M

Micromorphic Approach to Materials with Internal Length

Samuel Forest
Centre des Matériaux, Mines ParisTech CNRS, Evry, France

Synonyms

Continua with microstructure; Generalized continua; Higher order continua; Micromorphic approach to gradient elasticity; Micromorphic media; Plasticity and damage

Definitions

Micromorphic media are three-dimensional continua made of material points endowed with usual translational degrees of freedom and additional kinematical degrees of freedom accounting for the rotation and distortion of a triad of directors. The directors are related to an underlying microstructure (lattice directions in a crystal, fiber directions in a composite materials, etc.). Their transformation is represented by a generally noncompatible field of second rank generally nonsymmetric microdeformation tensors. More generally, the micromorphic approach consists in enriching the kinematics of the material point by additional degrees of freedom related to plastic strain, damage, or phase field variables. An essential feature of such theories is that the gradient of the micromorphic variable enters the constitutive functions which therefore include internal length parameters.

Overview

The micromophic model represents one of the most sophisticated generalized continuum theories since each material point is endowed with 12 independent degrees of freedom in the general three-dimensional case. It aims at incorporating some features of the underlying material's microstructure into the macroscopic continuum setting, namely the existence of priviledged microstructural directions whose orientation and curvature affects the material's response. It was invented simultaneously by Mindlin (1964) and Eringen and Suhubi (1964) but the seminal name *micromorphic* was coined by C.A. Eringen who also provided the complete theory at finite deformations including inertial effects. It is a first gradient theory in so far as only the first gradient of all degrees of freedom, i.e., displacement gradient and microdeformation gradient, are included in the theory in contrast to strain gradient models involving second and even higher order gradients of the displacement field. The micromorphic model overcomes the limitations of the Cosserat theory which relies on the rotation and curvature of microstructure directors only. For example, the Cosserat model is suitable for the prediction

H. Altenbach, A. Öchsner (eds.), *Encyclopedia of Continuum Mechanics*,
https://doi.org/10.1007/978-3-662-53605-6_150-1

of shear bands with finite width in plasticity but inappropriate for crushing bands in cellular materials due to the absence of rotation in such localization modes (Forest et al. 2005).

In the 1960s and 1970s, the micromorphic model was essentially confined to applications related to the elastic behavior of materials including the dispersion of elastic waves and the regularization of stress and strain singularities at the crack tip and dislocation core Eringen (1999). The modern interest in micromorphic approaches is due to the strong development of computational methods and experimental field measurement techniques allowing for proper identification of intrinsic length scales. The most promising applications of the micromorphic approach deal with the modeling and simulation of strain and damage localization phenomena in materials.

This entry deals only with the statics of micromorphic media. The reader is referred to the entries Waves and Generalized Continua in this Encyclopedia for the dynamical equations.

Intrinsic notations are used throughout the entry. Vectors and tensors of various order are denoted by boldface letters. Double and triple contractions are written as:

$$\boldsymbol{a} : \boldsymbol{b} = a_{ij} b_{ij}, \quad \boldsymbol{a} \,\vdots\, \boldsymbol{b} = a_{ijk} b_{ijk} \tag{1}$$

using the Einstein summation rule for repeated indices. The tensor product is denoted by $\otimes$. For example, the component $(\boldsymbol{a} \otimes \boldsymbol{b})_{ijkl}$ is $a_{ij} b_{kl}$. A modified tensor product $\boxtimes$ is also used: the component $(\boldsymbol{a} \boxtimes \boldsymbol{b})_{ijkl}$ is $a_{ik} b_{jl}$.

The gradient operators ∇ or ∇_X are introduced when the functions respectively depend on current spatial coordinates $\boldsymbol{x}$ or Lagrangian material coordinates $\boldsymbol{X}$. The following notation is used:

$$\boldsymbol{U} \otimes \nabla_X = U_{i,j} \, \boldsymbol{e}_i \otimes \boldsymbol{e}_j, \text{ with } U_{i,j} = \frac{\partial U_i}{\partial X_j} \tag{2}$$

$$\boldsymbol{u} \otimes \nabla = u_{i,j} \, \boldsymbol{e}_i \otimes \boldsymbol{e}_j, \text{ with } u_{i,j} = \frac{\partial u_i}{\partial x_j} \tag{3}$$

where $(\boldsymbol{e}_i)_{i=1,2,3}$ is a Cartesian orthonormal basis.

Kinematics

The degrees of freedom of the theory are the displacement vector $\boldsymbol{u}$ and the generally nonsymmetric second rank microdeformation tensor $\boldsymbol{\chi}$. The current position of the material point with reference position $\boldsymbol{X}$ is given by the transformation Φ according to $\boldsymbol{x} = \Phi(\boldsymbol{X}, t) = \boldsymbol{X} + \boldsymbol{u}(\boldsymbol{X}, t)$. The microdeformation describes the deformation of a triad of directors, $\boldsymbol{\Xi}^i$ attached to the material point

$$\boldsymbol{\xi}^i (\boldsymbol{X}, t) = \boldsymbol{\chi} (\boldsymbol{X}, t) \cdot \boldsymbol{\Xi}^i \tag{4}$$

As such, its determinant is taken as strictly positive. The polar decomposition of the generally incompatible microdeformation field $\boldsymbol{\chi}(\boldsymbol{X})$ into a pure rotation and a symmetric second rank tensor is written as

$$\boldsymbol{\chi} = \boldsymbol{R}^\sharp \cdot \boldsymbol{U}^\sharp \tag{5}$$

Internal constraints can be prescribed to the microdeformation. The micromorphic medium reduces to the Cosserat medium when the microdeformation is constrained to be a pure rotation: $\boldsymbol{\chi} \equiv \boldsymbol{R}^\sharp$. The microstrain medium is obtained when $\boldsymbol{\chi} \equiv \boldsymbol{U}^\sharp$. Finally, the second gradient theory is retrieved when the microdeformation coincides with the deformation gradient, $\boldsymbol{\chi} \equiv \boldsymbol{F}$. A hierarchy of higher order continua can be established by specializing the micromorphic theory and depending on the targeted material class, see Forest and Sievert (2006).

The following kinematical quantities are then introduced:

- The velocity field $\boldsymbol{v}(\boldsymbol{x}, t) := \dot{\boldsymbol{u}}\left(\Phi^{-1}(\boldsymbol{x}, t)\right)$
- The deformation gradient $\boldsymbol{F} = \boldsymbol{1} + \boldsymbol{u} \otimes \nabla_X$
- The velocity gradient $\boldsymbol{v} \otimes \nabla_x = \dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1}$
- The microdeformation rate $\dot{\boldsymbol{\chi}} \cdot \boldsymbol{\chi}^{-1}$
- The third rank Lagrangean microdeformation gradient $\boldsymbol{K} := \boldsymbol{\chi}^{-1} \cdot \boldsymbol{\chi} \otimes \nabla_X$
- The gradient of the microdeformation rate tensor

$$\left(\dot{\chi} \cdot \chi^{-1}\right) \otimes \nabla_x = \chi \cdot \dot{\boldsymbol{K}} : \left(\chi^{-1} \boxtimes \boldsymbol{F}^{-1}\right) \quad (6)$$

with the corresponding index notation:

$$\left(\dot{\chi}_{il}\ \chi_{lj}^{-1}\right), k = \chi_{ip} \dot{K}_{pqr}\ \chi_{qj}^{-1} F_{rk}^{-1}$$

General Micromorphic Media

Germain (1973) interpreted the microdeformation tensor as the first term in an expansion of the relative motion around the material volume element center of mass:

$$u_i'\left(\boldsymbol{x}'\right) = u_i + \chi_{ij}\chi_j' + \chi_{ijk}x_j'x_k' + \chi_{ijkl}\chi_j'x_k'x_j' + \cdots \quad (7)$$

where $\boldsymbol{x}'$ refers to the position of microscopic points around the center of mass, $\boldsymbol{x}$, and the coefficients $\chi_{ijkl\ldots}$, symmetric w.r.t. to all indices except the first one, are microdeformation tensors of higher orders. This concept generalizes Eringen's model to micromorphic media of order n, corresponding to the truncation of the previous expansion. This expansion is a Taylor series if the various micromorphic degrees of freedom coincide with gradients of the displacement field:

$$\chi_{ij} \equiv \delta_{i,j} + u_{i,j},\ \chi_{ijk} \equiv u_{i,jk}, \quad \chi_{ijkl} \equiv u_{i,jkl}, \quad \ldots \quad (8)$$

Germain's general micromorphic model has been recently extended to totally nonsymmetric microdeformation tensors by Forest and Sab (2017), who interpreted them as relaxed deformation gradients.

Balance Laws for Momentum and Generalized Moment of Momentum

The method of virtual power is used to introduce the generalized stress tensors and the field and boundary equations must satisfy following Germain (1973).

The virtual power of internal forces of a subdomain $\mathcal{D} \subset \mathcal{B}$ of the current body is

$$\mathcal{P}^{(i)}\left(\boldsymbol{v}^*, \dot{\chi}^* \cdot \chi^{*-1}\right) = \int_{\mathcal{D}} p^{(i)}\left(\boldsymbol{v}^*, \dot{\chi}^* \cdot \chi^{*-1}\right) dV$$

The virtual power density of internal forces is a linear form on the fields of virtual modeling variables:

$$\begin{aligned} p^{(i)} &= \boldsymbol{\sigma} : \left(\dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1}\right) + \boldsymbol{s} : \left(\dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1} - \dot{\chi} \cdot \chi^{-1}\right) \\ &\quad + \boldsymbol{M} \vdots \left(\left(\dot{\chi} \cdot \chi^{-1}\right) \otimes \nabla_x\right) \\ &= \boldsymbol{\sigma} : \left(\dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1}\right) + \boldsymbol{s} : \left(\chi \cdot \left(\chi^{-1} \cdot \boldsymbol{F}\right)^{\cdot} \cdot \boldsymbol{F}^{-1}\right) \\ &\quad + \boldsymbol{M} \vdots \left(\chi \cdot \dot{\boldsymbol{K}} : \left(\chi^{-1} \boxtimes \boldsymbol{F}^{-1}\right)\right) \end{aligned} \quad (9)$$

where the relative deformation rate $\dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1} - \dot{\chi} \cdot \chi^{-1}$ is introduced and expressed in terms of the rate of the relative deformation $\chi^{-1} \cdot \boldsymbol{F}$. The virtual power density of internal forces is invariant with respect to virtual rigid body motions so that $\boldsymbol{\sigma}$ must be symmetric. The generalized stress tensors conjugate to the velocity gradient, the relative deformation rate, and the gradient of the microdeformation rate respectively are the simple stress tensor $\boldsymbol{\sigma}$, the relative stress tensor $\boldsymbol{s}$, and the double stress tensor $\boldsymbol{M}$ of third rank. The Gauss theorem is then applied to the power of internal forces

$$\begin{aligned} \int_{\mathcal{D}} \mathcal{P}^{(i)}\, dV &= \int_{\partial\mathcal{D}} \boldsymbol{v}^* \cdot (\boldsymbol{\sigma} + \boldsymbol{s}) \cdot \boldsymbol{n}\, dS \\ &\quad + \int_{\partial\mathcal{D}} \left(\dot{\chi}^* \cdot \chi^{*-1}\right) : \boldsymbol{M} \cdot \boldsymbol{n}\, dS \\ &\quad - \int_{\mathcal{D}} \boldsymbol{v}^* \cdot (\boldsymbol{\sigma} + \boldsymbol{s}) \cdot \nabla\, dV \\ &\quad - \int_{\mathcal{D}} \left(\dot{\chi}^* \cdot \chi^{*-1}\right) : (\boldsymbol{M} \cdot \nabla + \boldsymbol{s})\, dV \end{aligned}$$

The form of the previous boundary integral dictates the possible form of the power of contact forces acting on the boundary $\partial\mathcal{D}$ of the subdomain $\mathcal{D} \subset \mathcal{B}$

$$\begin{aligned} \mathcal{P}^{(c)}\left(\boldsymbol{v}^*, \dot{\chi}^* \cdot \chi^{*-1}\right) &= \int_{\partial\mathcal{D}} p^{(c)}\left(\boldsymbol{v}^*, \dot{\chi}^* \cdot \chi^{*-1}\right) dV \\ p^{(c)}\left(\boldsymbol{v}^*, \dot{\chi}^* \cdot \chi^{*-1}\right) &= \boldsymbol{t} \cdot \boldsymbol{v}^* + \boldsymbol{m} : \left(\dot{\chi}^* \cdot \chi^{*-1}\right) \end{aligned}$$

M

where the simple traction $\boldsymbol{t}$ and double traction $\boldsymbol{m}$, a tensor of second rank, are introduced. The power of forces acting at a distance is defined as

$$\mathcal{P}^{(e)}\left(\boldsymbol{v}^*, \dot{\boldsymbol{\chi}}^* \cdot \boldsymbol{\chi}^{*-1}\right) = \int_{\mathcal{D}} p^{(e)}\left(\boldsymbol{v}^*, \dot{\boldsymbol{\chi}}^* \cdot \boldsymbol{\chi}^{*-1}\right) dV$$
$$p^{(e)}\left(\boldsymbol{v}^*, \dot{\boldsymbol{\chi}}^* \cdot \boldsymbol{\chi}^{*-1}\right) = \boldsymbol{f} \cdot \boldsymbol{v}^* + \boldsymbol{p} : \left(\dot{\boldsymbol{\chi}}^* \cdot \boldsymbol{\chi}^{*-1}\right)$$

including simple body forces $\boldsymbol{f}$ and double body forces $\boldsymbol{p}$. More general double and triple volume forces could also be incorporated according to Germain (1973).

The principle of virtual power is now stated in the static case,

$$\forall \boldsymbol{v}^*, \boldsymbol{\chi}^*, \forall \mathcal{D} \subset \mathcal{B}, \mathcal{P}^{(i)}\left(\boldsymbol{v}^*, \dot{\boldsymbol{\chi}}^* \cdot \boldsymbol{\chi}^{*-1}\right) = \mathcal{P}^{(c)}\left(\boldsymbol{v}^*, \dot{\boldsymbol{\chi}}^* \cdot \boldsymbol{\chi}^{*-1}\right) + \mathcal{P}^{(e)}\left(\boldsymbol{v}^*, \dot{\boldsymbol{\chi}}^* \cdot \boldsymbol{\chi}^{*-1}\right) \quad (10)$$

This variational formulation leads to the field equations of the problem (static case) (Germain 1973; Kirchner and Steinmann 2005; Lazar and Maugin 2007; Hirschberger et al. 2007):

- Balance of momentum equation

$$(\boldsymbol{\sigma} + \boldsymbol{s}) \cdot \nabla + \boldsymbol{f} = 0, \ \forall \boldsymbol{x} \in \mathcal{B} \quad (11)$$

- Balance of generalized moment of momentum equation

$$\boldsymbol{M} \cdot \nabla + \boldsymbol{s} + \boldsymbol{p} = 0, \ \forall \boldsymbol{x} \in \mathcal{B} \quad (12)$$

- Boundary conditions

$$(\boldsymbol{\sigma} + \boldsymbol{s}) \cdot \boldsymbol{n} = \boldsymbol{t}, \ \forall \boldsymbol{x} \in \partial\mathcal{B} \quad (13)$$

$$\boldsymbol{M} \cdot \boldsymbol{n} = \boldsymbol{m}, \ \forall \boldsymbol{x} \in \partial\mathcal{B} \quad (14)$$

Thermomechanical Setting and Constitutive Laws

This section is dedicated to the formulation of constitutive equations for micromorphic media. The general case of hyperelastic-viscoplastic materials is considered. According to Eringen (1999), the following Lagrangian strain measures are adopted:

$$\boldsymbol{C} := \boldsymbol{F}^T \cdot \boldsymbol{F}, \quad \boldsymbol{\Upsilon} := \boldsymbol{\chi}^{-1} \cdot \boldsymbol{F},$$
$$\boldsymbol{K} := \boldsymbol{\chi}^{-1} \cdot (\boldsymbol{\chi} \otimes \nabla_X)$$

i.e., the Cauchy–Green strain tensor, the relative deformation, and the microdeformation gradient, a third rank tensor.

In the presence of plastic deformation, the question arises of splitting the previous Lagrangian strain measures into elastic and plastic contributions. Following Mandel (1973), a multiplicative decomposition of the deformation gradient is postulated:

$$\boldsymbol{F} = \boldsymbol{F}^e \cdot \boldsymbol{F}^p = \boldsymbol{R}^e \cdot \boldsymbol{U}^e \cdot \boldsymbol{F}^p \quad (15)$$

which defines an intermediate local configuration at each material point. Uniqueness of the decomposition requires the suitable definition of directors. Such directors are available in any micromorphic theory. As an example, lattice directions in a single crystal are physically relevant directors for an elastoviscoplasticity micromorphic theory, see (Aslan et al. 2011). A multiplicative decomposition of the microdeformation is also considered:

$$\boldsymbol{\chi} = \boldsymbol{\chi}^e \cdot \boldsymbol{\chi}^p = \mathbf{R}^{e\sharp} \cdot \mathbf{U}^{e\sharp} \cdot \boldsymbol{\chi}^p \quad (16)$$

according to Forest and Sievert (2003, 2006). Finally, a partition rule must also be proposed for the third strain measure, namely the microdeformation gradient. Sansour (1998a, b) introduced an additive decomposition of curvature:

$$\boldsymbol{K} = \boldsymbol{K}^e + \boldsymbol{K}^p \quad (17)$$

A quasi-additive decomposition was proposed by Forest and Sievert (2003) with the objective of defining an intermediate local configuration for which all generalized stress tensor are simultaneously released, as it will become apparent in the next section:

$$\boldsymbol{K} = \boldsymbol{\chi}^{p-1} . \boldsymbol{K}^e : (\boldsymbol{\chi}^p \boxtimes \boldsymbol{F}^p) + \boldsymbol{K}^p \quad (18)$$

The continuum thermodynamic formulation is essentially unchanged in the presence of additional degrees of freedom provided that all functionals are properly extended to the new set of variables. The local equation of energy balance is written in its usual form:

$$\rho\dot{\epsilon} = \; p^{(i)} \; - \; \boldsymbol{q}.\nabla + r \tag{19}$$

where ε is the specific internal energy density, and $p^{(i)}$ is the power density of internal forces according to Eq. (9). The heat flux vector is $\boldsymbol{q}$ and r is a heat source term. The local form of the second principle of thermodynamics is written as

$$\rho\dot{\eta} + \left(\frac{\boldsymbol{q}}{T}\right)\cdot\nabla - \frac{r}{T} \geq 0$$

where η is the specific entropy density. Introducing the Helmholtz free energy function ψ, the second law becomes

$$p^{(i)} - \rho\,\dot{\Psi} - \eta\dot{T} - \frac{\boldsymbol{q}}{T}\cdot(\nabla T) \geq 0$$

The state variables of the elastoviscoplastic micromorphic material are all the elastic strain measures and a set of internal variables q. The free energy density is a function of the state variables:

$$\Psi\left(\boldsymbol{C}^e := \boldsymbol{F}^{eT}\cdot\boldsymbol{F}^e, \boldsymbol{\Upsilon}^e := \boldsymbol{\chi}^{e-1}.\boldsymbol{F}^e, \boldsymbol{K}^e, \quad q\right)$$

The exploitation of the entropy inequality leads to the definition of the hyperelastic state laws in the form:

$$\boldsymbol{\sigma} = 2\boldsymbol{F}^e\cdot\rho\frac{\partial\Psi}{\partial\boldsymbol{C}^e}\cdot\boldsymbol{F}^{eT}, \boldsymbol{s} = \boldsymbol{F}^{e-T}\cdot\rho\frac{\partial\Psi}{\partial\boldsymbol{\Upsilon}^e}\cdot\boldsymbol{F}^{eT}$$
$$\boldsymbol{M} = \boldsymbol{\chi}^{-T}\cdot\rho\frac{\partial\Psi}{\partial\boldsymbol{K}^e} : \left(\boldsymbol{\chi}^T \boxtimes \boldsymbol{F}^T\right) \tag{20}$$

while the entropy density is given by $\eta = -\frac{\partial\Psi}{\partial T}$. The thermodynamic force associated with the internal variable q is

$$R = \rho\frac{\partial\Psi}{\partial q}$$

The hyperelasticity law (20) for the double stress tensor was derived for the additive decomposition (17). The quasi-additive decomposition (18) leads to an hyperelastic constitutive equation for the conjugate stress $\boldsymbol{M}$ in the current configuration, that has also the same form as for pure hyperelastic behavior. One finds:

$$\boldsymbol{M} = \boldsymbol{\chi}^{e-T}\cdot\rho\frac{\partial\Psi}{\partial\boldsymbol{K}^e} : \left(\boldsymbol{\chi}^{eT} \boxtimes \boldsymbol{F}^{eT}\right) \tag{21}$$

The residual intrinsic dissipation is

$$D = \boldsymbol{\Sigma} : \left(\dot{\boldsymbol{F}}^p \boldsymbol{F}^{p-1}\right) + \boldsymbol{\mathcal{S}} : \left(\dot{\boldsymbol{\chi}}^p\cdot\boldsymbol{\chi}^{p-1}\right)$$
$$+ \boldsymbol{\mathcal{M}} \vdots \dot{\boldsymbol{K}}^p - R\dot{q} \geq 0$$

where generalized Mandel stress tensors have been defined

$$\Sigma = \boldsymbol{F}^{eT}\cdot(\boldsymbol{\sigma} + \boldsymbol{s})\cdot\boldsymbol{F}^{e-T},$$
$$\boldsymbol{\mathcal{S}} = -\boldsymbol{U}^{e\sharp}\cdot\boldsymbol{R}^{e\sharp T}\cdot\boldsymbol{s}\cdot\boldsymbol{R}^{e\sharp}\cdot\boldsymbol{U}^{e\sharp-1} \tag{22}$$

$$\boldsymbol{\mathcal{M}} = \boldsymbol{\chi}^T\cdot\boldsymbol{S} : \left(\boldsymbol{\chi}^{-T} \boxtimes \boldsymbol{F}^{-T}\right) \tag{23}$$

At this stage, one may define a dissipation potential, function of the Mandel stress tensors, from which the viscoplastic flow rule and the evolution equations for the internal variables are derived.

$$\Omega\;(\boldsymbol{\Sigma}, \boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{M}}, R)$$
$$\dot{\boldsymbol{F}}^p \boldsymbol{F}^{p-1} = \frac{\partial\Omega}{\partial\boldsymbol{\Sigma}},\quad \dot{\boldsymbol{\chi}}^p \boldsymbol{\chi}^{p-1} = \frac{\partial\Omega}{\partial\boldsymbol{\mathcal{S}}}, \tag{24}$$
$$\dot{\boldsymbol{K}}^p = \frac{\partial\Omega}{\partial\boldsymbol{\mathcal{M}}},\; \dot{q} = \frac{\partial\Omega}{\partial R}$$

Appropriate convexity properties of the dissipation potential with respect to its arguments ensure the positivity of the dissipation rate at each instant.

Explicit constitutive equations can be found in (Forest and Sievert 2003; Grammenoudis and Tsakmakis 2009; Grammenoudis et al. 2009; Regueiro 2010; Sansour et al. 2010), including extension of von Mises isotropic plasticity. Examples of application of elastoplastic

M

micromorphic media can be found in (Dillard et al. 2006) for plasticity and failure of metallic foams.

Linearized Constitutive Laws

The previous constitutive laws can now be linearized, thus providing a close set of equations for infinitesimal gradients of displacement, microdeformations, and gradients of microdeformation (the latter is not dimensionless and should therefore be compared to some characteristic curvature value). The gradient of the displacement and the microdeformation itself are additively split into elastic and plastic parts:

$$\boldsymbol{H} := \boldsymbol{u} \otimes \nabla = \boldsymbol{H}^e + \boldsymbol{H}^p, \ \boldsymbol{\chi} = \boldsymbol{\chi}^e + \boldsymbol{\chi}^p \tag{25}$$

all of these second rank tensors being generally nonsymmetric. In the linearized case, χ is written for the previous $\boldsymbol{\chi} - \mathbf{1}$. The microdeformation gradient is also split into elastic and plastic parts which are generally distinct from the elastic and plastic microdeformation gradients: $\boldsymbol{K} = \boldsymbol{K}^e + \boldsymbol{K}^p$. The state and evolution laws (20) and (24) are linearized into:

$$\boldsymbol{\sigma} = \rho \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}^e}, \boldsymbol{s} = \rho \frac{\partial \Psi}{\partial \left(\boldsymbol{H}^e - \boldsymbol{\chi}^e\right)}, \quad \boldsymbol{M} = \rho \frac{\partial \Psi}{\partial \boldsymbol{K}^e}, R = \rho \frac{\partial \psi}{\partial q} \tag{26}$$

$$\dot{\boldsymbol{H}}^p = \frac{\partial \Omega}{\partial \boldsymbol{\sigma}}, \quad \dot{\boldsymbol{\chi}}^p = \frac{\partial \Omega}{\partial \boldsymbol{s}}, \quad \dot{\boldsymbol{K}} = \frac{\partial \Omega}{\partial \boldsymbol{M}}, \dot{q} = \frac{\partial \Omega}{\partial R} \tag{27}$$

The skew-symmetric parts of $\dot{\boldsymbol{H}}^p$ and $\dot{\boldsymbol{\chi}}^p$ must be provided to determine the associated plastic spins that are essentials for the description of the elastic-plastic behavior of anisotropic materials like metal polycrystals or composites.

The Micromorphic Approach to Gradient Elasticity, Viscoplasticity, and Damage

The previous micromorphic model can be extended to other types of additional degrees of freedom. This leads to a systematic approach for the construction of generalized continuum models with enriched kinematics. The method of thermomechanics with additional degrees is presented within the small deformation framework, following (Forest 2009). The reader is referred to Forest (2016) for the formulation at finite deformations.

The method amounts to enhancing any elasto-viscoplasticity model formulation within classical continuum thermodynamics according to Germain et al. (1983), Maugin (1999), by introducing suitable additional degrees of freedom. The small strain tensor is denoted by ε, whereas q represents the whole set of internal variables of arbitrary tensorial order accounting for nonlinear processes at work inside the material volume element, like isotropic and kinematic hardening variables. The absolute temperature is T. Additional degrees of freedom ϕ_χ are then introduced in the original model. They may be of any tensorial order and of different physical nature (deformation, plasticity, or damage variable). The notation $_\chi$ indicates that these variables eventually represent some microstructural features of the material so that we will call them micromorphic variables or microvariables (*microdeformation, microdamage*, etc.). The spaces of degrees of freedom and state variables are the following, respectively:

$$\left\{\boldsymbol{u}, \phi_\chi\right\}, \left\{\boldsymbol{\varepsilon}, T, q, \phi_\chi, \nabla\phi_\chi\right\} \tag{28}$$

Depending on the physical nature of ϕ_χ, it may or may not be a state variable. For instance, if the microvariable is a microrotation as in the Cosserat model, it is not a state variable for objectivity reasons and will appear in the state space only in combination with the macrorotation. In contrast, if the microvariable is a microplastic equivalent strain, as in Aifantis model, it then explicitly appears in the state space.

The method of virtual power can be used in the same way as for regular micromorphic media in order to introduce the generalized stresses and the balance laws they must satisfy. The virtual power of internal forces is a linear form with respect to the virtual fields and their gradients:

$$p^{(i)}\left(\boldsymbol{v}^{\star},\dot{\phi}_{\chi}^{\star}\right)=\boldsymbol{\sigma}:\nabla\boldsymbol{v}^{\star}+a\,\dot{\phi}_{\chi}^{\star}+\boldsymbol{b}.\nabla\dot{\phi}_{\chi}^{\star} \tag{29}$$

The Cauchy stress tensor σ is symmetric and a and $\boldsymbol{b}$ are generalized stresses associated with the micromorphic variable and its first gradient. The method of virtual power is used then to derive the standard local balance of momentum equation in the static case:

$$\text{div }\boldsymbol{\sigma}+\rho\boldsymbol{f}=0,\ \forall\boldsymbol{x}\in\Omega \tag{30}$$

and the generalized balance of micromorphic momentum equation:

$$\text{div }\boldsymbol{b}-a=0,\ \forall\boldsymbol{x}\in\Omega \tag{31}$$

The associated boundary conditions for the simple and generalized tractions are:

$$\boldsymbol{t}=\boldsymbol{\sigma}.\boldsymbol{n},\ a^{c}=\boldsymbol{b}.\boldsymbol{n},\ \forall\boldsymbol{x}\in\partial\mathcal{D} \tag{32}$$

where t is the traction vector and a^c a generalized traction.

The thermodynamic formulation and the constitutive laws follow the same lines as exposed for the regular micromorphic media. As a special case, the constitutive formulation is given here in the case of nondissipative contributions of the generalized stresses, because it plays an essential role in the construction of regularization operatores for ill-posed boundary value problems. Total strain is split into elastic and plastic parts:

$$\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}^{e}+\boldsymbol{\varepsilon}^{p} \tag{33}$$

The following constitutive functional dependencies are then introduced

$$\begin{aligned}\psi&=\widehat{\psi}\left(\boldsymbol{\varepsilon}^{e},T,q,\phi_{\chi},\nabla\phi_{\chi}\right),\\ \boldsymbol{\sigma}&=\widehat{\boldsymbol{\sigma}}\left(\boldsymbol{\varepsilon}^{e},T,q,\phi_{\chi},\nabla\phi_{\chi}\right),\\ \eta&=\widehat{\eta}\left(\boldsymbol{\varepsilon}^{e},T,q,\phi_{\chi},\nabla\phi_{\chi}\right)\\ a&=\widehat{a}\left(\boldsymbol{\varepsilon}^{e},T,q,\phi_{\chi},\nabla\phi_{\chi}\right),\\ \boldsymbol{b}&=\widehat{\boldsymbol{b}}\left(\boldsymbol{\varepsilon}^{e},T,q,\phi_{\chi},\nabla\phi_{\chi}\right)\end{aligned} \tag{34}$$

The state laws take the form:

$$\boldsymbol{\sigma}=\rho\frac{\partial\psi}{\partial\boldsymbol{\varepsilon}^{e}},\ \eta=-\frac{\partial\psi}{\partial T},\ R=\rho\frac{\partial\psi}{\partial q} \tag{35}$$

$$a=\rho\frac{\partial\psi}{\partial\phi_{\chi}},\ \boldsymbol{b}=\rho\frac{\partial\psi}{\partial\boldsymbol{\nabla}\phi_{\chi}} \tag{36}$$

The residual dissipation is

$$D^{res}=\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}^{p}-R\dot{q}-\frac{\boldsymbol{q}}{T}\nabla T\geq 0 \tag{37}$$

where R the thermodynamic force associated with the internal variable q. The existence of a dissipation potential, $\Omega(\boldsymbol{\sigma}, R)$ depending on the thermodynamic forces can then be assumed from which the evolution rules for internal variables are derived, that identically fulfill the entropy inequality provided the dissipation potential possesses suitable convexity properties:

$$\dot{\boldsymbol{\varepsilon}}^{p}=\frac{\partial\Omega}{\partial\boldsymbol{\sigma}},\ \dot{q}=-\frac{\partial\Omega}{\partial R} \tag{38}$$

After presenting the general approach, we readily give the most simple example which provides a direct connection to several existing generalized continuum models. An element ϕ is selected among the state variables (example: the accumulated plastic strain variable, p, or damage variable, d), or among other variables present in the classical model. The presentation is limited to cases where ϕ and ϕ_χ are observer invariant quantities. The free energy density function ψ is chosen as a function of the generalized relative strain variable e defined as:

$$e=\phi-\phi_{\chi} \tag{39}$$

thus introducing a coupling between macro and micromorphic variables. Assuming isotropic material behavior for brevity, the additional con-

M

tributions to the free energy can be taken as quadratic functions of e and $\nabla\phi_\chi$:

$$\psi\left(\boldsymbol{\varepsilon}, T, q, \phi_\chi, \nabla\phi_\chi\right) = \psi^{(1)}\psi^1\left(\boldsymbol{\varepsilon}, T, q\right) + \psi^{(2)}\left(e, \nabla\phi_\chi, T\right) \tag{40}$$

with

$$\rho\psi^{(2)}\left(e, \nabla\phi_\chi, T\right) = \frac{1}{2}H_\chi\left(\phi - \phi_\chi\right)^2 + \frac{1}{2}A\nabla\phi_\chi.\nabla\phi_\chi \tag{41}$$

where H_χ and A are the additional moduli introduced by the micromorphic model. The function $\psi^{(1)}$ $(\boldsymbol{\varepsilon}, T, q)$ refers to any constitutive function in a classical continuum thermomechanical model with internal variables. After inserting the state laws (36)

$$a = \rho\frac{\partial\psi}{\partial\phi_\chi} = -H_\chi\left(\phi - \phi_\chi\right), \quad \boldsymbol{b} = \rho\frac{\partial\psi}{\partial\nabla\phi_\chi} = A\nabla\phi_\chi \tag{42}$$

into the additional balance equation (31), the following partial differential equation is obtained, at least for a homogeneous material under isothermal conditions:

$$\phi = \phi_\chi - \frac{A}{H_\chi}\Delta\phi_\chi \tag{43}$$

where Δ is the Laplace operator. This type of equation is encountered at several places in the mechanics of generalized continua especially in the linear micromorphic theory (Mindlin 1964; Eringen 1999; Dillard et al. 2006) and in the so-called implicit gradient theory of plasticity and damage (Peerlings et al. 2001, 2004; Engelen et al. 2003). Note however that this equation corresponds to a special quadratic potential and represents the simplest micromorphic extension of the classical theory. It involves a characteristic length scale defined by:

$$l_c^2 = \frac{A}{H_\chi} \tag{44}$$

This length is real for positive values of the ratio A/H_χ. The additional material parameters H_χ and A are assumed to be positive for stability reasons. This does not exclude a softening material behavior that can be induced by the proper evolution of the internal variables (including $\phi = q$ itself). As an example, if $\phi = p$, the accumulated plastic strain in an isotropic classical plasticity model, with R as the yield stress function, and $\phi_\chi = p_\chi$, is a plastic microstrain variable, the isotropic hardening rule is:

$$R = \rho\frac{\partial\psi}{\partial p} = R_0(p) + a = R_0(p) - H_\chi\left(p - p_\chi\right) = R_0(p) - A\Delta p_\chi \tag{45}$$

where $R_0(p)$ is any classical hardening function. The classical isotropic hardening law is therefore modified by a Laplace operator. When the penalty modulus H_χ is 12 large enough, p_χ cannot be distinguished from p itself so that the micromorphic model degenerates into Aifantis celebrated strain gradient plasticity model (Aifantis 1984, 1987), involving the Laplacian of the accumulated plastic strain field. Note that this contribution, with $A > 0$, can regularize softening functions $R_0(p)$.

The reader is referred to Aslan and Forest (2011) for considerations of dissipative terms induced by the generalized stresses and for the relation between the micromorphic approach and phase field model (see also the entry "▶ Phase Field Models and Mechanics" in this Encyclopedia).

Micromorphic Media and Heterogeneous Materials

The micromorphic models introduce new kinematical degrees of freedom and several additional constitutive parameters like higher order elastic moduli or the moduli H_χ, A arising in the previous section. Even though these parameters can be identified from strain field measurements in appropriate experimental tests (Geers et al. 1998; Mazière et al. 2017), one may ask for their microstructural origin and try to find ways of derivation of the micromorphic model from the underlying microstructural discrete or continuum models. Atomic systems can be approximated by micromorphic media as discussed by Eringen (1999) and Chen and Lee (2003) who derived associated virial theorems. Homogenization methods for composite media must be extended to derive effective micromorphic theories from periodic or random Cauchy microcontinua. Such techniques are still under development but accounts of these can be found in Alibert et al. (2003), Forest (2012, 2014), Forest and Trinh (2011), Trinh et al. (2012), Nassar et al. (2016), and Hütter (2017a).

Application to Strain and Damage Localization Phenomena

Strain localization phenomena are overwhelming in the plasticity of metals, alloys, polymers, and geomaterials in the form of static or propagating shear bands (Mühlhaus and Vardoulakis 1987). They result from unstable material behavior induced by material softening. They can be predicted based a bifurcation analysis of the boundary value problem, see Besson et al. (2009). Strain and damage localization are precursors of crack initiation and subsequent structural failure. The finite element simulation of such localization phenomena is known to be associated with spurious mesh-dependence which makes it impossible to make post-bifurcation predictions of the overall structural response and local strain field. The ill-posedness of such boundary value problems can be restored by introducing intrinsic length scale(s) into the continuum mechanical setting (Mühlhaus 1995).

A simple one-dimensional example can be given in the case of microstrain gradient plasticity based on the enhanced hardening law (45). In the 1D case, the stress is uniform and equal to $R(p,\ p_\chi)$ under plastic loading conditions. This provides a second order differential equation for p_χ involving p. At the strain gradient plasticity limit $p \equiv p_\chi$, this differential equation is solved by harmonic functions for plastic strain for a linear softening material ($R_0(p) = R_0 + Hp$, $H < 0$). The localization band is described by a sinus arc of finite width that is directly related to the parameters H, H_χ, A, see Forest et al. (2005); Mazière and Forest (2015).

Localization and damage up to fracture surely belong to the most-promising domain of application of the micromorphic approach. Internal lengths are necessary for the reliable prediction of crack initiation, propagation, and crack path including crack branching. Successful examples are available for example in the case of the ductile fracture of metals and alloys, see (Dillard et al. 2006; Lorentz et al. 2008; Enakoutsa and Leblond 2009; Feld-Payet et al. 2015; Hütter 2017b).

Cross-References

▶ Computational Mechanics of Generalized Continua
▶ Cosserat Media
▶ Dislocations and Cracks in Generalized Continua
▶ Non Local Theories
▶ Phase Field Models and Mechanics
▶ Strain Gradient Plasticity
▶ Strain Gradient Theories
▶ Waves and Generalized Continua

M

References

Aifantis E (1984) On the microstructural origin of certain inelastic models. J Eng Mater Technol 106:326–330
Aifantis E (1987) The physics of plastic deformation. Int J Plast 3:211–248
Alibert J, Seppecher P, dell'Isola F (2003) Truss modular beams with deformation energy depending on higher displacement gradients. Math Mech Solids 8:51–73
Aslan O, Forest S (2011) The micromorphic versus phase field approach to gradient plasticity and damage with application to cracking in metal single crystals. In: de Borst R, Ramm E (eds) Multiscale methods in computational mechanics. Lecture notes in applied and computational mechanics, vol 55. Springer, pp 135–154
Aslan O, Cordero NM, Gaubert A, Forest S (2011) Micromorphic approach to single crystal plasticity and damage. Int J Eng Sci 49:1311–1325

Besson J, Cailletaud G, Chaboche JL, Forest S, Blétry M (2009) Non–linear mechanics of materials. Solid mechanics and its applications, vol 167. Springer, Berlin/Heidelberg

Chen Y, Lee J (2003) Connecting molecular dynamics to micromorphic theory. (I) Instantaneous and averaged mechanical variables. Phys A 322:359–376

Dillard T, Forest S, Ienny P (2006) Micromorphic continuum modelling of the deformation and fracture behaviour of nickel foams. Eur J Mech A Solids 25:526–549

Enakoutsa K, Leblond J (2009) Numerical implementation and assessment of the GLPD micromorphic model of ductile rupture. Eur J Mech A Solids 28:445–460

Engelen R, Geers M, Baaijens F (2003) Nonlocal implicit gradient-enhanced elasto-plasticity for the modelling of softening behaviour. Int J Plast 19:403–433

Eringen A (1999) Microcontinuum field theories. Springer, New York

Eringen A, Suhubi E (1964) Nonlinear theory of simple microelastic solids. Int J Eng Sci 2(189–203):389–404

Feld-Payet S, Chiaruttini V, Besson J, Feyel F (2015) A new marching ridges algorithm for crack path tracking in regularized media. Int J Solids Struct 71:57–69

Forest S (2009) The micromorphic approach for gradient elasticity, viscoplasticity and damage. ASCE J Eng Mech 135:117–131

Forest S (2012) Micromorphic media. In: Altenbach H, Eremeyev V (eds) Generalized continua – from the theory to engineering applications. CISM International Centre for Mechanical Sciences, courses and lectures, no. 541. Springer, pp 249–300

Forest S (2014) Asymptotic analysis of heterogeneous micromorphic elastic solids. In: Hetnarski R (ed) Encyclopedia of thermal stresses. Springer, Dordrecht, pp 239–251

Forest S (2016) Nonlinear regularisation operators as derived from the micromorphic approach to gradient elasticity, viscoplasticity and damage. Proc R Soc A 472:20150755. https://doi.org/10.1098/rspa.2015.0755

Forest S, Sab K (2017) Finite deformation second order micromorphic theory and its relations to strain and stress gradient models. Math Mech Solids. https://doi.org/10.1177/1081286517720844

Forest S, Sievert R (2003) Elastoviscoplastic constitutive frameworks for generalized continua. Acta Mech 160:71–111

Forest S, Sievert R (2006) Nonlinear microstrain theories. Int J Solids Struct 43:7224–7245

Forest S, Trinh DK (2011) Generalized continua and non–homogeneous boundary conditions in homogenization methods. ZAMM 91:90–109

Forest S, Blazy J, Chastel Y, Moussy F (2005) Continuum modelling of strain localization phenomena in metallic foams. J Mater Sci 40:5903–5910

Geers M, Rd B, Brekelmans W, Peerlings R (1998) On the use of local strain fields for the determination of the intrinsic length scale. J Phys IV 8:Pr8–167–174

Germain P (1973) The method of virtual power in continuum mechanics. Part 2: microstructure. SIAM J Appl Math 25:556–575

Germain P, Nguyen Q, Suquet P (1983) Continuum thermodynamics. J Appl Mech 50:1010–1020

Grammenoudis P, Tsakmakis C (2009) Micromorphic continuum part I: strain and stress tensors and their associated rates. Int J Non Linear Mech 44:943–956

Grammenoudis P, Tsakmakis C, Hofer D (2009) Micromorphic continuum part II: finite deformation plasticity coupled with damage. Int J Non Linear Mech 44: 957–974

Hirschberger C, Kuhl E, Steinmann P (2007) On deformational and configurational mechanics of micromorphic hyperelasticity – theory and computation. Comput Methods Appl Mech Eng 196:4027–4044

Hütter G (2017a) Homogenization of a cauchy continuum towards a micromorphic continuum. J Mech Phys Solids 99:394–408. https://doi.org/10.1016/j.jmps.2016.09.010

Hütter G (2017b) A micromechanical gradient extension of Gurson's model of ductile damage within the theory of microdilatational media. Int J Solids Struct 110–111:15–23. https://doi.org/10.1016/j.ijsolstr.2017.02.007

Kirchner N, Steinmann P (2005) A unifying treatise on variational principles for gradient and micromorphic continua. Philos Mag 85:3875–3895

Lazar M, Maugin GA (2007) On microcontinuum field theories: the eshelby stress tensor and incompatibility conditions. Philos Mag 87:3853–3870

Lorentz E, Besson J, Cano V (2008) Numerical simulation of ductile fracture with the Rousselier constitutive law. Comput Methods Appl Mech Eng 197:1965–1982

Mandel J (1973) Equations constitutives et directeurs dans les milieux plastiques et viscoplastiques. Int J Solids Struct 9:725–740

Maugin G (1999) Thermomechanics of nonlinear irreversible behaviors. World Scientific, Samuel Forest, Singapore

Mazière M, Forest S (2015) Strain gradient plasticity modeling and finite element simulation of lüders band formation and propagation. Contin Mech Thermodyn 27:83–104. https://doi.org/10.1007/s00161-013-0331-8

Mazière M, Luis C, Marais A, Forest S, Gaspérini M (2017) Experimental and numerical analysis of the Lüders phenomenon in simple shear. Int J Solids Struct 106–107:305–314

Mindlin R (1964) Micro–structure in linear elasticity. Arch Ration Mech Anal 16:51–78

Mühlhaus H (1995) Continuum models for materials with microstructure. Wiley, Chichester

Mühlhaus H, Vardoulakis I (1987) The thickness of shear bands in granular materials. Géotechnique 37:271–283

Nassar H, He QC, Auffray N (2016) A generalized theory of elastodynamic homogenization for periodic media. Int J Solids Struct 84:139–146. https://doi.org/10.1016/j.ijsolstr.2016.01.022

Peerlings R, Geers M, Rd B, Brekelmans W (2001) A critical comparison of nonlocal and gradient–enhanced softening continua. Int J Solids Struct 38:7723–7746

Peerlings R, Massart T, Geers M (2004) A thermodynamically motivated implicit gradient damage framework and its application to brick masonry cracking. Comput Methods Appl Mech Eng 193:3403–3417

Regueiro R (2010) On finite strain micromorphic elastoplasticity. Int J Solids Struct 47:786–800

Sansour C (1998a) A theory of the elastic–viscoplastic cosserat continuum. Arch Mech 50:577–597

Sansour C (1998b) A unified concept of elastic–viscoplastic Cosserat and micromorphic continua. J Phys IV 8:Pr8–341–348

Sansour C, Skatulla S, Zbib H (2010) A formulation for the micromorphic continuum at finite inelastic strains. Int J Solids Struct 47:1546–1554

Trinh DK, Jänicke R, Auffray N, Diebels S, Forest S (2012) Evaluation of generalized continuum substitution models for heterogeneous materials. Int J Multiscale Comput Eng 10:527–549. https://doi.org/10.1615/IntJMultCompEng.2012003105