim}{\boldsymbol{\varepsilon}} : \underset{\approx}{\boldsymbol{\Lambda}} : \underset{\sim}{\boldsymbol{\varepsilon}} - H_{\chi}(D - {}^{\chi}D) \tag{151}$$

The generalized stresses are

$$a = \rho\frac{\partial\psi}{\partial {}^{\chi}D} = -H_{\chi}(D - {}^{\chi}D) \quad \underset{\sim}{\mathbf{b}} = \rho\frac{\partial\psi}{\partial \nabla {}^{\chi}D} = A\,\nabla {}^{\chi}D \tag{152}$$

The additional partial differential equation is obtained as

$$a = \mathrm{div}\,\underset{\sim}{\mathbf{b}} \Rightarrow {}^{\chi}D - \frac{A}{H_{\chi}}\Delta {}^{\chi}D = D \tag{153}$$

This additional partial differential equation has been used successfully to model the development of damage in composites in Germain et al. (2007).

For a linear evolution of the damage threshold (at least as long as D remains smaller than 1)

$$\kappa(D) = Y_0 + HD \tag{154}$$

the damage level can be determined explicitly from the continuing damaging condition $f=0$

$$D = \frac{\underset{\sim}{\boldsymbol{\varepsilon}} : \underset{\approx}{\boldsymbol{\Lambda}} : \underset{\sim}{\boldsymbol{\varepsilon}} - 2Y_0 + 2H_{\chi}{}^{\chi}D}{2(H + H_{\chi})} \tag{155}$$

More general evolution laws for the damage threshold have been considered in Geers et al. (1998) to ensure a suitable description down to zero stress levels corresponding to final fracture of the material, especially avoiding unrealistic growth of the damage zone.

The previous example involves the damage variable D for brittle fracture. For ductile damage, the relevant variable is the void density. The gradient of porosity has been considered for modeling ductile fracture as early as Bammann and Aifantis (1989).

Damage and Scalar Microstrain

In the original strain-based gradient damage models presented in Geers et al. (1998) and Peerlings et al. (2001), the selected variable is not the damage variable itself but an equivalent strain measure $\varepsilon_{\mathrm{eq}}$

$$\phi \equiv \varepsilon_{\mathrm{eq}} \quad {}^{\chi}\phi \equiv {}^{\chi}\varepsilon \quad \mathrm{STATE} = \{\underset{\sim}{\boldsymbol{\varepsilon}}, \varepsilon_{\mathrm{eq}}, D, {}^{\chi}\varepsilon, \nabla {}^{\chi}\varepsilon\} \tag{156}$$

The additional partial differential equation postulated in Geers et al. (1998)

$$^{\chi}\varepsilon - \frac{A}{H_\chi}\Delta^{\chi}\varepsilon = \varepsilon_{eq} \quad (157)$$

is retrieved for the following form of the free energy density function

$$\rho\psi(\underset{\sim}{\varepsilon},\varepsilon_{eq},D,{}^{\chi}\varepsilon,\nabla^{\chi}\varepsilon) = \frac{1}{2}(1-D)\underset{\sim}{\varepsilon}:\underset{\approx}{\Lambda}:\underset{\sim}{\varepsilon} + \frac{1}{2}H_\chi(\varepsilon_{eq} - {}^{\chi}\varepsilon)^2 + \frac{1}{2}A\,\nabla\,{}^{\chi}\varepsilon\,.\,\nabla^{\chi}\varepsilon \quad (158)$$

Such a coupling between the elastic energy and the cost for a deviation of the microstrain from the macrostrain leads to a modification of the stress-strain relation compared to the original classical model

$$\underset{\sim}{\sigma} = (1-D)\underset{\approx}{\Lambda}:\underset{\sim}{\varepsilon} + H_\chi(\varepsilon_{eq} - {}^{\chi}\varepsilon)\frac{\partial\varepsilon_{eq}}{\partial\underset{\sim}{\varepsilon}} \quad (159)$$

The same modification has been found in Peerlings et al. (2004) according to a different thermodynamical framework. In their work, this modification arises as a sufficient condition to identically fulfill the entropy inequality in its global form. Following the same argument, the authors derive the partial differential Eq. (157) and a specialized form of the boundary condition Eq. (63). In their numerical applications, they study the impact of values of the coupling modulus ranging from 0 to E (classical Young's modulus).

The main difference between the derivation of the strain-based gradient model according to the micromorphic approach and the derivation proposed in Peerlings et al. (2004) lies in the fact that the generalized stresses a and $\underline{b}$ fulfilling an additional balance equation and contributing explicitly to the local energy balance (local formulation of the first principle of thermodynamics) are introduced in the present approach but not in Peerlings et al. (2004). In contrast, the latter reference is based on the exploitation of the entropy principle in its global form and on the choice of sufficient (local) conditions that ensure the positivity of global dissipation.

Multimechanism and Anisotropic Micromorphic Plasticity and Damage

The paradigm for anisotropic elastoplastic modeling is represented by crystal plasticity because the kinematics of deformation and the evolution of anisotropy are dictated by the crystallography of the physical deformation mechanisms. That is why the crystal plasticity framework is chosen here to illustrate how the micromorphic approach can be applied to multimechanism plasticity and eventually damage. The infinitesimal deformation framework is adopted here for the sake of conciseness.

We start from a formulation of single crystal behavior proposed by Diard et al. (2002) and Marchal et al. (2006) in which the total strain increment is due to the contributions of elastic, plastic deformation, and damage mechanism

$$\dot{\underset{\sim}{\varepsilon}} = \dot{\underset{\sim}{\varepsilon}}^e + \dot{\underset{\sim}{\varepsilon}}^p + \dot{\underset{\sim}{\varepsilon}}^d \quad (160)$$

The plastic deformation is due to slip processes with respect to N slip systems represented by the normal to the slip plane $\underline{n}^s$ and the slip direction $\underline{m}^s$

$$\dot{\underset{\sim}{\varepsilon}}^p = \sum_{s=1}^{N}\dot{\gamma}^s(\underline{m}^s \otimes \underline{n}^s)^{sym} \quad (161)$$

where $(\;)^{sym}$ denotes the symmetric part of the orientation tensor. The driving force for activating plastic slip is the resolved shear stress obtained by projecting the stress tensor on the orientation tensor. Cleavage damage is represented by the opening δ^s of M crystallographic cleavage planes

$$\dot{\underset{\sim}{\varepsilon}}^d = \sum_{s=1}^{M}\dot{\delta}^s\underline{n}^s \otimes \underline{n}^s \quad (162)$$

Other damage systems must be introduced to accommodate in-plane deformation once cleavage has started. Cleavage is primarily driven by the normal stress with respect to the plane orthogonal to $\underline{n}^s$. Finite-element simulations of the deformation and damage of single crystals and grains at high temperature were performed in Diard et al. (2002) and Marchal et al. (2006). The question arises of the generalized continuum extension of such a model to include size effects in the behavior and/or ensure mesh-independent results in the damaging response of the material.

Regarding size effects in crystal plasticity, the authors in Gurtin (2000), Gurtin and Needleman (2005) and Bayley et al. (2006) propose to select the slip variable

$$\phi^s \equiv \gamma^s \quad (163)$$

and to incorporate the effect of the gradient of slip $\nabla\gamma^s$ into the modeling. A generalized stress $\underline{b}^s$ is associated with $\nabla\gamma^s$. Regarding damage in laminate composites, it was proposed in Germain et al. (2007) to select the variable δ^s and to associate it with a microdamage variable ${}^{\chi}\delta^s$

$$\phi^s \equiv \delta^s \qquad {}^{\chi}\phi^{\varepsilon} \equiv {}^{\chi}\delta^s \quad (164)$$

Similarly, a generalized stress $\underline{b}^s$ must be associated with ${}^{\chi}\delta^s$. In the finite-element implementation of this model, M additional degrees-of-freedom are attributed to each node. In Bayley et al. (2006), 18 dislocation densities were treated as additional degrees-of-freedom. The number of degrees-of-freedom, thus, increases with the number of involved deformation or damage mechanisms.

Boundary and Interface Conditions

Additional boundary and interface conditions must be taken into account as a consequence of the previous choices. For an ideal interface with normal $\underline{n}$, they read

$$[\![{}^{\chi}\phi^s]\!] = 0 \quad \text{and} \quad [\![\underline{b}^s\,.\,\underline{n}]\!] = 0 \quad (165)$$

where $[\![f]\!]$ represents the jump of f across the boundary. For the size-dependent plasticity models (Gurtin 2000), ${}^{\chi}\phi^s$ has to be replaced by ϕ^s itself. Such conditions have to be enforced at grain boundaries separating crystals or at the interface between laminates. They are unambiguous for single slip ($N=1$) or cleavage ($M=1$). But for multislip situations, a difficulty arises considering that the variables γ^s are defined only up to a symmetry belonging to the crystal symmetry group of the material. As a result, jump conditions at an interface between two such materials (such as two grains) cannot be uniquely defined. Indeed, the labeling of slip or cleavage systems is not intrinsic and is independent in different grains so that it is impossible to say whether γ^1 on one side of the grain boundary should be continuous with γ^1 or, let say, γ^2 in the other grain, according to Eq. (165). Similar difficulties arise in the anisotropic damage of composites (Germain et al. 2007).

The alternative choice consists of selecting that intrinsic total plastic or damage strain tensors, instead of the individual slip or cleavage contribution

$$\phi \equiv \underset{\sim}{\boldsymbol{\varepsilon}}^p \quad {}^\chi\phi \equiv {}^\chi\underset{\sim}{\boldsymbol{\varepsilon}}^p \quad \text{or} \quad \phi \equiv \underset{\sim}{\boldsymbol{\varepsilon}}^d \quad {}^\chi\phi \equiv {}^\chi\underset{\sim}{\boldsymbol{\varepsilon}}^d \tag{166}$$

If a quadratic potential is chosen with respect to this microstrain variable and its gradient, the additional balance equation leads to the following partial differential equations relating macro and microstrain:

$${}^\chi\underset{\sim}{\boldsymbol{\varepsilon}}^d - \text{div}(\underset{\approx}{\mathbf{A}} \;\vdots\; \boldsymbol{\nabla}{}^\chi\underset{\sim}{\boldsymbol{\varepsilon}}^d) = \underset{\sim}{\boldsymbol{\varepsilon}}^d \tag{167}$$

where $\underset{\approx}{\mathbf{A}}$ is the sixth order tensor of generalized stiffness.

Internal Constraints

At several places, it was noted that models based on the micromorphic approach are related to existing gradient models by imposing the internal constraint that the micromorphic variable coincides with the selected variable of the original classical model

$${}^\chi\phi \equiv \phi \tag{168}$$

The microstrain gradient $\underset{\sim}{\mathbf{K}} = \boldsymbol{\nabla}{}^\chi\phi$ then coincides with the gradient of ϕ. In this context, the coupling modulus H_χ that penalizes the departure of the microvariable from the macrovariable according to the micromorphic approach, becomes a Lagrange multiplier in the constrained theory.

It was already mentioned that the constrained full micromorphic theory is nothing but Mindlin second gradient theory. Aifantis-like models involving the Laplacian of plastic strain, damage, or dislocation densities are retrieved by constraining the micromorphic theories involving plastic or damage microstrains. The gradient of plastic strain theory proposed by Gurtin (2003) can also be regarded as a constrained micromorphic theory. The relation to the so-called "explicit approach" (Engelen et al. 2003) appears when the hardening behavior is linked to the difference $\varepsilon^{p,d} - A\Delta\varepsilon^{p,d}$ where $\varepsilon^{p,d}$ is an equivalent measure of plastic or damage strain.

The constrained micromorphic approach delivers models that belong to the class of gradient of internal variable or internal degree-of-freedom models as initially proposed in Maugin (1990), Maugin and Muschik (1994), and Papenfuss and Forest (2006).

From the computational point of view, the constrained micromorphic approach can be used to formulate a finite-element implementation of these gradient plasticity or damage models, in which the micromorphic variable and the Lagrange multipliers are treated as additional nodal degrees-of-freedom.

Conclusions

The micromorphic approach to plasticity and damage consists of introducing additional internal degrees-of-freedom associated with the state or internal variables present in elastoviscoplasticity models of materials. Generalized stresses are attributed to the micromorphic variable and its first gradients. They must fulfill a balance equation in addition of the usual balance of momentum. The coupling between the macro and microvariables goes through the introduction of the relative generalized strain $\phi - {}^\chi\phi$ as an argument of the free energy density function. When the free energy is assumed to be a quadratic function of the generalized relative strain and of the gradient of the micromorphic variable, one invariably obtains a Helmholtz type of additional partial differential equation fulfilled by the micromorphic variables, including a source term, according to the terminology in Lazar et al. (2006). This equation obtained by substituting constitutive equations in the generalized balance of momentum is akin to the so-called *implicit gradient* approach of plasticity and damage, which postulates a priori this form of partial differential equation.

The suitable form of the additional boundary conditions was derived from the generalized principle of virtual power. The contact generalized force a^c can in general take any value. At a free surface, it must be set to zero. In other cases, the boundary conditions in the Dirichlet or Neumann forms may be more difficult to settle. Far from a zone of strong gradients in the mechanical fields, the condition $a^c = 0$ will still be acceptable.

The fundamental difference between the so-called implicit gradient approach and the micromorphic approach lies in the fact that, in the micromorphic model, the status of balance equation, resulting from the extension of the principle of virtual work, is fully acknowledged for the additional partial differential equation, in the spirit of Eringen (1999) and Frémond and Nedjar (1996). The power expanded by the generalized stress tensors contributes to local balance of the energy equation. As a result, the detailed form of this balance equation can be derived from the usual thermomechanical principles in the presence of any coupling and anisotropy effects. The modification of its form has been illustrated for the thermal coupling. In contrast, this partial differential equation arises according to the thermodynamical framework in Peerlings et al. (2004) as a sufficient condition to fulfill the entropy inequality in its global form. In the micromorphic approach, the second principle is still considered in its local form and the constitutive laws are selected to fulfill it identically in the spirit of Coleman and Noll (1963).

The general structure of the partial differential Eq. (33) results from the combination of an additional balance equation and of specific linear constitutive equations. It is the merit of the micromorphic approach to clearly separate universal balance equations from specific constitutive equations. This is the main argument put forward by Gurtin (1996) and Maugin (2006) for introducing generalized stresses in diffusion theory also (see the last line of Table 1). The additional balance equation and associated boundary conditions take the general form [Eqs. (32) and (63)]. The micromorphic approach can give access to more general additional partial differential equations resulting from more general constitutive equations than Eqs. (30) and (31). In particular, anisothermal and anisotropic theories can be developed in an unambiguous way, as exemplified in earlier sections.

The thermodynamical formulation of generalized continuum mechanics is a necessary step to establish the well-suited thermomechanical coupling required for realistic structural computations, which represent the ultimate objective of the approach. Few attempts to derive such thermomechanical effects exist in the literature. In Lorentz and Andrieux (2003), for example, a thermodynamical framework for strain and damage gradient models was proposed but the thermal effects were not derived. In Cardona et al. (1999), Forest et al. (2000b), Ireman and Nguyen (2004), and Forest and Amestoy (2008), the effect of gradient of temperature was investigated.

Table 1. Application of the Micromorphic Approach to the Elasticity, Plasticity, Damage, and Diffusion in Solids; the Additional Balance Equation Is Explained in the Case of a Linearized Simplified Theory

Name	Selected variable	Micromorphic variable	Number of additional degrees-of-freedom	Additional balance equation	Reference for related model	Associated constrained model
Micromorphic continuum	$\nabla \underline{\mathbf{u}}$	$\underset{\sim}{\boldsymbol{\chi}}$	9	$\underset{\sim}{\boldsymbol{\chi}} - l_c^2 \Delta \underset{\sim}{\chi} = \nabla \underline{\mathbf{u}}$	(Eringen and Suhubi 1964a,b) (Mindlin 1964)	(Mindlin and Eshel 1968)
Microstrain	$\underset{\sim}{\varepsilon}$	${}^{\chi}\underset{\sim}{\varepsilon}$	6	${}^{\chi}\underset{\sim}{\varepsilon} - l_c^2 \Delta {}^{\chi}\underset{\sim}{\varepsilon} = \underset{\sim}{\varepsilon}$	(Forest and Sievert 2006)	(Mindlin and Eshel 1968)
Microstrain gradient plasticity	p	${}^{\chi}p$	1	${}^{\chi}p - \dfrac{A}{H_\chi} \Delta {}^{\chi}p = p$	(Peerlings et al. 2004)	(Aifantis 1987) (Fleck and Hutchinson 2001)
	ϱ	${}^{\chi}\varrho$	1	${}^{\chi}\varrho - \dfrac{A}{bQ_\chi} \Delta {}^{\chi}\varrho = 1 - \exp(-\mathrm{bp})$	(Dorgan and Voyiadjis 2003)	(Walgraef and Aifantis 1988)
	$\underset{\sim}{\varepsilon}^p$	${}^{\chi}\underset{\sim}{\varepsilon}^p$	5	${}^{\chi}\underset{\sim}{\varepsilon}^p - \dfrac{3A}{2C_\chi} \Delta {}^{\chi}\underset{\sim}{\varepsilon}^p = \underset{\sim}{\varepsilon}^p$		(Forest and Sievert 2003) (Gurtin 2003) (Abu Al-Rub et al. 2007)
	$\underset{\sim}{\boldsymbol{\alpha}}$	${}^{\chi}\underset{\sim}{\boldsymbol{\alpha}}$	5	${}^{\chi}\underset{\sim}{\boldsymbol{\alpha}} - \dfrac{3A}{2C_\chi} \Delta {}^{\chi}\underset{\sim}{\boldsymbol{\alpha}} = \underset{\sim}{\boldsymbol{\alpha}}$	(Dorgan and Voyiadjis 2003)	(Steinmann 1996) (Forest and Sievert 2003)
Microdamage	D	${}^{\chi}D$	1	${}^{\chi}D - \dfrac{A}{H_\chi} \Delta {}^{\chi}D = D$	(Germain et al. 2007)	(Frémond and Nedjar 1996) (Bammann and Aifantis 1989)
	$\varepsilon_{\mathrm{eq}}$	${}^{\chi}\varepsilon$	1	${}^{\chi}\varepsilon - \dfrac{A}{H_\chi} \Delta {}^{\chi}\varepsilon = \varepsilon_{\mathrm{eq}}$	(Peerlings et al. 1996)	
	$\underset{\sim}{\varepsilon}^d$	${}^{\chi}\underset{\sim}{\varepsilon}^d$	6	${}^{\chi}\underset{\sim}{\varepsilon}^d - \dfrac{A}{H_\chi} \Delta {}^{\chi}\underset{\sim}{\varepsilon}^d = \underset{\sim}{\varepsilon}^d$		
Microdiffusion	c	${}^{\chi}c$	1	${}^{\chi}c - \lambda^2 \Delta {}^{\chi}c = c$	(Ubachs et al. 2004)	(Cahn and Hilliard 1958)

References

Abu Al-Rub, K., Voyiadjis, G. Z., and Bammann, D. J. (2007). "A thermodynamic based higher-order gradient theory for size dependent plasticity." *Int. J. Solids Struct.*, 44, 2888–2923.

Aifantis, E. C. (1987). "The physics of plastic deformation." *Int. J. Plast.*, 3, 211–248.

Bammann, D. J., and Aifantis, E. C. (1989). "A damage model for ductile metals." *Nucl. Eng. Des.*, 116, 355–362.

Bayley, C. J., Brekelmans, W. A. M., and Geers, M. G. D. (2006). "A comparison of dislocation induced back stress formulations in strain gradient crystal plasticity." *Int. J. Solids Struct.*, 43, 7268–7286.

Besson, J. (2004). *Local approach to fracture*, Ecole des Mines de Paris-Les Presses, Paris.

Besson, J., Cailletaud, G., Chaboche, J.-L., and Forest, S. (2001). *Mécanique non linéaire des matériaux*, Hermès, France.

Cahn, J. W., and Hilliard, J. E. (1958). "Free energy of a nonuniform system. I: Interfacial free energy." *J. Chem. Phys.*, 28, 258–267.

Cardona, J.-M., Forest, S., and Sievert, R. (1999). "Towards a theory of second grade thermoelasticity." *Extr. Math.*, 14(2), 127–140.

Coleman, B. D., and Noll, W. (1963). "The thermodynamics of elastic materials with heat conduction and viscosity." *Arch. Ration. Mech. Anal.*, 13, 167–178.

de Borst, R. (1993). "A generalization of J_2-flow theory for polar continua." *Comput. Methods Appl. Mech. Eng.*, 103, 347–362.

de Borst, R., Sluys, L. J., Mühlhaus, H. B., and Pamin, J. (1993). "Fundamental issues in finite-element analyses of localization of deformation." *Eng. Comput.*, 10, 99–121.

Diard, O., Leclercq, S., Rousselier, G., and Cailletaud, G. (2002). "Distribution of normal stress at grain boundaries in multicrystals: Application to an intergranular damage modeling." *Comput. Mater. Sci.*, 25, 73–84.

Dillard, T., Forest, S., and Ienny, P. (2006). "Micromorphic continuum modelling of the deformation and fracture behaviour of nickel foams." *Eur. J. Mech. A/Solids*, 25, 526–549.

Dorgan, J. D., and Voyiadjis, G. Z. (2003). "Nonlocal dislocation based plasticity incorporating gradients of hardening." *Mech. Mater.*, 35, 721–732.

Engelen, R. A. B., Fleck, N. A., Peerlings, R. H. J., and Geers, M. G. D. (2006). "An evaluation of higher-order plasticity theories for predicting size effects and localisation." *Int. J. Solids Struct.*, 43, 1857–1877.

Engelen, R. A. B., Geers, M. G. D., and Baaijens, F. P. T. (2003). "Nonlocal implicit gradient-enhanced elasto-plasticity for the modelling of softening behaviour." *Int. J. Plast.*, 19, 403–433.

Eringen, A. C. (1999). *Microcontinuum field theories*, Springer, New York.

Eringen, A. C., and Suhubi, E. S. (1964a). "Nonlinear theory of simple microelastic solids: I." *Int. J. Eng. Sci.*, 2, 189–203.

Eringen, A. C., and Suhubi, E. S. (1964b). "Nonlinear theory of simple microelastic solids: II." *Int. J. Eng. Sci.*, 2, 389–404.

Fleck, N. A., and Hutchinson, J. W. (2001). "A reformulation of strain gradient plasticity." *J. Mech. Phys. Solids*, 49, 2245–2271.

Forest, S., and Amestoy, M. (2008). "Hypertemperature in thermoelastic solids." *C. R. Mec.*, 336, 347–353.

Forest, S., Barbe, F., and Cailletaud, G. (2000a). "Cosserat modelling of size effects in the mechanical behaviour of polycrystals and multiphase materials." *Int. J. Solids Struct.*, 37, 7105–7126.

Forest, S., Cardona, J.-M., and Sievert, R. (2000b). "Thermoelasticity of second-grade media." *Continuum thermomechanics, the art and science of modelling material behaviour, Paul Germain's anniversary volume*, G. A. Maugin, R. Drouot, and F. Sidoroff, eds., Kluwer Academic, Boston, 163–176.

Forest, S., and Sievert, R. (2003). "Elastoviscoplastic constitutive frameworks for generalized continua." *Acta Mech.*, 160, 71–111.

Forest, S., and Sievert, R. (2006). "Nonlinear microstrain theories." *Int. J. Solids Struct.*, 43, 7224–7245.

Forest, S., Sievert, R., and Aifantis, E. C. (2002c). "Strain gradient crystal plasticity: Thermomechanical formulations and applications." *J. Mech. Behav. Mater.*, 13, 219–232.

Frémond, M., and Nedjar, B. (1996). "Damage, gradient of damage and principle of virtual power." *Int. J. Solids Struct.*, 33, 1083–1103.

Geers, M. G. D., de Borst, R., Brekelmans, W. A. M., and Peerling, R. H. J. (1998). "Strain-based transient-gradient damage model for failure analysis." *Comput. Methods Appl. Mech. Eng.*, 160, 133–153.

Germain, N., Besson, J., and Feyel, F. (2007). "Simulation of laminate composites degradation using mesoscopic non-local damage model and non-local layered shell element." *Modell. Simul. Mater. Sci. Eng.*, 15, S425–S434.

Germain, P. (1973). "La méthode des puissances virtuelles en mécanique des milieux continus, première partie: Théorie du second gradient." *J. de Mécanique*, 12, 235–274.

Germain, P., Nguyen, Q. S., and Suquet, P. (1983). "Continuum thermodynamics." *J. Appl. Mech.*, 50, 1010–1020.

Gurtin, M. E. (1996). "Generalized Ginzburg-Landau and Cahn-Hilliard equations based on a microforce balance." *Physica D*, 92, 178–192.

Gurtin, M. E. (2000). "On the plasticity of single crystals: Free energy, microforces, plastic-strain gradients." *J. Mech. Phys. Solids*, 48, 989–1036.

Gurtin, M. E. (2003). "On a framework for small-deformation viscoplasticity: Free energy, microforces, strain gradients." *Int. J. Plast.*, 19, 47–90.

Gurtin, M. E., and Needleman, A. (2005). "Boundary conditions in small-deformation, single-crystal plasticity that account for the Burgers vector." *J. Mech. Phys. Solids*, 53, 1–31.

Hutchinson, J. W. (2000). "Plasticity at the micron scale." *Int. J. Solids Struct.*, 37, 225–238.

Ireman, P., and Nguyen, Q.-S. (2004). "Using the gradients of temperature and internal parameters in continuum mechanics." *C. R. Mec.*, 332, 249–255.

Lazar, M., Maugin, G. A., and Aifantis, E. C. (2006). "On a theory of nonlocal elasticity of bi-Helmholtz type and some applications." *Int. J. Solids Struct.*, 43, 1404–1421.

Lemaitre, J., and Chaboche, J.-L. (1994). *Mechanics of solid materials*, Univ. Press, Cambridge, U.K.

Lemaitre, J., and Desmorat, R. (2004). *Engineering damage mechanics: Ductile, creep, fatigue and brittle failures*, Springer, Berlin.

Liebe, T., Menzel, A., and Steinmann, P. (2003). "Theory and numerics of geometrically nonlinear gradient plasticity." *Int. J. Eng. Sci.*, 41, 1603–1629.

Liebe, T., Steinmann, P., and Benallal, A. (2001). "Theoretical and computational aspects of a thermodynamically consistent framework for geometrically linear gradient damage." *Comput. Methods Appl. Mech. Eng.*, 190, 6555–6576.

Lorentz, E., and Andrieux, S. (2003). "Analysis of nonlocal models through energetic formulations." *Int. J. Solids Struct.*, 40, 2905–2936.

Marchal, N., Forest, S., Rémy, L., and Duvinage, S. (2006). "Simulation of fatigue crack growth in single crystal superalloys using local approach to fracture." *Local approach to fracture, Proc., 9th European Mechanics of Materials Conf. Euromech-Mecamat*, J. Besson, D. Moinereau, and D. Steglich, eds., Presses de l'Ecole des Mines de Paris, Paris, 353–358.

Marigo, J. J. (1981). "Formulation d'une loi d'endommagement d'un matériau élastique." *C. R. Seances Acad. Sci., Ser. 2*, 292, 1309–1312.

Maugin, G. A. (1980). "The method of virtual power in continuum mechanics: Application to coupled fields." *Acta Mech.*, 35, 1–70.

Maugin, G. A. (1990). "Internal variables and dissipative structures." *J. Non-Equil. Thermodyn.*, 15, 173–192.

Maugin, G. A. (1992). *The thermomechanics of plasticity and fracture*, Cambridge University Press, Cambridge, U.K.

Maugin, G. A. (1999). *Thermomechanics of nonlinear irreversible behaviors*, World Scientific, River Edge, N.J.

Maugin, G. A. (2006). "On the thermomechanics of continuous media with diffusion and/or weak nonlocality." *Arch. Appl. Mech.*, 75, 723–738.

Maugin, G. A., and Muschik, W. (1994). "Thermodynamics with internal variables. Part I: General concepts." *J. Non-Equil. Thermodyn.*, 19, 217–249.

Mindlin, R. D. (1964). "Micro-structure in linear elasticity." *Arch. Ration. Mech. Anal.*, 16, 51–78.

Mindlin, R. D., and Eshel, N. N. (1968). "On first strain gradient theories in linear elasticity." *Int. J. Solids Struct.*, 4, 109–124.

Mühlhaus, H. B., and Aifantis, E. C. (1991). "A variational principle for gradient plasticity." *Int. J. Solids Struct.*, 28, 845–857.

Mühlhaus, H. B., and Vardoulakis, I. (1987). "The thickness of shear bands in granular materials." *Geotechnique*, 37, 271–283.

Papenfuss, C., and Forest, S. (2006). "Thermodynamical frameworks for higher grade material theories with internal variables or additional degrees-of-freedom." *J. Non-Equil. Thermodyn.*, 31, 319–353.

Peerlings, R. H. J., de Borst, R., Brekelmans, W. A. M., and de Vree, J. H. P. (1996). "Gradient-enhanced damage for quasi-brittle materials." *Int. J. Numer. Methods Eng.*, 39, 3391–3403.

Peerlings, R. H. J., Geers, M. G. D., de Borst, R., and Brekelmans, W. A. M. (2001). "A critical comparison of nonlocal and gradient-enhanced softening continua." *Int. J. Solids Struct.*, 38, 7723–7746.

Peerlings, R. H. J., Massart, T. J., and Geers, M. G. D. (2004). "A thermodynamically motivated implicit gradient damage framework and its application to brick masonry cracking." *Comput. Methods Appl. Mech. Eng.*, 193, 3403–3417.

Sansour, C. (1998a). "A theory of the elastic-viscoplastic Cosserat continuum." *Arch. Mech.*, 50, 577–597.

Sansour, C. (1998b). "A unified concept of elastic-viscoplastic Cosserat and micromorphic continua." *J. Phys. IV*, 8, 341–348.

Sidoroff, F. (1975). "Microstructure and plasticity." *Mech. Res. Commun.*, 2, 73–77.

Steinmann, P. (1996). "Views on multiplicative elastoplasticity and the continuum theory of dislocations." *Int. J. Eng. Sci.*, 34, 1717–1735.

Svendsen, B. (1999). "On the thermodynamics of thermoelastic materials with additional scalar degrees-of-freedom." *Continuum Mech. Thermodyn.*, 4, 247–262.

Teodosiu, C. (1997). *Large plastic deformation of crystalline aggregates, CISM Courses and Lectures No. 376, Udine*, Springer, Berlin.

Ubachs, R. L. J. M., Schreurs, P. J. G., and Geers, M. G. D. (2004). "A nonlocal diffuse interface model for microstructure evolution of tin-lead solder." *J. Mech. Phys. Solids*, 52, 1763–1792.

Voyiadjis, G. Z., and Abu Al-Rub, K. (2005). "Gradient plasticity theory with a variable length scale parameter." *Int. J. Solids Struct.*, 42, 3998–4029.

Voyiadjis, G. Z., Abu Al-Ruba, R. K., and Palazotto, A. N. (2004). "Thermodynamic framework for coupling of non-local viscoplasticity and non-local anisotropic viscodamage for dynamic localization problems using gradient theory." *Int. J. Plast.*, 20, 981–1038.

Voyiadjis, G. Z., and Dorgan, J. D. (2004). "Bridging of length scales through gradient theory and diffusion equations of dislocations." *Comput. Methods Appl. Mech. Eng.*, 193, 1671–1692.

Walgraef, D., and Aifantis, E. C. (1988). "Plastic instabilities, dislocation patterns and nonequilibrium phenomena." *Res. Mech.*, 23, 161–195.