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Asymptotic Analysis of Heterogeneous Micromorphic Elastic Solids

Samuel Forest
Mines ParisTech/CNRS, Centre des Matériaux/ CNRS UMR 7633, Evry Cedex, France

Overview

Heterogeneous materials like metal polycrystals and metal matrix composites exhibit a size-dependent mechanical elastoplastic and fracture behavior. Generalized continuum theories can be used for the constitutive behavior of each constituent in order to predict such size effects. Extended homogenization methods are then needed to compute the effective properties of composite higher-order materials. Higher-order continua include the Cosserat medium for which the material point is endowed with independent translational and rotation degrees of freedom and the micromorphic continuum which accounts for the full microdeformation of a triad of directors attached to the material point. An asymptotic multiscale expansion method is used here to derive the effective properties of heterogeneous linear elastic micromorphic media. The type of continuum theory representing the effective medium is shown to be either a Cauchy, Cosserat, microstrain, or full micromorphic model, depending on the ratio between the characteristic lengths of the micromorphic constituents and the size of the heterogeneities. Applications deal with fiber size effects in metal matrix composites and with the grain-size effect in polycrystals.

Introduction

The mechanics of generalized continua represents extensions of the classical Cauchy continuum mechanics that incorporate some aspects of the microstructure underlying the material point. Directors can be attached to each material point that evolve in a different way than the material lines. They account for privileged physical directions existing in the microstructure like lattice or fiber directions. In addition to the usual motion of the material point, the associated directors can rotate or even deform with straining. The microrotation case corresponds to the Cosserat continuum, whereas microdeformation is possible in the micromorphic continuum [6]. The Cosserat and micromorphic media are examples of higher-order continuum theories that are characterized by additional degrees of freedom of the material points. In the micromorphic continuum designed by Eringen and Mindlin [7, 13], the directors can also be distorted, so that a second-order tensor is attributed to each material point. Such higher-order media are sometimes called continua with *microstructure*. This name has now become misleading in the sense that even Cauchy material models can integrate some aspects of the underlying microstructure as illustrated by classical homogenization methods used to derive the effective properties of composites. However, generalized continua incorporate a feature of the microstructure which is not accounted for by standard homogenization methods, namely, their size-dependent material response. They involve intrinsic lengths directly stemming from the microstructure of the material.

The links between the micromorphic continuum and the behavior of crystalline solids have been recognized very early by Eringen himself [4]. Lattice directions in a single crystal can be regarded as directors that rotate and deform. The fact that lattice directions can be rotated and stretched in a different way than material lines connecting individual atoms, especially in the presence of static or moving dislocations, illustrates the independence between directors and material lines in a micromorphic continuum, even though their deformations can be related at the constitutive level.

The identification of a micromorphic continuum from the discrete atomic single-crystal model is possible based on suitable averaging relations proposed in [3]. These works contain virial formula for the higher-order stress tensors arising in the micromorphic theory. This atomistic-based approach can be used to predict phonon dispersion relations; see for instance [4] for the study of dispersion of waves in a dislocated crystal.

If single-crystalline materials can be regarded as micromorphic media, then polycrystalline materials must be seen as a mixture of micromorphic media. The effective behavior of such materials can therefore be obtained by means of homogenization methods well known in the mechanics of heterogeneous materials [16, 18]. Classical homogenization methods can be used to account for the influence of the volume fraction, distribution, and morphology of the different constituents of the heterogeneous material, but they are not able to predict size effects. The authors in [20] propose to incorporate intrinsic length scales in the constitutive behavior of the constituents by means of a strain-gradient theory of plasticity. Reasons for introducing generalized continuum models in the mechanics of heterogeneous materials are twofold. Firstly, it is a natural way to obtain an explicit dependence of the effective properties of composites or multiphase materials on the absolute size of the constituents within a continuum model and to account for size effects observed for instance in materials strengthened by inclusions, fibers, or precipitates [1]. On the other hand, generalized continua can be used to limit strain localization phenomena that may occur in one constituent when it exhibits a strain-softening behavior [14]. If the constituents of a heterogeneous material are described by a generalized continuum like second grade, Cosserat, or micromorphic media, specific homogenization methods must be designed to derive its effective behaviour. The questions are the following: Does a homogeneous substitute medium exist? Under which conditions does it still have a nonlocal character? What is the relation between the effective characteristic length and that of the constituents? Bounds and estimates of the overall properties of heterogeneous linear couple stress media have been proposed for instance in [17]. Although most physically relevant applications deal with plasticity or damage phenomena, a first step is to develop homogenization methods for generalized continua in the case of linear elasticity [9]. These methods can then be applied to nonlinear behavior by introducing some linear comparison solids.

In this entry, the attention is focused on the case of heterogeneous micromorphic media with periodic microstructure. For that purpose, asymptotic methods classically used for periodic heterogeneous materials [15] are applied to linear elastic micromorphic constituents. The main interest of asymptotic methods in homogenization theory lies in the fact that it can provide the form of the balance and constitutive equations of an effective medium without any assumption on their nature and form. In particular, the nature of the effective medium for a mixture of micromorphic media will not be assumed a priori but rather will be an essential outcome of the asymptotic analysis. Asymptotic methods have been used in [2] to get solutions of higher orders to the problem of the effective properties of periodic heterogeneous classical media. In contrast, the present analysis is restricted to the first orders in the asymptotic developments, but the method is applied to the case of periodic heterogeneous micromorphic media.

Homogenization of Cosserat composites is considered in the reference [9, 12, 19]. It is a special case of the situation envisaged in this entry. Note that this situation is different from that of a classical heterogeneous Cauchy material that can be homogenized into a Cosserat continuum by suitable homogenization techniques [11].

Regarding notations, the tensor product of two vectors is $\otimes$, with $\overset{s}{\otimes}$ and $\overset{a}{\otimes}$ respectively delivering the symmetric and skew–symmetric parts of the tensor product of two vectors. A wide use of the nabla operator ∇ is made in the sequel. The notation used for the gradient and divergence operators are the following:

$$a\nabla = a_{,i}\underline{e}_i, \ \ \underline{a} \otimes \nabla = a_{i,j}\, \underline{e}_i \otimes \underline{e}_j, \ \ \underset{\sim}{a}.\nabla = a_{ij,j}\, \underline{e}_i$$

where a, $\underline{a}$ and $\underset{\sim}{a}$ respectively denote scalar, first- and second-rank tensors. The $(\underline{e}_i)_{i=1,2,3}$ are the vectors of an orthonormal basis of space, and the associated Cartesian coordinates have been used. Third-, fourth-, fifth-, and sixth-rank tensors are respectively denoted by $\underset{\sim}{\underline{a}}$ (or $\underset{\approx}{a}$), $\underset{\approx}{a}$, $\underset{\approx}{\underline{a}}$, and $\underset{\approx}{\underset{\approx}{a}}$. Indices can be contracted as follows: $\underset{\sim}{a} : \underset{\sim}{b} = a_{ij}b_{ij}$,

$$\underset{\sim}{\underline{a}} : \underset{\sim}{b} = a_{ijk}b_{jk}\underline{e}_i, \quad \underset{\approx}{a} : \underset{\sim}{b} = a_{ijkl}b_{kl}\underline{e}_i \otimes \underline{e}_j,$$

$$\underset{\sim}{a} : \underset{\approx}{A} : \underset{\sim}{b} = a_{ij}A_{ijkl}b_{kl}, \quad \underset{\sim}{\underline{a}} \,\vdots\, \underset{\sim}{\underline{b}} = a_{ijk}b_{ijk}$$

Linear Elastic Micromorphic Media

The balance and constitutive equations of the micromorphic continuum are recalled briefly in the linear elastic framework. The motion of a micromorphic body Ω is described by two independent sets of degrees of freedom: the displacement $\underline{u}$ and the microdeformation $\underset{\sim}{\chi}$ attributed to each material point. The microdeformation accounts for the rotation and distortion of a triad associated with the underlying microstructure [6]. The microdeformation field is generally not compatible. The microdeformation can be split into its symmetric and skew–symmetric parts:

$$\underset{\sim}{\chi} = \underset{\sim}{\chi}^s + \underset{\sim}{\chi}^a \tag{1}$$

that are called respectively the microstrain and the Cosserat rotation. The associated deformation fields are the classical strain tensor $\underset{\sim}{\varepsilon}$, the relative deformation $\underset{\sim}{e}$, and the microdeformation gradient tensor $\underset{\approx}{\kappa}$ defined by:

$$\underset{\sim}{\varepsilon} = \underline{u} \overset{s}{\otimes} \nabla, \quad \underset{\sim}{e} = \underline{u} \otimes \nabla - \underset{\sim}{\chi}, \quad \underset{\approx}{\kappa} = \underset{\sim}{\chi} \otimes \nabla \tag{2}$$

The symmetric part of $\underset{\sim}{e}$ corresponds to the difference of material strain and microstrain, whereas its skew–symmetric part accounts for the relative rotation of the material with respect to microstructure. The analysis is restricted to small deformations, small micro-rotations, small microstrains, and small microdeformation gradients. The microdeformation gradient can be split into two contributions:

$$\underset{\approx}{\kappa} = \underset{\approx}{\kappa}^s + \underset{\approx}{\kappa}^a, \quad \text{with} \quad \underset{\approx}{\kappa}^s = \underset{\sim}{\chi}^s \otimes \nabla, \underset{\approx}{\kappa}^a = \underset{\sim}{\chi}^a \otimes \nabla \tag{3}$$

The statics of the micromorphic continuum is described by the symmetric simple stress tensor $\underset{\sim}{\sigma}$, the generally non-symmetric relative force–stress tensor $\underset{\sim}{s}$, and the third-rank double stress tensor $\underset{\approx}{m}$. These tensors must fulfill the local form of the balance equations in the static case, in the absence of body simple nor double forces for simplicity:

$$(\underset{\sim}{\sigma} + \underset{\sim}{s}) \cdot \nabla = 0, \quad \underset{\approx}{m} \cdot \nabla + \underset{\sim}{s} = 0 \quad \text{on } \Omega \tag{4}$$

The constitutive equations for linear elastic centrosymmetric micromorphic materials read

$$\underset{\sim}{\sigma} = \underset{\approx}{a} : \underset{\sim}{\varepsilon}, \qquad \underset{\sim}{s} = \underset{\approx}{b} : \underset{\sim}{e}, \qquad \underset{\approx}{m} = \underset{\approx}{\underset{\approx}{c}} \,\vdots\, \underset{\approx}{\kappa} \tag{5}$$

The elasticity tensors display the major symmetries:

$$a_{ijkl} = a_{klij}, \qquad b_{ijkl} = b_{klij}, \quad c_{ijkpqr} = c_{pqrijk} \tag{6}$$

and $\underset{\approx}{a}$ has also the usual minor symmetries. The last constitutive law can be written in the form

$$\underset{\sim}{m} = \underset{\approx}{c}^s \vdots \underset{\sim}{\kappa}^s + \underset{\approx}{c}^a \vdots \underset{\sim}{\kappa}^a \tag{7}$$

For the sake of simplicity, the tensors $\underset{\approx}{c}^s$ and $\underset{\approx}{c}^a$ are supposed to fulfill the following conditions:

$$c^s_{ijkpqr} = c^s_{jikpqr}, c^a_{ijkpqr} = -c^a_{jikpqr} \tag{8}$$

thus assuming that there is no coupling between the contributions of the symmetric and skew–symmetric parts of $\underset{\sim}{\kappa}$ to the third-rank stress tensor.

The setting of the boundary value problem on body Ω is then closed by the boundary conditions. In the following, Dirichlet boundary conditions are considered of the form

$$\underline{u}(\underline{x}) = 0, \quad \underset{\sim}{\chi}(\underline{x}) = 0, \quad \forall \underline{x} \in \partial\Omega \tag{9}$$

where $\partial\Omega$ denotes the boundary of Ω. The equations (2), (4), (5), and (9) define the boundary value problem $\mathcal{P}$.

Multiscale Asymptotic Expansion Method

The multiscale asymptotic expansion method is exposed in details in the case of heterogeneous micromorphic media so that the reader will be in the position of applying it readily to other similar situations.

The heterogeneous material under study is a mixture of micromorphic constituents, i.e., a heterogeneous micromorphic medium. One investigates the nature of the resulting homogeneous equivalent medium by means of asymptotic methods. The multiscale asymptotic method from [15] is especially adequate for this purpose since the nature of the effective medium is not postulated a priori but rather is the result of the analysis. The microstructure of the material is assumed to be periodic. The heterogeneous material is then obtained by space tessellation with cells translated from a single cell Y^l. The period of the microstructure is described by three dimensionless independent vectors $(\underline{a}_1, \underline{a}_2, \underline{a}_3)$ such that

$$Y^l = \left\{ \underline{x} = x_i \underline{a}_i, |x_i| < \frac{l}{2} \right\}$$

where l is the characteristic size of the cell. We call $\underset{\approx}{a}^l$, $\underset{\approx}{b}^l$ and $\underset{\approx}{c}^l$ the elasticity tensor fields of the periodic micromorphic material. They are such that

$$\begin{aligned}
&\forall \underline{x} \in \Omega, \forall (n_1, n_2, n_3) \in Z^3 / \underline{x} \\
&\qquad + l(n_1 \underline{a}_1 + n_2 \underline{a}_2 + n_3 \underline{a}_3) \in \Omega \\
&\underset{\approx}{a}^l(\underline{x}) = \underset{\approx}{a}^l(\underline{x} + l(n_1 \underline{a}_1 + n_2 \underline{a}_2 + n_3 \underline{a}_3)) \\
&\underset{\approx}{b}^l(\underline{x}) = \underset{\approx}{b}^l(\underline{x} + l(n_1 \underline{b}_1 + n_2 \underline{b}_2 + n_3 \underline{b}_3)) \\
&\underset{\approx}{c}^l(\underline{x}) = \underset{\approx}{c}^l(\underline{x} + l(n_1 \underline{a}_1 + n_2 \underline{a}_2 + n_3 \underline{a}_3))
\end{aligned}$$

Dimensional Analysis

The first step of a multiscale expansion analysis is the dimensional analysis which is necessary to identify the small parameters of the problem. The size L of body Ω is defined for instance as the maximum distance between two points. Dimensionless coordinates and displacements are introduced:

$$\underline{x}^* = \frac{\underline{x}}{L}, \underline{u}^*(\underline{x}^*) = \frac{\underline{u}(\underline{x})}{L}, \underset{\sim}{\chi}^*(\underline{x}^*) = \underset{\sim}{\chi}(\underline{x}) \tag{10}$$

The corresponding strain measures are

$$\begin{aligned}
&\underset{\sim}{\varepsilon}^*(\underline{x}^*) = \underline{u}^* \overset{s}{\otimes} \nabla^* = \underset{\sim}{\varepsilon}(\underline{x}), \\
&\underset{\sim}{e}^*(\underline{x}^*) = \underline{u}^* \otimes \nabla^* - \underset{\sim}{\chi}^* = \underset{\sim}{e}(\underline{x})
\end{aligned} \tag{11}$$

$$\underset{\sim}{\kappa}^*(\underline{x}^*) = \underset{\sim}{\chi}^* \otimes \nabla^* = L\, \underset{\sim}{\kappa}(\underline{x}) \tag{12}$$

with $\nabla^* = \left(\frac{\partial \cdot}{\partial x_i^*}\right) \underline{e}_i = L\nabla$. Similarly,

$$\begin{aligned} \underset{\sim}{\boldsymbol{\kappa}}^{s*}(\underline{\boldsymbol{x}}^*) &= \underset{\sim}{\boldsymbol{\chi}}^{s*} \otimes \nabla^* = L\, \underset{\sim}{\boldsymbol{\kappa}}^{s}(\underline{\boldsymbol{x}}) \\ \underset{\sim}{\boldsymbol{\kappa}}^{a*}(\underline{\boldsymbol{x}}^*) &= \underset{\sim}{\boldsymbol{\chi}}^{a*} \otimes \nabla^* = L\, \boldsymbol{\kappa}^{a}(\underline{\boldsymbol{x}}) \end{aligned} \qquad (13)$$

It is necessary to introduce next a norm of the elasticity tensors:

$$A = \underset{\underline{\boldsymbol{x}}\in Y^l}{\mathrm{Max}} \left(\left| a^l_{ijkl}(\underline{\boldsymbol{x}}) \right|, \left| b^l_{ijkl}(\underline{\boldsymbol{x}}) \right| \right)$$

$$C^s = \underset{\underline{\boldsymbol{x}}\in Y^l}{\mathrm{Max}} \left| c^{sl}_{ijkpqr}(\underline{\boldsymbol{x}}) \right|$$

$$C^a = \underset{\underline{\boldsymbol{x}}\in Y^l}{\mathrm{Max}} \left| c^{al}_{ijkpqr}(\underline{\boldsymbol{x}}) \right|$$

whereby characteristic lengths l_s and l_a can be defined as $C^s = Al_s^2, C^a = Al_a^2$.

The definition of dimensionless stress and elasticity tensors is as follows:

$$\begin{aligned} &\underset{\sim}{\boldsymbol{\sigma}}^*(\underline{x}^*) = A^{-1}\underset{\sim}{\boldsymbol{\sigma}}(\underline{\boldsymbol{x}}), \quad \boldsymbol{s}^*(\underline{\boldsymbol{x}}^*) = A^{-1}\boldsymbol{s}(\underline{\boldsymbol{x}}), \\ &\underset{\sim}{\boldsymbol{m}}^*(\underline{\boldsymbol{x}}^*) = (AL)^{-1}\underset{\sim}{\boldsymbol{m}}(\underline{\boldsymbol{x}}) \\ &\underset{\approx}{\boldsymbol{a}}^*(\underline{\boldsymbol{x}}^*) = A^{-1}\underset{\approx}{\boldsymbol{a}}^l(\underline{x}), \; \underset{\approx}{\boldsymbol{b}}^*(\underline{x}^*) = A^{-1}\underset{\approx}{\boldsymbol{b}}^l(\underline{\boldsymbol{x}}), \\ &\underset{\approx}{\boldsymbol{c}}^{s*}(\underline{\boldsymbol{x}}^*) = (Al_s^2)^{-1}\underset{\approx}{\boldsymbol{c}}^{sl}(\underline{x}), \quad \underset{\approx}{\boldsymbol{c}}^{a*}(\underline{\boldsymbol{x}}^*) = (Al_a^2)^{-1}\underset{\approx}{\boldsymbol{c}}^{al}(\underline{\boldsymbol{x}}) \end{aligned}$$

Since the initial tensors $\underset{\approx}{\boldsymbol{a}}^l, \underset{\approx}{\boldsymbol{b}}^l$ and $\underset{\approx}{\boldsymbol{c}}^l$ are Y^l-periodic, the dimensionless counterparts are Y^*-periodic:

$$Y^* = \frac{1}{l}Y, Y = \left\{ \underline{\boldsymbol{y}} = y_i\underline{\boldsymbol{a}}_i, \; |y_i| < \frac{1}{2} \right\} \qquad (14)$$

Y is the (dimensionless) unit cell used in the following asymptotic analyses. As a result, the dimensionless stress and strain tensors are related by the following constitutive equations:

$$\begin{aligned} &\underset{\sim}{\boldsymbol{\sigma}}^* = \underset{\approx}{\boldsymbol{a}}^* : \underset{\sim}{\boldsymbol{\varepsilon}}^*, \quad \underset{\sim}{\boldsymbol{s}}^* = \underset{\approx}{\boldsymbol{b}}^* : \underset{\sim}{\boldsymbol{e}}^*, \\ &\underset{\sim}{\boldsymbol{m}}^* = \left(\frac{l_s}{L}\right)^2 \underset{\approx}{\boldsymbol{c}}^{s*} \vdots \underset{\sim}{\boldsymbol{\kappa}}^{s*} + \left(\frac{l_a}{L}\right)^2 \underset{\approx}{\boldsymbol{c}}^{a*} \vdots \underset{\sim}{\boldsymbol{\kappa}}^{a*} \end{aligned} \qquad (15)$$

The dimensionless balance equations read

$$\forall \underline{\boldsymbol{x}}^* \in \Omega^*, (\underset{\sim}{\boldsymbol{\sigma}}^* + \underset{\sim}{\boldsymbol{s}}^*)\cdot\nabla^* = 0, \underset{\sim}{\boldsymbol{m}}^*\cdot\nabla^* + \underset{\sim}{\boldsymbol{s}}^* = 0 \qquad (16)$$

A boundary value problem $\mathcal{P}^*$ can be defined using equations (12), (15), and (16), complemented by the boundary conditions:

$$\forall \underline{\boldsymbol{x}}^* \in \partial\Omega^*, \underline{\boldsymbol{u}}^*(\underline{\boldsymbol{x}}^*) = 0, \underset{\sim}{\boldsymbol{\chi}}^*(\underline{\boldsymbol{x}}^*) = 0 \qquad (17)$$

The Homogenization Problem

The boundary value problem $\mathcal{P}^*$ is treated here as an element of a series of problems $(\mathcal{P}_{\boldsymbol{\varepsilon}})_{\boldsymbol{\varepsilon}>0}$ on Ω^*. The homogenization problem consists in the determination of the limit of this series when the dimensionless parameter $\boldsymbol{\epsilon}$, regarded as small, tends towards 0. The series is chosen such that

$$\mathcal{P}_{\boldsymbol{\epsilon}=\frac{l}{L}} = \mathcal{P}^*$$

The unknowns of boundary value problem $\mathcal{P}_{\boldsymbol{\varepsilon}}$ are the displacement and microdeformation fields $\underline{\boldsymbol{u}}^{\boldsymbol{\epsilon}}$ and $\underset{\sim}{\boldsymbol{\chi}}^{\boldsymbol{\epsilon}}$ satisfying the following field equations on Ω^*:

$$\begin{aligned} &\underset{\sim}{\boldsymbol{\sigma}}^{\boldsymbol{\epsilon}} = \underset{\approx}{\boldsymbol{a}}^{\boldsymbol{\epsilon}} : (\underline{\boldsymbol{u}}^{\boldsymbol{\epsilon}} \overset{s}{\otimes} \nabla^*), \quad \underset{\sim}{\boldsymbol{s}}^{\boldsymbol{\epsilon}} = \underset{\approx}{\boldsymbol{b}}^{\boldsymbol{\epsilon}} : (\underline{\boldsymbol{u}}^{\boldsymbol{\epsilon}} \otimes \nabla^* - \underset{\sim}{\boldsymbol{\chi}}^{\boldsymbol{\epsilon}}), \\ &\underset{\sim}{\boldsymbol{m}}^{\boldsymbol{\epsilon}} = \underset{\approx}{\boldsymbol{c}}^{\boldsymbol{\epsilon}} \vdots (\underset{\sim}{\boldsymbol{\chi}}^{\boldsymbol{\epsilon}} \otimes \nabla^*) \end{aligned} \qquad (18)$$

$$(\underset{\sim}{\boldsymbol{\sigma}}^{\boldsymbol{\epsilon}} + \underset{\sim}{\boldsymbol{s}}^{\boldsymbol{\epsilon}})\cdot\nabla^* = 0, \quad \underset{\sim}{\boldsymbol{m}}^{\boldsymbol{\epsilon}}\cdot\nabla^* + \underset{\sim}{\boldsymbol{s}}^{\boldsymbol{\epsilon}} = 0 \qquad (19)$$

Different cases must now be distinguished depending on the relative position of the constitutive lengths l_s and l_a with respect to the characteristic lengths l and L of the problem. Four special cases can be distinguished for the present asymptotic analysis. The first case corresponds to a limiting process for which l_s/l and l_a/l remain constant when l/L goes to zero. The second case corresponds to the situation for which l_s/L and l_a/L remain constant when l/L goes to zero. The third (resp. fourth) situation assumes that l_s/l and l_a/L (resp. l_s/L and l_a/l) remain constant when l/L goes to zero. These assumptions lead to four different homogenization schemes labeled $HS1$ to $HS4$ in the sequel. The homogenization scheme 1 (resp. 2) will be relevant when the ratio l/L is small enough and when l_s, l_a and l (resp. L) have the same order of magnitude.

Accordingly, the following tensors of elastic moduli are defined:

$$\underset{\approx}{a}^{(0)}(\underline{y}) = \underset{\approx}{a}^*\left(\tfrac{l}{L}\underline{y}\right), \quad \underset{\approx}{b}^{(0)}(\underline{y}) = \underset{\approx}{b}^*\left(\tfrac{l}{L}\underline{y}\right) \quad (20)$$

$$\underset{\approx}{c}^{(1)}(\underline{y}) = \left(\tfrac{l_s}{l}\right)^2 \underset{\approx}{c}^*\left(\tfrac{l}{L}\underline{y}\right), \quad \underset{\approx}{c}^{(2)}(\underline{y}) = \left(\tfrac{l_s}{L}\right)^2 \underset{\approx}{c}^*\left(\tfrac{l}{L}\underline{y}\right) \quad (21)$$

$$\underset{\approx}{c}^{s(1)}(\underline{y}) = \left(\tfrac{l_s}{l}\right)^2 \underset{\approx}{c}^{s*}\left(\tfrac{l}{L}\underline{y}\right), \quad \underset{\approx}{c}^{a(1)}(\underline{y}) = \left(\tfrac{l_a}{l}\right)^2 \underset{\approx}{c}^{a*}\left(\tfrac{l}{L}\underline{y}\right) \quad (22)$$

$$\underset{\approx}{c}^{s(2)}(\underline{y}) = \left(\tfrac{l_s}{L}\right)^2 \underset{\approx}{c}^{s*}\left(\tfrac{l}{L}\underline{y}\right), \quad \underset{\approx}{c}^{a(2)}(\underline{y}) = \left(\tfrac{l_a}{L}\right)^2 \underset{\approx}{c}^{a*}\left(\tfrac{l}{L}\underline{y}\right) \quad (23)$$

They are Y-periodic since $\underset{\approx}{a}^*, \underset{\approx}{b}^*$ and $\underset{\approx}{c}^*$ are Y^*-periodic. Four different hypotheses will be made concerning the constitutive tensors of problem $\mathcal{P}_{\varepsilon}$:

$$\begin{aligned}
\textit{Assumption}\ 1 :\ & \underset{\approx}{a}^{\epsilon}(\underline{x}^*) = \underset{\approx}{a}^{(0)}(\epsilon^{-1}\underline{x}^*) \\
& \underset{\approx}{b}^{\epsilon}(\underline{x}^*) = \underset{\approx}{b}^{(0)}(\epsilon^{-1}\underline{x}^*) \text{ and} \\
& \underset{\approx}{c}^{\epsilon}(\underline{x}^*) = \epsilon^2 \underset{\approx}{c}^{(1)}(\epsilon^{-1}\underline{x}^*) \\
\textit{Assumption}\ 2 :\ & \underset{\approx}{a}^{\epsilon}(\underline{x}^*) = \underset{\approx}{a}^{(0)}(\epsilon^{-1}\underline{x}^*) \\
& \underset{\approx}{b}^{\epsilon}(\underline{x}^*) = \underset{\approx}{b}^{(0)}(\epsilon^{-1}\underline{x}^*) \text{ and} \\
& \underset{\approx}{c}^{\epsilon}(\underline{x}^*) = \underset{\approx}{c}^{(2)}(\epsilon^{-1}\underline{x}^*) \\
\textit{Assumption}\ 3 :\ & \underset{\approx}{a}^{\epsilon}(\underline{x}^*) = \underset{\approx}{a}^{(0)}(\epsilon^{-1}\underline{x}^*) \\
& \underset{\approx}{b}^{\epsilon}(\underline{x}^*) = \underset{\approx}{b}^{(0)}(\epsilon^{-1}\underline{x}^*) \text{ and} \\
& \underset{\approx}{c}^{s\epsilon}(\underline{x}^*) = \epsilon^2 \underset{\approx}{c}^{s(1)}(\epsilon^{-1}\underline{x}^*) \\
& \underset{\approx}{c}^{a\epsilon}(\underline{x}^*) = \underset{\approx}{c}^{a(2)}(\epsilon^{-1}\underline{x}^*) \\
\textit{Assumption}\ 4 :\ & \underset{\approx}{a}^{\epsilon}(\underline{x}^*) = \underset{\approx}{a}^{(0)}(\epsilon^{-1}\underline{x}^*) \\
& \boldsymbol{b}^{\epsilon}(\underline{x}^*) = \underset{\approx}{b}^{(0)}(\epsilon^{-1}\underline{x}^*) \text{ and} \\
& \underset{\approx}{c}^{s\epsilon}(\underline{x}^*) = \underset{\approx}{c}^{s(2)}(\epsilon^{-1}\underline{x}^*) \\
& \underset{\approx}{c}^{a\epsilon}(\underline{x}^*) = \epsilon^2 \underset{\approx}{c}^{a(1)}(\epsilon^{-1}\underline{x}^*)
\end{aligned}$$

Assumptions 1 and 2 respectively correspond to the homogenization schemes HS1 and HS2. Both choices meet the requirement that

$$\left(\epsilon = \frac{l}{L}\right) \Rightarrow \left(\underset{\approx}{a}^{\epsilon} = \underset{\approx}{a}^* \quad \text{and} \quad \underset{\approx}{c}^{\epsilon} = \left(\frac{l}{L}\right)^2 \underset{\approx}{c}^*\right)$$

Assumptions 3 and 4 respectively correspond to the homogenization schemes HS3 and HS4. Both choices meet the requirement that

$$\left(\epsilon = \frac{l}{L}\right) \Rightarrow \left(\underset{\approx}{a}^{\epsilon} = \underset{\approx}{a}^*, \quad \underset{\approx}{c}^{s\epsilon} = \left(\frac{l_s}{L}\right)^2 \underset{\approx}{c}^{s*} \quad \text{and} \quad \underset{\approx}{c}^{a\epsilon} = \left(\frac{l_a}{L}\right)^2 \underset{\approx}{c}^{a*}\right)$$

It must be noted that, in our presentation of the asymptotic analysis, the lengths l, l_s, l_a and L are given and fixed, whereas parameter ε is allowed to tend to zero in the limiting process. In the sequel, the stars $*$ are dropped for conciseness.

Multiscale Asymptotic Expansion of the Fields

In the setting of the homogenization problems, two space variables have been distinguished: $\underline{x}$ describes the macroscopic scale and $\underline{y}$ is the local variable in the unit Y. According to the method of multiscale asymptotic developments, all fields are regarded as functions of both variables $\underline{x}$ and $\underline{y}$. It is assumed that they can be expanded in a series of powers of small parameter ε. In particular, the displacement, microdeformation, and simple and double stress fields are supposed to take the form

$$\begin{aligned}
\underline{u}^{\epsilon}(\underline{x}) &= \underline{u}_0(\underline{x},\underline{y}) + \epsilon \underline{u}_1(\underline{x},\underline{y}) + \epsilon^2 \underline{u}_2(\underline{x},\underline{y}) + \dots \\
\underset{\sim}{\mathcal{X}}^{\epsilon}(\underline{x}) &= \underset{\sim}{\mathcal{X}}_1(\underline{x},\underline{y}) + \epsilon \underset{\sim}{\mathcal{X}}_2(\underline{x},\underline{y}) + \epsilon \underset{\sim}{\mathcal{X}}_3(\underline{x},\underline{y}) + \dots \\
\underset{\sim}{\sigma}^{\epsilon}(\underline{x}) &= \underset{\sim}{\sigma}_0(\underline{x},\underline{y}) + \epsilon \underset{\sim}{\sigma}_1(\underline{x},\underline{y}) + \epsilon \underset{\sim}{\sigma}_2(\underline{x},\underline{y}) + \dots \\
\underset{\sim}{s}^{\epsilon}(\underline{x}) &= \underset{\sim}{s}_0(\underline{x},\underline{y}) + \epsilon \underset{\sim}{s}_1(\underline{x},\underline{y}) + \epsilon^2 \underset{\sim}{s}_2(\underline{x},\underline{y}) + \dots \\
\underset{\approx}{m}^{\epsilon}(\underline{x}) &= \underset{\approx}{m}_0(\underline{x},\underline{y}) + \epsilon \underset{\approx}{m}_1(\underline{x},\underline{y}) + \epsilon^2 \underset{\approx}{m}_2(\underline{x},\underline{y}) + \dots
\end{aligned}$$

where the coefficients $\underline{u}_i(\underline{x},\underline{y}), \underset{\sim}{\chi}_i(\underline{x},\underline{y})$, $\underset{\sim}{\sigma}_i(\underline{x},\underline{y}), \underset{\sim}{s}_i(\underline{x},\underline{y})$ and $\underset{\approx}{m}_i(\underline{x},\underline{y})$ are assumed to have the same order of magnitude and to be Y-periodic with respect to variable $\underline{y}\left(\underline{y} = \underline{x}/\epsilon\right)$. The average operator over the unit cell Y is denoted by

$$\langle\cdot\rangle = \frac{1}{|Y|}\int_Y \cdot dV$$

As a result,

$$<\underline{\boldsymbol{u}}^{\epsilon}> = \underline{\boldsymbol{U}}_0 + \epsilon \underline{\boldsymbol{U}}_1 + \dots \quad \text{and} \quad <\underset{\approx}{\boldsymbol{\chi}}^{\epsilon}> = \epsilon \underset{\approx}{\Xi}_2 + \dots \tag{24}$$

where $\underline{\boldsymbol{U}}_i = <\underline{\boldsymbol{u}}_i>$ and $\underset{\approx}{\Xi}_i = <\underset{\approx}{\boldsymbol{\chi}}_i>$. The gradient operator can be split into partial derivatives with respect to $\underline{\boldsymbol{x}}$ and $\underline{\boldsymbol{y}}$:

$$\nabla = \nabla_x + \frac{1}{\epsilon}\nabla_y \tag{25}$$

This operator is used to compute the strain measures and balance equations:

$$\begin{aligned}
\underset{\approx}{\boldsymbol{\varepsilon}}^{\epsilon} &= \epsilon^{-1}\underset{\approx}{\boldsymbol{\varepsilon}}_{-1} + \underset{\approx}{\boldsymbol{\varepsilon}}_0 + \epsilon^1 \underset{\approx}{\boldsymbol{\varepsilon}}_1 + \dots \\
&= \epsilon^{-1}\underline{\boldsymbol{u}}_0 \overset{s}{\otimes} \nabla_y + (\underline{\boldsymbol{u}}_0 \overset{s}{\otimes} \nabla_x + \underline{\boldsymbol{u}}_1 \overset{s}{\otimes} \nabla_y) \\
&\quad + \epsilon(\underline{\boldsymbol{u}}_1 \overset{s}{\otimes} \nabla_x + \underline{\boldsymbol{u}}_2 \overset{s}{\otimes} \nabla_y) + \dots \\
\underset{\approx}{e}^{\epsilon} &= \epsilon^{-1}\underset{\approx}{e}_{-1} + \underset{\approx}{e}_0 + \epsilon^1 \underset{\approx}{e}_{-1} + \dots \\
&= \epsilon^{-1}\underline{\boldsymbol{u}}_0 \otimes \nabla_y + (\underline{\boldsymbol{u}}_0 \otimes \nabla_x + \underline{\boldsymbol{u}}_1 \otimes \nabla_y - \underset{\approx}{\boldsymbol{\chi}}_1) \\
&\quad + \epsilon(\underline{\boldsymbol{u}}_1 \otimes \nabla_x + \underline{\boldsymbol{u}}_2 \otimes \nabla_y - \underset{\approx}{\boldsymbol{\chi}}_2) + \dots \\
\underset{\approx}{K}^{\epsilon} &= \epsilon^{-1}\underset{\approx}{\boldsymbol{\kappa}}_{-1} + \underset{\approx}{\boldsymbol{\kappa}}_0 + \epsilon^1 \underset{\approx}{\boldsymbol{\kappa}} + \dots \\
&= \epsilon^{-1}\underset{\approx}{\boldsymbol{\chi}}_1 \otimes \nabla_y + (\underset{\approx}{\boldsymbol{\chi}}_1 \otimes \nabla_x + \underset{\approx}{\boldsymbol{\chi}}_2 \otimes \nabla_y) \\
&\quad + \epsilon(\underset{\approx}{\boldsymbol{\chi}}_2 \otimes \nabla_x \underset{\approx}{\boldsymbol{\chi}}_3 \otimes \nabla_y) + \dots \\
&(\underset{\approx}{\boldsymbol{\sigma}}^{\epsilon} + \underset{\approx}{s}^{\epsilon}) \cdot \nabla_x + \epsilon^{-1}(\underset{\approx}{\boldsymbol{\sigma}}^{\epsilon} + \underset{\approx}{s}^{\epsilon}) \cdot \nabla_y = 0, \\
&\underset{\approx}{\boldsymbol{m}}^{\epsilon} \cdot \nabla_x + \epsilon^{-1}\underset{\approx}{\boldsymbol{m}}^{\epsilon} \cdot \nabla_y + \underset{\approx}{s}^{\epsilon} = 0
\end{aligned} \tag{26}$$

Similar expansions are valid for the tensors $\underset{\approx}{\boldsymbol{\kappa}}^s, \underset{\approx}{\boldsymbol{\kappa}}^a$. The expansions of the stress tensors are then introduced in the balance equations (26), and the terms can be ordered with respect to the powers of ϵ. Identifying the terms of same order, we are lead to the following set of equations:

- *order* ϵ^{-1}, $\quad (\underset{\approx}{\boldsymbol{\sigma}}_0 + \underset{\approx}{s}_0) \cdot \nabla_y = 0$ and $\underset{\approx}{\boldsymbol{m}}_0 \cdot \nabla_y = 0$
- *order* ϵ^{0}, $\quad (\underset{\approx}{\boldsymbol{\sigma}}_0 + \underset{\approx}{s}_0) \cdot \nabla_x + (\underset{\approx}{\boldsymbol{\sigma}}_1 + \underset{\approx}{s}_1) \cdot \nabla_y = 0$ and
  $\underset{\approx}{\boldsymbol{S}}_0 \cdot \nabla_x + \underset{\approx}{\boldsymbol{S}}_1 \cdot \nabla_y + \underset{\approx}{s}_1 = 0$

The effective balance equations follow from the first above equation by averaging over the unit cell Y and, at the order ϵ^0, one gets

$$(\underset{\approx}{\boldsymbol{\Sigma}}_0 + \underset{\approx}{\boldsymbol{S}}_0) \cdot \nabla = 0 \qquad \text{and} \qquad \underset{\approx}{\boldsymbol{M}}_0 \cdot \nabla + \underset{\approx}{\boldsymbol{S}}_0 = 0 \tag{27}$$

where effective stress tensors are defined as the following averages $\underset{\approx}{\boldsymbol{\Sigma}}_0 = <\underset{\approx}{\boldsymbol{\sigma}}_0>, \underset{\approx}{\boldsymbol{S}}_0 = <\underset{\approx}{s}_0>$ and $\underset{\approx}{\boldsymbol{M}}_0 = <\underset{\approx}{\boldsymbol{m}}_0>$.

Homogenization Scheme HS1

For the first homogenization scheme HS1 previously defined, the equations describing the local behavior are

$$\begin{aligned}
&\underset{\approx}{\boldsymbol{\sigma}}^{\epsilon} = \underset{\approx}{\boldsymbol{a}}^{(0)}(\underline{\boldsymbol{y}}) : \underset{\approx}{\boldsymbol{\varepsilon}}^{\epsilon}, \quad \underset{\approx}{s}^{\epsilon} = \underset{\approx}{\boldsymbol{b}}^{(0)}(\underline{\boldsymbol{y}}) : \underset{\approx}{e}^{\epsilon} \quad \text{and} \\
&\underset{\approx}{\boldsymbol{m}}^{\epsilon} = \epsilon^2 \underset{\approx}{\boldsymbol{c}}^{(1)}(\underline{\boldsymbol{y}}) \vdots \underset{\approx}{\boldsymbol{\kappa}}^{\epsilon}
\end{aligned} \tag{28}$$

At this stage, the expansion (26) can be substituted into the constitutive equations (28). Identifying the terms of same order, one gets

- *order* ϵ^{-1},

$$\begin{aligned}
\underset{\approx}{\boldsymbol{a}}^{(0)} : \underset{\approx}{\boldsymbol{\varepsilon}}_{-1} &= \underset{\approx}{\boldsymbol{a}}^{(0)} : \left(\underline{\boldsymbol{u}}_0 \overset{s}{\otimes} \nabla_y\right) = 0 \\
\underset{\approx}{\boldsymbol{b}}^{(0)} : \underset{\approx}{e}_0 &= \underset{\approx}{\boldsymbol{b}}^{(0)} : (\underline{\boldsymbol{u}}_0 \otimes \nabla_y) = 0
\end{aligned} \tag{29}$$

- *order* ϵ^{0},

$$\underset{\approx}{\boldsymbol{\sigma}}_0 = \underset{\approx}{\boldsymbol{a}}^{(0)} : \boldsymbol{\epsilon}_0, \quad \underset{\approx}{s}_0 = \underset{\approx}{\boldsymbol{b}}^{(0)} : \underset{\approx}{e}_0, \quad \underset{\approx}{\boldsymbol{m}}_0 = 0 \tag{30}$$

- *order* ϵ^{1},

$$\underset{\approx}{\boldsymbol{\sigma}}_1 = \underset{\approx}{\boldsymbol{a}}^{(0)} : \boldsymbol{\epsilon}_1, \quad \underset{\approx}{s}_1 = \underset{\approx}{\boldsymbol{b}}^{(0)} : \underset{\approx}{e}_1, \quad \underset{\approx}{\boldsymbol{m}}_1 = \underset{\approx}{\boldsymbol{c}}^{(1)} \vdots \underset{\approx}{\boldsymbol{\kappa}}_{-1} \tag{31}$$

The equation (21) implies that $\underline{\boldsymbol{u}}_0$ does not depend on the local variable $\underline{\boldsymbol{y}}$:

$$\underline{\boldsymbol{u}}_0(\underline{\boldsymbol{x}}, \underline{\boldsymbol{y}}) = \underline{\boldsymbol{U}}_0(\underline{\boldsymbol{x}})$$

At the order ϵ^0, the higher-order stress tensor vanishes, $\underset{\approx}{\boldsymbol{M}}_0 = <\underset{\approx}{\boldsymbol{m}}_0> = 0$.

Finally, the fields $(\underline{\boldsymbol{u}}_1, \underset{\approx}{\boldsymbol{\chi}}_1, \underset{\sim}{\boldsymbol{\sigma}}_0, \underset{\sim}{\boldsymbol{s}}_0, \underset{\approx}{\boldsymbol{m}}_1)$ are solutions of the following auxiliary boundary value problem defined on the unit cell:

$$\begin{cases} \underset{\sim}{\boldsymbol{\varepsilon}}_0 = \underline{\boldsymbol{U}}_0 \overset{s}{\otimes} \nabla_x + \underline{\boldsymbol{u}}_1 \overset{s}{\otimes} \nabla_y, \underset{\sim}{\boldsymbol{e}}_0 = \underline{U}_0 \otimes \nabla_x \\ \quad + \underline{\boldsymbol{u}}_1 \otimes \nabla_y - \boldsymbol{\chi}_1 \\ \underset{\approx}{\boldsymbol{\kappa}}_{-1} = \boldsymbol{\chi}_1 \otimes \nabla_y \\ \underset{\sim}{\boldsymbol{\sigma}}_0 = \underset{\approx}{\boldsymbol{a}}^{(0)} : \boldsymbol{\epsilon}_0, \quad \underset{\sim}{\boldsymbol{s}}_0 = \underset{\approx}{\boldsymbol{b}}^{(0)} : \underset{\sim}{\boldsymbol{e}}_0, \quad \boldsymbol{m}_1 = \underset{\approx}{\boldsymbol{c}}^{(1)} : \boldsymbol{\kappa}_{-1} \\ (\underset{\sim}{\boldsymbol{\sigma}}_0 + \underset{\sim}{\boldsymbol{s}}_0) \cdot \nabla_y = 0, \quad \underset{\approx}{\boldsymbol{m}}_1 \cdot \nabla_y + \underset{\sim}{\boldsymbol{s}}_0 = 0 \end{cases} \tag{32}$$

The boundary conditions of this problem are given by the periodicity requirements for the unknown fields. A series of auxiliary problems similar to (32) can be defined to obtain the solutions at higher orders. It must be noted that these problems must be solved in cascade since, for instance, the solution of (32) requires the knowledge of $\underline{\boldsymbol{U}}_0$. A particular solution $\underset{\sim}{\boldsymbol{\chi}}$ for a vanishing prescribed $\underline{\boldsymbol{U}}_0 \overset{s}{\otimes} \nabla_x$ is $\underset{\sim}{\boldsymbol{\chi}} = \underline{\boldsymbol{U}}_0 \overset{a}{\otimes} \nabla_x$. It follows that the solution $(\underline{\boldsymbol{u}}_1, \underline{\boldsymbol{U}}_0 \overset{a}{\otimes} \nabla_x - \underset{\sim}{\boldsymbol{\chi}}_1)$ to problem (32) depends linearly on $\underline{\boldsymbol{U}}_0 \overset{s}{\otimes} \nabla_x$, up to a translation term, so that

$$\underline{\boldsymbol{u}}^\epsilon = \underline{\boldsymbol{U}}_0(\underline{x}) + \epsilon(\underline{\boldsymbol{U}}_1(\underline{x}) + \underset{\approx}{\boldsymbol{\chi}}_u^{(1)}(\underline{y}) : (\underline{\boldsymbol{U}}_0 \overset{s}{\otimes} \nabla)) + \ldots \tag{33}$$

$$\underset{\sim}{\boldsymbol{\chi}}^\epsilon = \underline{\boldsymbol{U}}_0 \overset{a}{\otimes} \nabla_x + \underset{\approx}{\boldsymbol{\chi}}_{\boldsymbol{\chi}}^{(1)}(\underline{y}) : \underline{\boldsymbol{U}}_0 \overset{s}{\otimes} \nabla + \ldots \tag{34}$$

where concentration tensors $\underset{\approx}{\boldsymbol{X}}_u^{(1)}$ and $\underset{\approx}{\boldsymbol{X}}_{\boldsymbol{\chi}}^{(1)}$ have been introduced, the components of which are determined by the successive solutions of the auxiliary problem for unit values of the components of $\underline{\boldsymbol{U}}_0 \overset{s}{\otimes} \nabla$. Concentration tensor $\underset{\approx}{\boldsymbol{X}}_u^{(1)}$ is such that its mean value over the unit cell vanishes.

The macroscopic stress tensor is given by

$$\underset{\sim}{\boldsymbol{\Sigma}}_0 = <\underset{\sim}{\boldsymbol{\sigma}}_0> = <\underset{\approx}{\boldsymbol{a}}^{(0)} : (\underset{\approx}{\boldsymbol{1}} + \nabla_x \overset{s}{\otimes} \underset{\approx}{\boldsymbol{X}}_u^{(1)})> : (\underline{\boldsymbol{U}}_0 \overset{s}{\otimes} \nabla) = \underset{\approx}{\boldsymbol{A}}_0^{(1)} : (\underline{\boldsymbol{U}}_0 \overset{s}{\otimes} \nabla) \tag{35}$$

Accordingly, the tensor of effective moduli possesses all symmetries of classical elastic moduli for a Cauchy medium: $A^{(1)}_{0ijkl} = A^{(1)}_{0klij} = A^{(1)}_{0jikl} = A^{(1)}_{0ijlk}$.

The additional second-rank stress tensor can be shown to vanish:

$$\underset{\sim}{\boldsymbol{S}}_0 = <\underset{\sim}{\boldsymbol{s}}_0> = < -\underset{\approx}{\boldsymbol{m}}_1 \cdot \nabla_y> = 0 \tag{36}$$

The effective medium is therefore governed by the single equation:

$$\underset{\sim}{\boldsymbol{\Sigma}}_0 \cdot \nabla = 0 \tag{37}$$

The effective medium turns out to be a Cauchy continuum with symmetric stress tensor.

Homogenization Scheme HS2

For the second homogenization scheme HS2, the equations describing the local behavior are

$$\underset{\sim}{\boldsymbol{\sigma}}^\epsilon = \underset{\approx}{\boldsymbol{a}}^{(0)}(\underline{\boldsymbol{y}}) : \underset{\sim}{\varepsilon}^\epsilon, \quad \underset{\sim}{\boldsymbol{s}}^\epsilon = \underset{\approx}{\boldsymbol{b}}^{(0)}(\underline{\boldsymbol{y}}) : \underset{\sim}{\boldsymbol{e}}^\epsilon, \quad \text{and} \quad \underset{\approx}{\boldsymbol{m}} = \underset{\approx}{\boldsymbol{c}}^{(2)}(\underline{\boldsymbol{y}}) \vdots \underset{\approx}{\boldsymbol{\kappa}}^\epsilon \tag{38}$$

The different steps of the asymptotic analysis are the same as in the previous section for HS1. We will only focus here on the main results. At the order ϵ^{-1}, one gets

$$\underset{\approx}{\boldsymbol{a}}^{(0)} : \underset{\sim}{\boldsymbol{\epsilon}}_{-1} = 0, \quad \underset{\approx}{\boldsymbol{b}}^{(0)} : \underset{\sim}{\boldsymbol{e}}_{-1} = 0, \quad \underset{\approx}{\boldsymbol{c}}^{(2)} \vdots \underset{\approx}{\boldsymbol{\kappa}}_{-1} = 0 \tag{39}$$

This implies that the gradients of $\underline{\boldsymbol{u}}_0$ and $\underset{\sim}{\boldsymbol{\chi}}_1$ with respect to $\underline{\boldsymbol{y}}$ vanish, so that

$$\underline{\boldsymbol{u}}_0(\underline{x}, \underline{y}) = \underline{\boldsymbol{U}}_0(\underline{x}), \qquad \underset{\sim}{\boldsymbol{\chi}}_1(\underline{x}, \underline{y}) = \underset{\sim}{\Xi}_1(\underline{x}) \tag{40}$$

The fields $(\underline{u}_1, \underset{\sim}{\boldsymbol{\chi}}_1, \underset{\sim}{\sigma}_0, \underset{\approx}{m}_0)$ are solutions of the two following auxiliary boundary value problems defined on the unit cell:

$$\begin{cases} \underset{\sim}{\varepsilon}_0 = \underline{U}_0 \overset{s}{\otimes} \nabla_x + \underline{u}_1 \overset{s}{\otimes} \nabla_y \\ \underset{\sim}{e}_0 = \underline{U}_0 \otimes \nabla_x + \underline{u}_1 \otimes \nabla_y - \underset{\sim}{\Xi}_1 \\ \underset{\sim}{\sigma}_0 = \underset{\approx}{a}^{(0)} : \underset{\sim}{\epsilon}_0, \quad \underset{\sim}{s}_0 = \underset{\approx}{b}^{(0)} : \underset{\sim}{e}_0 \\ (\underset{\sim}{\sigma}_0 + \underset{\sim}{s}_0) \cdot \nabla_y = 0 \end{cases}$$

$$\begin{cases} \underset{\approx}{\kappa}_0 = \underset{\sim}{\Xi}_1 \otimes \nabla_x + \underset{\sim}{\chi}_2 \nabla_y \\ \underset{\approx}{m}_0 = \underset{\approx}{c}^{(2)} \vdots \underset{\approx}{\kappa}_0, \quad \underset{\approx}{m}_0 \cdot \nabla_y = 0 \end{cases}$$

We are therefore left with two decoupled boundary value problems: the first one with main unknown $\underline{u}_1$ depends linearly on $\underline{U}_0 \overset{s}{\otimes} \nabla_x$ and $\underline{U}_0 \otimes \nabla_x - \underset{\sim}{\Xi}_1$, whereas the second one with unknown $\underset{\sim}{\chi}_2$ is linear in $\underset{\sim}{\Xi}_1 \otimes \nabla_x$. The solutions take the form

$$\begin{aligned} \underline{u}^\epsilon &= \underline{U}_0(\underline{x}) + \epsilon(\underline{U}_1(\underline{x}) + \underset{\approx}{X}_u^{(2)}(\underline{y}) : (\underline{U}_0 \overset{s}{\otimes} \nabla) \\ &\quad + \underset{\approx}{X}_\epsilon^{(2)}(\underline{y}) : (\underline{U}_0 \otimes \nabla - \underset{\sim}{\Xi}_1)) + \dots, \\ \underset{\sim}{\chi}^\epsilon &= \underset{\sim}{\Xi}_1(\underline{x}) + \epsilon(\underset{\sim}{\Xi}_2(\underline{x}) + \underset{\approx}{X}_\kappa^{(2)}(\underline{y}) \vdots (\underset{\sim}{\Xi}_1 \otimes \nabla)) + \dots \end{aligned} \quad (41)$$

where concentration tensors $\underset{\approx}{X}_u^{(2)}, \underset{\approx}{X}_e^{(2)}$ and $\underset{\approx}{X}_\kappa^{(2)}$ have been introduced. Their components are determined by the successive solutions of the auxiliary problem for unit values of the components of $\underline{U}_0 \overset{s}{\otimes} \nabla, \underline{U}_0 \otimes \nabla - \underset{\sim}{\Xi}_1$ and $\underset{\sim}{\Xi}_1 \otimes \nabla_y$. They are such that their mean value over the unit cell vanishes.

The macroscopic stress tensors and effective elastic properties are given by

$$\begin{aligned} \underset{\sim}{\Sigma}_0 &= <\underset{\approx}{a}^{(0)} : (\underset{\approx}{1} + \nabla_y \overset{s}{\otimes} \underset{\approx}{X}_u^{(2)})> : (\underline{U}_0 \overset{s}{\otimes} \nabla) \\ &\quad + <\underset{\approx}{a}^{(0)} : (\nabla_y \overset{s}{\otimes} \underset{\approx}{X}_e^{(2)})> : (\underline{U}_0 \otimes \nabla - \underset{\sim}{\Xi}_1) \\ \underset{\sim}{S}_0 &= <\underset{\sim}{s}_0> = <\underset{\approx}{b}^{(0)} : (\nabla_y \otimes \underset{\approx}{X}_u^{(2)})> : (\underline{U}_0 \overset{s}{\otimes} \nabla) \\ &\quad + <\underset{\approx}{b}^{(0)} : (\nabla_y \otimes \underset{\approx}{X}_u^{(2)})> : (\underline{U}_0 \otimes \nabla - \underset{\sim}{\Xi}_1) \\ \underset{\approx}{M}_0 &= <\underset{\approx}{m}_0> = <\underset{\approx}{c}^{(2)} \vdots \left(\underset{\approx}{1} + \nabla_y \otimes \underset{\approx}{X}_\kappa^{(2)}\right)> \vdots \underset{\sim}{\Xi}_1 \otimes \nabla \end{aligned}$$

None of these tensors vanishes in general, which means that the effective medium is a full micromorphic continuum governed by the balance equations (27).

Homogenization Scheme HS3

In the case HS3, the equations describing the local behavior are

$$\begin{aligned} \underset{\sim}{\sigma}^\epsilon &= \underset{\approx}{a}^{(0)}(\underline{y}) : \underset{\sim}{\varepsilon}^\epsilon, \underset{\sim}{s}^\epsilon = \underset{\approx}{b}^{(0)}(\underline{y}) : \underset{\sim}{e}^\epsilon \\ \underset{\sim}{m}^\epsilon &= \epsilon^2 \underset{\approx}{c}^{s(1)}(\underline{y}) \vdots \underset{\sim}{\kappa}^{s\epsilon} + \underset{\approx}{c}^{s(2)}(\underline{y}) \underset{\sim}{\kappa}^{a\epsilon} \end{aligned} \quad (42)$$

At the order ϵ^{-1}, one gets $\underset{\approx}{a}^{(0)} : \underset{\sim}{\varepsilon}_{-1} = 0, \quad \underset{\approx}{b}^{(0)} : \underset{\sim}{e}_{-1} = 0, \quad \underset{\approx}{c}^{a(2)} \vdots \underset{\approx}{\kappa}^a_{-1} = 0.$

This implies that the gradients of $\underline{u}_0$ and $\underset{\sim}{\chi}_1^0$ with respect to $\underline{y}$ vanish, so that

$$\underline{u}_0(\underline{x}, \underline{y}) = \underline{U}_0(\underline{x}), \quad \underset{\sim}{\chi}_1^a(\underline{x}, \underline{y}) = \underset{\sim}{\Xi}_1^a(\underline{x}) \quad (43)$$

The fields $(\underline{u}_1, \underset{\sim}{\chi}_1^s, \underset{\sim}{\chi}_2^a, \underset{\sim}{\chi}_3^a, \underset{\sim}{\sigma}_0, \underset{\sim}{s}_0, \underset{\approx}{m}_0, \underset{\approx}{m}_1)$ are solutions of the following auxiliary boundary value problem defined on the unit cell:

$$\begin{cases} \epsilon_0 = \underline{U}_0 \overset{s}{\otimes} \nabla_x + \underline{u}_1 \overset{s}{\otimes} \nabla_y, \underset{\sim}{e}_0 \underline{U}_0 \otimes \nabla_x \\ \qquad + \underline{u}_1 \otimes \nabla_y - \underset{\sim}{\Xi}_1^a - \underset{\sim}{\chi}_1^s \\ \underset{\approx}{\kappa}^s_{-1} = \underset{\sim}{\chi}_1^s \otimes \nabla_y, \underset{\approx}{\kappa}^s_{-1} = \underset{\sim}{\Xi}_1^a \otimes \nabla_x \underset{\sim}{\chi}_2^a \otimes \nabla_y \\ \underset{\approx}{\kappa}^a_1 = \underset{\sim}{\chi}_3^a \otimes \nabla_y \\ \underset{\sim}{\sigma}_0 = \underset{\approx}{a}^{(0)} : \epsilon_0, \quad \underset{\sim}{s}_0 = \underset{\approx}{b}^{(0)} : \underset{\sim}{e}_0 \\ \underset{\approx}{m}_0 = \underset{\approx}{c}^{a(2)} \vdots \underset{\approx}{\kappa}^a_0, \quad \underset{\approx}{m}_1 = \underset{\approx}{c}^{s(1)} \vdots \underset{\approx}{\kappa}^s_{-1} + \underset{\approx}{c}^{s(2)} \vdots \underset{\approx}{\kappa}^a_1 \\ (\underset{\sim}{\sigma}_0 + \underset{\sim}{s}_0) \cdot \nabla_y = 0, \quad \underset{\approx}{m}_0 \cdot \nabla_y = 0, \\ \underset{\approx}{m}_0 \cdot \nabla_x + \underset{\approx}{m}_1 \cdot \nabla_y + \underset{\sim}{s}_0 = 0 \end{cases}$$

This complex problem can be seen to depend linearly on
$\underline{U}_0 \overset{s}{\otimes} \nabla, \underline{U}_0 \overset{a}{\otimes} - \underset{\sim}{\Xi}_1^a$ and $\underset{\sim}{\Xi}_1^a \otimes \nabla$. The solutions take the form

$$\begin{aligned} \underline{u}^\epsilon &= \underline{U}_0(\underline{x}) + \epsilon(\underline{U}_1(\underline{x}) + \underset{\approx}{X}_u^{(3)}(\underline{y}) : \\ &\quad (\underline{U}_0 \overset{s}{\otimes} \nabla + \underset{\approx}{X}_e^{(3)}(\underline{y}) : \underline{U}_0 \overset{a}{\otimes} \nabla - \underset{\sim}{\Xi}_1^a)) + \dots \end{aligned} \quad (44)$$

$$\underset{\sim}{\chi}^{\epsilon} = \underset{\sim}{\Xi}_1(\underline{x}) + \epsilon(\underset{\sim}{\Xi}_2(\underline{x}) + \underset{\approx}{X}^{(3)}_{\kappa}(\underline{y})\vdots\underset{\sim}{\Xi}^a_1 \otimes \nabla)) + \ldots \tag{45}$$

where concentration tensors $\underset{\approx}{X}^{(3)}_u, \underset{\approx}{X}^{(3)}_e$ and $\underset{\approx}{X}^{(3)}_{\kappa}$ have been introduced. Their components are determined by the successive solutions of the auxiliary problem for unit values of the components of $\underline{U}_0 \overset{s}{\otimes} \nabla, \underline{U}_0 \overset{a}{\otimes} \nabla - \underset{\sim}{\Xi}^a_1$ and $\underset{\sim}{\Xi}^a_1 \otimes \nabla_y$. They are such that their mean value over the unit cell vanishes.

The macroscopic stress tensors and effective elastic properties are given by

$$\begin{aligned}
\underset{\sim}{\Sigma}_0 &= <\underset{\approx}{a}^{(0)} : (\underset{\sim}{1}\nabla_x \overset{s}{\otimes} \underset{\approx}{X}^{(3)}_u)> : (\underline{U}_0 \overset{s}{\otimes} \nabla) + <\underset{\approx}{a}^{(0)} : \\
&\quad (\nabla_x \overset{s}{\otimes} \underset{\approx}{X}^{(3)}_u)> : (\underline{U}_0 \overset{a}{\otimes} \nabla - \underset{\sim}{\Xi}^a_1)\\
\underset{\sim}{S}_0 &= <\underset{\sim}{s}_0> = <\underset{\approx}{b}^{(0)} : (\nabla_x \otimes \underset{\approx}{X}^{(3)}_u)> : (\underline{U}_0 \overset{s}{\otimes} \nabla)\\
&\quad + <\underset{\approx}{b}^{(0)} : (\nabla_x \otimes \underset{\approx}{X}^{(3)}_e)> : (\underline{U}_0 \overset{a}{\otimes} \nabla - \underset{\sim}{\Xi}^a_1)\\
\underset{\sim}{M}_0 &= <\underset{\sim}{m}_0> = <\underset{\approx}{c}^{a(2)}\vdots(\underset{\sim}{1}\nabla_y \overset{s}{\otimes} \underset{\approx}{X}^{(3)}_{\kappa})>\vdots\underset{\sim}{\Xi}^a_1 \otimes \nabla
\end{aligned}$$

They must fulfill the balance equations (27). Note that $\underset{\sim}{m}_0$ and therefore $\underset{\sim}{M}_0$ are skew symmetric with respect to their first two indices. The averaged equation of balance of moment of momentum implies that $\underset{\sim}{S}_0$ is symmetric. The macroscopic degrees of freedom are the displacement field $\underline{U}_0$ and the symmetric strain tensor $\underset{\sim}{\Xi}^a_1$. The found balance and constitutive equations are therefore that of a Cosserat effective medium. The more classical form of the Cosserat theory is retrieved once one rewrites the previous equations using the axial vector associated to the skew–symmetric tensor $\underset{\sim}{\Xi}^a$ [6].

Homogenization Scheme HS4

In the last considered case, the equations describing the local behavior are

$$\underset{\sim}{\sigma}^{\epsilon} = \underset{\approx}{a}^{(0)}(\underline{y}) : \underset{\sim}{\epsilon}^{\epsilon}, \qquad \underset{\sim}{s}^{\epsilon} = \underset{\approx}{b}^{(0)}(\underline{y}) : \underset{\sim}{e}^{\epsilon},$$
$$\underset{\sim}{m}^{\epsilon} = \underset{\approx}{c}^{s(2)}(\underline{y})\vdots\underset{\sim}{\kappa}^{a\epsilon} \tag{46}$$

At the order ϵ^{-1}, one gets $\underset{\approx}{a}^{(0)} : \underset{\sim}{\epsilon}_{-1} = 0$, $\underset{\approx}{b}^{(0)} : \underset{\sim}{e}_{-1} = 0, \quad \underset{\approx}{c}^{s(2)}\vdots\underset{\sim}{\kappa}^s_{-1} = 0$

This implies that the gradients of $\underline{u}_0$ and $\underset{\sim}{\chi}^s_1$ with respect to $\underline{y}$ vanish, so that

$$\underline{u}_0(\underline{x}, \underline{y}) = \underline{U}_0(\underline{x}), \qquad \underset{\sim}{\chi}^s_1(\underline{x}, \underline{y}) = \underset{\sim}{\Xi}^s_1(\underline{x}) \tag{47}$$

The fields $(\underline{u}_1, \underset{\sim}{\chi}^a_1, \underset{\sim}{\chi}^s_2, \underset{\sim}{\chi}^s_3, \underset{\sim}{\sigma}_0, \underset{\sim}{s}_0, \underset{\sim}{m}_0, \underset{\sim}{m}_1)$ are solutions of the following auxiliary boundary value problem defined on the unit cell:

$$\begin{cases}
\underset{\sim}{\epsilon}_0 = \underline{U}_0 \overset{s}{\otimes} \nabla_x + \underline{u}_1 \overset{s}{\otimes} \nabla_y, \underset{\sim}{e}_0 = \underline{U}_0 \otimes \nabla_x\\
\quad + \underline{u}_1 \otimes \nabla_y - \underset{\sim}{\Xi}^s_1 - \underset{\sim}{\chi}^a_1\\
\underset{\sim}{\kappa}^a_{-1} = \underset{\sim}{\chi}^a_1 \otimes \nabla_y, \underset{\sim}{\kappa}^s_0 = \underset{\sim}{\Xi}^s_1 \otimes \nabla_x + \underset{\sim}{\chi}^s_2 \otimes \nabla_y, \underset{\sim}{\kappa}^a_{-1}\\
\quad = \underset{\sim}{\chi}^s_2 \otimes \nabla_x + \underset{\sim}{\chi}^s_3 \otimes \nabla_y\\
\underset{\sim}{\sigma}_0 = \underset{\approx}{a}^{(0)} : \underset{\sim}{\epsilon}_0, \quad \underset{\sim}{s}_0 = \underset{\approx}{b}^{(0)} : \underset{\sim}{e}_0\\
\underset{\sim}{m}_0 = \underset{\approx}{c}^{s(2)}\vdots\underset{\sim}{\kappa}^s_0, \quad \underset{\sim}{m}_1 = \underset{\approx}{c}^{a(1)}\vdots\underset{\sim}{\kappa}^a_{-1} + \underset{\approx}{c}^{s(2)}\underset{\sim}{\kappa}^s_1\\
(\underset{\sim}{\sigma}_0 + \underset{\sim}{s}_0) \cdot \nabla_y = 0, \underset{\sim}{m}_0 \cdot \nabla_y = 0, \underset{\sim}{m}_0 \cdot \nabla_x\\
\quad + \underset{\sim}{m}_1 \cdot \nabla_y + \underset{\sim}{s}_0 = 0
\end{cases}$$

This complex problem can be seen to depend linearly on $\underline{U}_0 \overset{s}{\otimes} \nabla, \underline{U}_0 \overset{s}{\otimes} \nabla - \underset{\sim}{\Xi}^s_1$ and $\underset{\sim}{\Xi}^s_1 \otimes \nabla$. The solutions take the form

$$\begin{aligned}
\underline{u}^{\epsilon}\underline{U}_0(\underline{x}) &+ \epsilon(\underline{U}_1(\underline{x})\underset{\approx}{X}^{(4)}_u(\underline{y}) : (\underline{U}_1 \overset{s}{\otimes} \nabla)\\
&+ \underset{\approx}{X}^{(4)}_e : (\underline{U}_1 \overset{s}{\otimes} \nabla - \underset{\sim}{\Xi}^s_1)) + \ldots\\
\underset{\sim}{\chi}^{\epsilon} &= \underset{\sim}{\Xi}_1(\underline{x}) + \epsilon(\underset{\sim}{\Xi}_2(\underline{x}) + \underset{\approx}{X}^{(4)}_{\kappa}(\underline{y})\vdots(\underset{\sim}{\Xi}^s_1 \otimes \nabla)) + \ldots
\end{aligned}$$

where concentration tensors $\underset{\approx}{X}^{(4)}_u, \underset{\approx}{X}^{(4)}_e$ and $\underset{\approx}{X}^{(4)}_{\kappa}$ have been introduced. Their components are determined by the successive solutions of the auxiliary problem for unit values of the components of $\underline{U}_0 \overset{s}{\otimes} \nabla, \underline{U}_0 \overset{s}{\otimes} \nabla - \underset{\sim}{\Xi}^s_1$ and $\underset{\sim}{\Xi}^s_1 \otimes \nabla_y$. They are such that their mean value over the unit cell vanishes.

The macroscopic stress tensors and effective elastic properties are given by

$$\underset{\approx}{\Sigma}_0 = <\underset{\approx}{a}^{(0)} : (\underset{\approx}{1} + \nabla_x \overset{s}{\otimes} \underset{\approx}{X}_u^{(4)})> : (\underline{U}_0 \overset{s}{\otimes} \nabla)$$
$$+ <\underset{\approx}{a}^{(0)} : (\nabla_y \overset{s}{\otimes} \underset{\approx}{X}_e^{(4)})> : (\underline{U}_0 \overset{s}{\otimes} \nabla - \underset{\sim}{\Xi}_1^s)$$
$$\underset{\approx}{S}_0 = <\underset{\approx}{s}_0> = <\underset{\approx}{b}^{(0)} : (\nabla_y \overset{s}{\otimes} \underset{\approx}{X}_u^{(4)})> : (\underline{U}_0 \overset{s}{\otimes} \nabla)$$
$$+ <\underset{\approx}{b}^{(0)}> : (\underline{U}_0 \overset{s}{\otimes} \nabla - \underset{\sim}{\Xi}_1^s)$$
$$\underset{\approx}{M}_0 = <\underset{\approx}{m}_0> = <\underset{\approx}{c}^{s(2)} \vdots (\underset{\approx}{1} + \nabla_y \otimes \underset{\approx}{X}_\kappa^{(4)})> \vdots (\underset{\sim}{\Xi}_1^s \otimes \nabla)$$

They must fulfill the balance equations (27). Note that $\underset{\approx}{m}_0$ and therefore $\underset{\approx}{M}_0$ are symmetric with respect to their first two indices. The averaged equation of balance of moment of momentum implies that $\underset{\approx}{S}_0 = -<\underset{\approx}{m}_0> \cdot \nabla$ is symmetric. The macroscopic degrees of freedom are the displacement field $\underline{U}_0$ and the symmetric strain tensor $\underset{\sim}{\Xi}_1^s$.

Such a continuum is called a microstrain medium [8].

As a conclusion, depending on the relative contributions of the various intrinsic length scales of the micromorphic continuum, different effective media are obtained, as summarized in Table 1. The effective medium can be of micromorphic, microstrain, Cosserat, or Cauchy type. A similar situation is found in the case of the homogenization of heterogeneous Cosserat media. Depending on the ratio between the Cosserat characteristic length l_a and the sizes l, L, the effective medium will be a Cauchy continuum with body couples or a full Cosserat continuum [9].

Applications

The approach is applied to two important classes of materials, namely, composite and polycrystalline materials. The auxiliary problems evidenced in the previous homogenization method are solved by means of the finite element method with well-suited boundary conditions.

Fiber or Particle Composites

The reinforcement induced by fibers and particles embedded in a matrix material depends on their volume fraction and arrangement but also on their size compared to the characteristic size of the microstructure elements of the matrix. The former effect is satisfactorily accounted for by standard homogenization methods. The latter can be described by considering that both the matrix and inclusions are Cosserat materials having different intrinsic length l_a. The effective properties of such a composite are found by solving auxiliary problems of the unit cell. The unit cell corresponding to a square arrangement of fibers with a volume fraction of 0.4 is shown in Fig. 1. According to scheme HS3, the displacement microrotation fields are searched for in the following form in the unit cell:

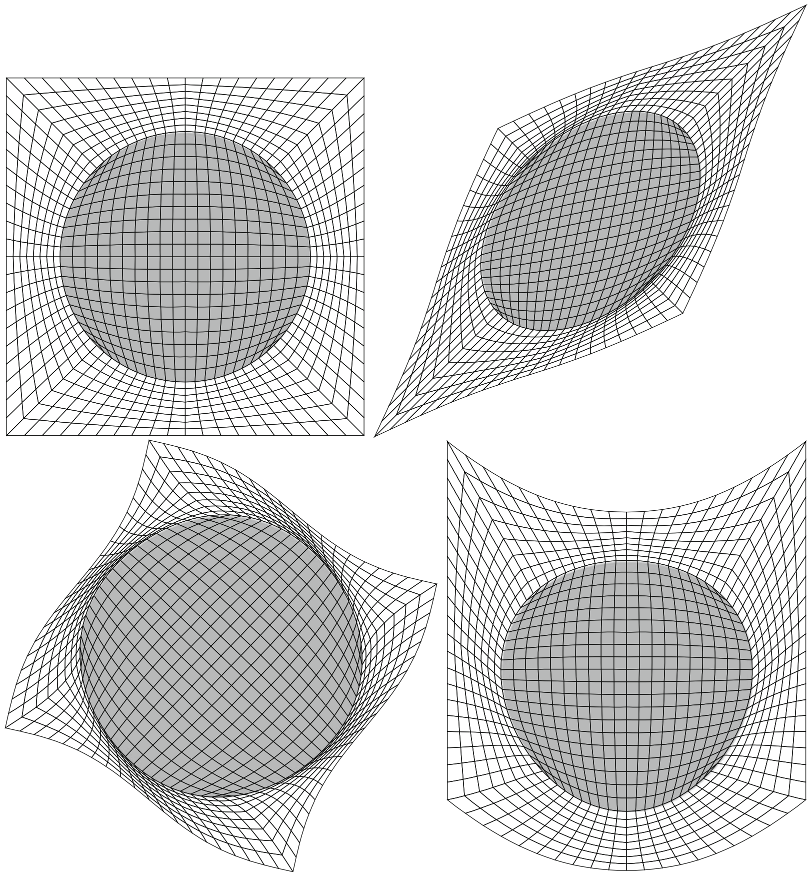
Asymptotic Analysis of Heterogeneous Micromorphic Elastic Solids, Table 1 Homogenization of heterogeneous micromorphic media: nature of the homogeneous equivalent medium depending on the values of the intrinsic lengths of the constituents

Homogenization scheme	Characteristic lengths	Effective medium
HS1	$l_s \sim l, l_a \sim l$	Cauchy
HS2	$l_s \sim L, l_a \sim L$	Micromorphic
HS3	$l_s \sim l, l_a \sim L$	Cosserat
HS4	$l_s \sim L, l_a \sim l$	Microstrain

$$\underline{u}(\underline{y}) = \underset{\sim}{E} \cdot \underline{y} + \underline{v}(\underline{y})$$
$$\underset{\approx}{\chi}^a(\underline{y}) = \underset{\approx}{K} \cdot \underline{y} + \underset{\approx}{\xi}^a(\underline{y})$$

The fluctuation displacement $\underline{v}$ and the skew–symmetric microrotation fluctuation $\underset{\approx}{\xi}^a$ are periodic. The macroscopic deformation $\underset{\sim}{E}$ and curvature $\underset{\approx}{K}$ are prescribed to the unit cell. The computation of the mean elastic energy contained in the deformed unit cell is used to identify the microscopic elastic moduli. According to Hill–Mandel's lemma that can be derived from the previous homogenization procedure, the macroscopic strain energy is the mean value of the local one over the volume element:

$$\underset{\sim}{\Sigma} : \underset{\sim}{E} + \underset{\approx}{M} \vdots \underset{\approx}{M} = <\underset{\sim}{\sigma} : \underset{\sim}{e} + \underset{\approx}{m} \vdots \underset{\approx}{\kappa}> \qquad (48)$$



Asymptotic Analysis of Heterogeneous Micromorphic Elastic Solids, Fig. 1 Solution of the auxiliary problem in the homogenization of Cosserat fiber composites: unit cell of the composite material (*top right*), simple shear (*top right*), mean relative rotation (*bottom left*), and mean curvature (*bottom right*), under plane strain conditions

Figure 1 shows how mean shear, relative rotation, and curvature can be applied successively to a unit cell.

Polycrystalline Materials

The previous homogenization method can be extended, at least in a heuristic way, to nonlinear micromorphic constitutive equations in order to predict size effects in the plasticity of polycrystals. The reader is referred to [5] for a detailed presentation of such models and a more complete description of polycrystal homogenization. The computation of polycrystalline aggregates based on standard crystal plasticity models follows the rule of classical homogenization theory in the sense that a mean strain is prescribed to a volume element of polycrystalline materials using suitable boundary conditions like strain-based, stress-based, or periodic ones. The structure of the boundary value problem is modified if a generalized continuum approach is used inside the considered volume element. The grain boundary conditions represent an important new feature of the theory. At any interface of a micromorphic continuum, there may exist some jump conditions for the degrees of freedom of the theory and the associated reactions, namely, the simple and double tractions. As a first approximation, however, the displacement vector and the microdeformation tensor can be assumed to be continuous at grain boundaries. As a result, the simple and double tractions also are continuous. The continuity of microdeformation is a new grain boundary condition that does not exist in classical crystal plasticity. It will generate boundary layers at grain boundaries which are essential for the observed size effects [5, 10]. In that way, material parameters of the micromorphic model can be identified in order to quantitatively describe the well-known Hall–Petch relationship which is a direct correspondence between the overall stress and the grain size at a given plastic strain.

References

1. Ashby M (1970) The deformation of plastically non–homogeneous materials. Philos Mag 21:399–424
2. Boutin C (1996) Microstructural effects in elastic composites. Int J Solids Struct 33:1023–1051
3. Chen Y, Lee J (2003) Connecting molecular dynamics to micromorphic theory. (I) instantaneous and averaged mechanical variables. Physica A 322:359–376
4. Claus W, Eringen A (1971) Dislocation dispersion of elastic waves. Int J Engin Sci 9:605–610
5. Cordero NM, Forest S, Busso EP, Berbenni S, Cherkaoui M (2012) Grain size effects on plastic strain and dislocation density tensor fields in metal polycrystals. Comput Mater Sci 52:7–13
6. Eringen A (1999) Microcontinuum field theories. Springer, New York
7. Eringen A, Suhubi E (1964) Nonlinear theory of simple microelastic solids. Int J Engng Sci 2(189–203):389–404
8. Forest S, Sievert R (2006) Nonlinear microstrain theories. Int J Solids Struct 43:7224–7245
9. Forest S, Pradel F, Sab K (2001) Asymptotic analysis of heterogeneous Cosserat media. Int J Solids Struct 38:4585–4608
10. Gurtin M (2002) A gradient theory of single–crystal viscoplasticity that accounts for geometrically necessary dislocations. J Mech Physics Solids 50:5–32
11. Lebée A, Sab K (2010) A cosserat multiparticle model for periodically layered materials. I J Solids Struct 37:293–297
12. Liu X, Hu G (2003) Inclusion problem of microstretch continuum. Int J Engin Sci 42:849–860
13. Mindlin R (1964) Micro–structure in linear elasticity. Arch Rat Mech Anal 16:51–78
14. Mühlhaus H (1995) Continuum models for materials with microstructure. Wiley, Chichester
15. Sanchez-Palencia E (1974) Comportement local et macroscopique d'un type de milieux physiques hétérogènes. Int J Engin Sci 12:331–351
16. Sanchez-Palencia E, Zaoui A (1987) Homogenization techniques for composite media, vol 272, Lecture Notes in Physics. Springer, Berlin
17. Smyshlyaev V, Fleck N (1994) Bounds and estimates for linear composites with strain gradient effects. J Mech Physics Solids 42:1851–1882
18. Suquet P (1997) Continuum micromechanics, vol 377, CISM courses and lectures. Springer, Udine \Berlin
19. Xun F, Hu G, Huang Z (2004) Size–dependence of overall in–plane plasticity for fiber composites. Int J Solids Struct 41:4713–4730
20. Zhu H, Zbib H, Aifantis E (1997) Strain gradients and continuum modeling of size effect in metal matrix composites. Acta Mechanica 121:165–176

Asymptotic Behavior in Time

Maria Grazia Naso
Università degli Studi di Brescia, Brescia, Italy

Synonyms

Decay rate; Longtime behavior in thermoelasticity

Overview

We are interested to study the longtime behavior for a linear one-dimensional thermoelastic system where the hyperbolic elastic system is joined with the parabolic heat equation. By some results in semigroup theory, we prove the exponential decay of the solutions related to the associated initial boundary value problem. For a detailed study in more general cases, some references are given at the end of this section.

A Simple Model in Thermoelasticity

The One-Dimensional Linear Thermoelastic System

For $T > 0$, we consider the following one-dimensional linear thermoelastic system:

$$u_{tt} - \alpha\, u_{xx} + \gamma\, \theta_x = 0 \qquad \text{in } (0,\ell) \times (0,T) \quad (1)$$

$$\theta_t - k\, \theta_{xx} + \gamma\, u_{xt} = 0 \qquad \text{in } (0,\ell) \times (0,T) \quad (2)$$

supplemented with initial conditions