

Thermoelasticity of second–grade media

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Abstract : The method of virtual power and continuum thermodynamics are used to incorporate temperature and temperature gradients into the theory of second grade solids settled by [Germain, 1973] in the isothermal case. In a second part, it is shown that heterogeneous classical materials submitted to slowly-varying mean fields can be replaced by a homogeneous equivalent medium including higher order gradients of displacement and temperature. For that purpose, an asymptotic analysis of thermoelastic heterogeneous periodic materials is performed. The form of the derived effective properties are compared to the previous phenomenological framework.

Keywords : second gradient, Cattaneo equation, virtual power, homogenization theory, asymptotic method.

1. INTRODUCTION AND NOTATIONS

Although the mechanical and constitutive framework of second gradient theory has been settled by Mindlin, the work of Germain [Germain, 1973] shows that the principle of virtual power is a powerful and elegant tool to derive the balance equations and boundary conditions for a medium modelled by the first and second gradients of the displacement field.

In several works including recent ones like [Boutin, 1996] and [Triantafyllidis, Bardenhagen, 1996], the need for such a theory arises in the mechanics of heterogeneous materials and in particular homogenization theory. The usual assumption that the size of the heterogeneities is much smaller than the size of the considered structure inevitably leads to a classical first gradient Cauchy medium to model a homogeneous equivalent medium (HEM). As soon as slowly-varying mean fields over the heterogeneities are possible (due to strong deformation gradients in a structure for instance), the HEM, if it exists, must be regarded as a generalized continuum

[Pideri, Seppecher, 1997] [Forest, 1998]. In the case of periodic media, asymptotic methods can be used to derive the effective properties of a second grade medium [Boutin, 1996] [Triantafyllidis, Bardenhagen, 1996].

The need for higher grade theories appears also in the thermomechanical framework. For instance, strong stress gradients can develop in a structure made of a heterogeneous material as a result of strong thermal gradients. Accordingly, a thermoelasticity theory must be developed for second grade media. This has been undertaken in [Cardona et al., 1998], where motivations and examples based on the mechanics of heterogeneous materials are given. Part 2 of the present work shows how the principle of virtual power combined with the thermodynamical principles can be used to settle a thermoelasticity theory of second grade media. The linear case is presented in part 3. In the last part, the asymptotic analysis of periodic heterogeneous classical media of [Boutin, 1996] is extended to the thermomechanical case, in order to show that the additional constitutive tensors arising in the phenomenological theory of part 2 can be computed explicitly as functions of the classical thermomechanical properties of the constituents. In this sense, the mechanics of heterogeneous materials provides an example of thermoelastic second grade effective medium, for which balance and constitutive equations have the form predicted by the basic principles of continuum thermomechanics.

An invariant notation is used throughout this work. First, second, third and fourth order tensors are respectively denoted by $\underline{\mathbf{a}}$, $\underline{\underline{\mathbf{a}}}$, $\underline{\underline{\underline{\mathbf{a}}}}$ (or $\underline{\underline{\underline{\mathbf{a}}}}$) and $\underline{\underline{\underline{\underline{\mathbf{a}}}}}$. The gradient and divergence operators are defined as follows

$$\nabla \underline{\mathbf{U}} = U_{i,j} \mathbf{e}_i \otimes \mathbf{e}_j, \quad \nabla \nabla \underline{\mathbf{U}} = U_{i,jk} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \quad (1)$$

$$\nabla \cdot \underline{\underline{\Sigma}} = \Sigma_{ij,j} \mathbf{e}_i, \quad \nabla \cdot \underline{\underline{\underline{\mathbf{S}}}} = S_{ijk,k} \mathbf{e}_i \otimes \mathbf{e}_j \quad (2)$$

where the nabla operator and an orthonormal basis $(\mathbf{e}_i)_{i=1,3}$ have been introduced : $\nabla = \frac{\partial}{\partial x_i} \mathbf{e}_i$. The notations for the contraction of tensors are :

$$\underline{\underline{\mathbf{A}}} : \underline{\underline{\mathbf{B}}} = A_{ij} B_{ij}, \quad \underline{\underline{\underline{\mathbf{K}}}} : \underline{\underline{\underline{\mathbf{L}}}} = K_{ijk} L_{ijk} \quad (3)$$

The gradient operator can be decomposed into its normal and tangent parts D_n and D_t , $\underline{\mathbf{n}}$ being the normal to a surface :

$$\nabla \underline{\mathbf{u}} = D_n \underline{\mathbf{u}} + D_t \underline{\mathbf{u}} \quad \text{with} \quad D_n \underline{\mathbf{u}} = (\nabla \underline{\mathbf{u}}) \cdot \underline{\mathbf{n}} \quad (4)$$

The gradient of the velocity field $\dot{\underline{\mathbf{U}}}$ on a solid Ω can be decomposed into a symmetric and a skew-symmetric part :

$$\nabla \dot{\underline{\mathbf{U}}} = \underline{\underline{\mathbf{D}}} + \underline{\underline{\mathbf{W}}} \quad (5)$$

For simplicity, the small perturbation framework is adopted, so that Lagrangian and Eulerian configurations are not distinguished. That is why tensor $\underline{\underline{\mathbf{D}}}$ will be replaced by $\dot{\underline{\underline{\mathbf{E}}}} = (\nabla \dot{\underline{\mathbf{U}}} + (\nabla \dot{\underline{\mathbf{U}}})^T)/2$. The boundary $\partial\Omega$ of body Ω will be regarded as twice differentiable and thus admits at each point a unique mean curvature R . The reader is referred to [Germain, 1973] for the treatment of edges and corners.

2.1 Principle of virtual power

The method of virtual power has proved to be an efficient tool for deriving balance equations and boundary conditions and has been applied to several coupled mechanical problems. Following [Maugin, 1980], the first step consists in defining the set \mathcal{V}° of virtual motions relevant for the considered physical situation. Within the present thermomechanical framework, velocity fields $\dot{\underline{\mathbf{U}}}$ and temperature rates \dot{T}^* are regarded as generalized virtual motions. It means that temperature is treated formally as an additional degree of freedom. The set \mathcal{V} then contains the variables that have to be introduced in a second grade theory :

$$\mathcal{V}^\circ = \{\dot{\underline{\mathbf{U}}}, \dot{T}\}, \quad \mathcal{V} = \{\dot{\underline{\mathbf{U}}}, \nabla \dot{\underline{\mathbf{U}}}, \nabla \nabla \dot{\underline{\mathbf{U}}}, \dot{T}, \nabla \dot{T}\} \quad (6)$$

The latter may be restricted to objective virtual fields \mathcal{V}^{obj} and an additional set \mathcal{V}^c is defined :

$$\mathcal{V}^{\text{obj}} = \{\underline{\mathbf{D}}, \nabla \underline{\mathbf{D}}, \dot{T}, \nabla \dot{T}\}, \quad \mathcal{V}^c = \{\dot{\underline{\mathbf{U}}}, D_n \dot{\underline{\mathbf{U}}}, \dot{T}\} \quad (7)$$

Since the small perturbation framework is used in this work, $\underline{\mathbf{D}}$ will be replaced by $\dot{\underline{\mathbf{E}}}$ in the following. The principle of virtual power states that, in a Galilean frame, the virtual power of inertial forces balances the virtual power of internal and external acting forces, for all generalized virtual motions ϑ^* and for all subdomain \mathcal{D} of body Ω :

$$\mathcal{P}^{(i)}(\vartheta^* \in \mathcal{V}^{\text{obj}}) + \mathcal{P}^{(d)}(\vartheta^* \in \mathcal{V}) + \mathcal{P}^{(c)}(\vartheta^* \in \mathcal{V}^c) = \mathcal{P}^{(a)}(\vartheta^* \in \mathcal{V}^\circ) \quad (8)$$

The virtual power of internal (i), volume (d), contact (c) and inertial forces (a) are supposed to admit densities according to :

$$\mathcal{P}^{(i)} = - \int_{\mathcal{D}} p^{(i)} dV, \quad \mathcal{P}^{(c)} = \int_{\partial \mathcal{D}} p^{(c)} dS, \quad \mathcal{P}^{(d,a)} = \int_{\mathcal{D}} p^{(d,a)} dV \quad (9)$$

The densities are then taken as linear forms on the appropriate set of generalized virtual motions :

$$p^{(i)}(\dot{\underline{\mathbf{U}}}, \dot{T}^*) = \underline{\underline{\Sigma}} : \dot{\underline{\mathbf{E}}}^* + \underline{\underline{\mathbf{S}}}\underline{\underline{\mathbf{}}} : \nabla \dot{\underline{\mathbf{E}}}^* + a^{(i)} \dot{T}^* + \underline{\mathbf{b}}^{(i)} \cdot \nabla \dot{T}^* \quad (10)$$

$$p^{(d)}(\dot{\underline{\mathbf{U}}}, \dot{T}^*) = \underline{\mathbf{f}} \cdot \dot{\underline{\mathbf{U}}}^* + \underline{\mathbf{C}} : \underline{\mathbf{W}}^* + \underline{\mathbf{F}} : \dot{\underline{\mathbf{E}}}^* + \underline{\underline{\mathbf{P}}}\underline{\underline{\mathbf{}}} : \nabla \dot{\underline{\mathbf{E}}}^* + a^{(d)} \dot{T}^* + \underline{\mathbf{b}}^{(d)} \cdot \nabla \dot{T}^* \quad (11)$$

$$p^{(c)}(\dot{\underline{\mathbf{U}}}, \dot{T}^*) = \underline{\mathbf{T}} \cdot \dot{\underline{\mathbf{U}}}^* + \underline{\underline{\mathbf{M}}}\underline{\underline{\mathbf{}}} \cdot (D_n \dot{\underline{\mathbf{U}}}^*) + a^{(c)} \dot{T}^*; \quad p^{(a)}(\dot{\underline{\mathbf{U}}}) = \rho \ddot{\underline{\mathbf{U}}} \cdot \dot{\underline{\mathbf{U}}}^* \quad (12)$$

The quantities dual to the strain rate and strain rate gradient in the power density of internal forces are the symmetric stress tensor $\underline{\underline{\Sigma}}$ and the hyperstress tensor $\underline{\underline{\mathbf{S}}}\underline{\underline{\mathbf{}}}$ ($S_{ijk} = S_{jik}$). Volume forces $\underline{\mathbf{f}}$, volume couples $\underline{\mathbf{C}}$ and volume double and triple forces $\underline{\mathbf{F}}$ and $\underline{\underline{\mathbf{P}}}\underline{\underline{\mathbf{}}}$ may exist in general. A traction vector $\underline{\mathbf{T}}$ and a normal surface double force $\underline{\underline{\mathbf{M}}}\underline{\underline{\mathbf{}}}$ may act on the domain surface. The acceleration at a given material point reads $\ddot{\underline{\mathbf{U}}}$. For the sake of generality, additional terms associated with virtual temperature rates and their gradients have been systematically introduced. Note that a similar

term should in principle appear in (12)₂. However it will be assumed that, if such additional terms exist in the power of acceleration forces and in the expression of the kinetic energy \mathcal{K} , they are still such that :

$$\mathcal{P}^{(a)} = \dot{\mathcal{K}} \quad (13)$$

Note that the homogenization procedure developed in section 4 and restricted to the static case, should be applied to the full dynamical case in order to justify the existence or not of additional terms in $\mathcal{P}^{(a)}$ and \mathcal{K} .

The application of the principle of virtual power and the successive use of Stokes theorem for volumes and surfaces as shown in [Germain, 1973] and in [Cardona et al., 1998] lead to the balance equations

$$\nabla \cdot \underline{\boldsymbol{\tau}} + \underline{\mathbf{f}} = \rho \ddot{\underline{\mathbf{U}}} \quad \text{with} \quad \underline{\boldsymbol{\tau}} = \underline{\boldsymbol{\Sigma}} - \underline{\mathbf{F}} - \underline{\mathbf{C}} - \nabla \cdot \underline{\boldsymbol{\underline{\underline{S}}}} + \nabla \cdot \underline{\boldsymbol{\underline{\underline{P}}}} \quad (14)$$

$$\nabla \cdot \underline{\mathbf{b}} - a = 0 \quad \text{with} \quad a = a^{(i)} - a^{(e)} \quad \text{and} \quad \underline{\mathbf{b}} = \underline{\mathbf{b}}^{(i)} - \underline{\mathbf{b}}^{(e)} \quad (15)$$

and associated boundary conditions :

$$\underline{\mathbf{T}} = \underline{\boldsymbol{\tau}} \cdot \underline{\mathbf{n}} + 2R\underline{\boldsymbol{\underline{\underline{S}}}} : (\underline{\mathbf{n}} \otimes \underline{\mathbf{n}}) - D_t(\underline{\boldsymbol{\underline{\underline{S}}}} \cdot \underline{\mathbf{n}}); \underline{\mathbf{M}} = \underline{\boldsymbol{\underline{\underline{S}}}} : (\underline{\mathbf{n}} \otimes \underline{\mathbf{n}}); a^{(c)} = \underline{\mathbf{b}} \cdot \underline{\mathbf{n}} \quad (16)$$

Note that the equations (15) and (16)₃ can be regarded as a definition of a and $a^{(c)}$ depending on $\underline{\mathbf{b}}$, and thus the formulation does not require any additional partial differential equation to be solved. In the classical first gradient theory, the term $a^{(i)}\dot{T}$ could also be introduced in the power density of internal forces, but, for vanishing $a^{(e)}$, it would have no counterpart $\underline{\mathbf{b}}^{(i)}$ so that the application of the principle finally implies $a^{(i)} = 0$. As a result, the proposed thermomechanical framework is relevant for a second grade theory and reduces to the classical case if higher order gradients are excluded. Similarly, the subsequent developments will show that a constitutive dependence on temperature gradient that seems to be necessary for instance in the modelling on some thermal treatments, can be introduced only within the framework of a second grade theory for the displacement.

2.2 Energy and entropy principles

The global form of the energy balance equation on Ω takes the forms :

$$\dot{\mathcal{E}} + \dot{\mathcal{K}} = \mathcal{P}^{(d)} + \mathcal{P}^{(c)} + \mathcal{Q}_e; \quad \dot{\mathcal{E}} = -\mathcal{P}^{(i)} + \mathcal{Q}_e \quad (17)$$

where \mathcal{E} is the internal energy of the system and \mathcal{Q}_e the total heat supply. In the second form, the kinetic energy theorem (13) and the principle of virtual power (8) have been applied. Introducing the specific internal energy ϵ , the heat flux vector $\underline{\mathbf{Q}}$, and an inner heat production rate r ,

$$\dot{E} = \int_{\Omega} \rho \dot{\epsilon} dV; \quad \mathcal{Q} = - \int_{\partial\Omega} \underline{\mathbf{Q}} \cdot \underline{\mathbf{n}} dS + \int_{\Omega} r dV \quad (18)$$

a local form of the energy balance is obtained :

$$\rho \dot{\epsilon} = p^{(i)} - \nabla \cdot \underline{\mathbf{Q}} + r \quad (19)$$

where the expression (10) of the power density of internal forces is to be substituted.

The global form of the second principle reads : $\dot{\mathcal{S}} \geq \mathcal{Q}_s$, where \mathcal{S} is the global entropy of the system and \mathcal{Q}_s is the total flux of entropy. Introducing the local specific entropy s , the following relations are assumed to hold :

$$\mathcal{S} = \int_{\Omega} \rho s \, dV, \quad \mathcal{Q}_s = - \int_{\partial\Omega} \underline{\mathbf{J}}_s \cdot \underline{\mathbf{n}} \, dS + \int_{\Omega} \frac{r}{T} \, dV \quad \text{and} \quad \underline{\mathbf{J}}_s = \frac{\underline{\mathbf{Q}}}{T} \quad (20)$$

where $\underline{\mathbf{J}}_s$ is the entropy flux vector. A local form of the entropy imbalance is adopted :

$$\rho \dot{s} + \nabla \cdot \left(\frac{\underline{\mathbf{Q}}}{T} \right) - \frac{r}{T} \geq 0 \quad (21)$$

Combining (19) and (21) and introducing the Helmholtz free energy $\Psi = \epsilon - Ts$, we get the Clausius-Duhem inequality :

$$-\rho(\dot{\Psi} + s\dot{T}) + p^{(i)} - \frac{1}{T} \underline{\mathbf{Q}} \cdot \nabla T \geq 0 \quad (22)$$

In the case of hyperelastic material behaviour, the specific free energy is a function of $(\underline{\mathbf{E}}, \underline{\mathbf{K}} = \nabla \underline{\mathbf{E}}, T, \nabla T)$. The Clausius-Duhem inequality (22) can then be expanded as follows :

$$\begin{aligned} (\underline{\underline{\Sigma}} - \rho \frac{\partial \Psi}{\partial \underline{\mathbf{E}}}) : \dot{\underline{\mathbf{E}}} &+ (\underline{\underline{\mathbf{S}}} - \rho \frac{\partial \Psi}{\partial \underline{\mathbf{K}}}) : \dot{\underline{\mathbf{K}}} + (a^{(i)} - \rho s - \rho \frac{\partial \Psi}{\partial T}) \dot{T} \\ &+ (\underline{\mathbf{b}}^{(i)} - \rho \frac{\partial \Psi}{\partial \nabla T}) \cdot (\nabla \dot{T}) - \frac{\underline{\mathbf{Q}}}{T} \cdot (T \underline{\nabla}) \geq 0 \end{aligned} \quad (23)$$

from which the state laws can be deduced :

$$\underline{\underline{\Sigma}} = \rho \frac{\partial \Psi}{\partial \underline{\mathbf{E}}}; \quad \underline{\underline{\mathbf{S}}} = \rho \frac{\partial \Psi}{\partial \underline{\mathbf{K}}}; \quad \underline{\mathbf{b}}^{(i)} = \rho \frac{\partial \Psi}{\partial (\nabla T)}; \quad \rho s = -\rho \frac{\partial \Psi}{\partial T} + a^{(i)} \quad (24)$$

When compared to the isothermal second grade theory in [Germain, 1973], the proposed thermodynamical framework takes full account of the introduction of variable ∇T in the constitutive modelling. The main consequence is the modification of the entropy by the term $a^{(i)}$ which, according to the balance equation (15), is nothing but the divergence of the generalized thermodynamical force $\underline{\mathbf{b}}$ associated with the temperature gradient. As a result, dissipation is reduced to its thermal part :

$$D = -\frac{\underline{\mathbf{Q}}}{T} \cdot \nabla T \quad (25)$$

2.4 Alternative formulations

Alternative thermodynamical formulations of second grade thermoelasticity exist and some of them have been described in [Cardona *et al.*, 1998]. The common feature of the alternative formulations is to avoid the introduction of additional terms in the power density of internal forces (10) and the modifications may then

appear in the energy balance (thus extending the treatment of [Dunn and Serrin, 1985] in the isothermal case) or in the entropy flux as recommended in [Maugin, 1990] for the introduction of gradients of internal variables in the constitutive framework. According to the latter procedure, the \dot{T} and ∇T terms can be dropped in (10) and an extra-entropy flux must be defined :

$$\underline{\mathbf{J}}_s = \frac{\underline{\mathbf{Q}}}{T} + \underline{\mathbf{k}}, \quad \text{with } \underline{\mathbf{k}} = \frac{\dot{T}}{T} \underline{\mathbf{b}}, \quad \text{and } \underline{\mathbf{b}} = \rho \partial \Psi / \partial (\nabla T) \quad (26)$$

This framework however leads to the same state laws as (24), but the classical heat conduction inequality is changed leading to a non-classical Fourier's law modified by the extra-entropy flux $\underline{\mathbf{k}}$.

3. SECOND GRADE LINEAR THERMOELASTICITY

The previous general framework is explicitated in the case of linear thermoelasticity in the static case and the associated constitutive properties are derived. The additional terms arising then in the heat equation are discussed.

3.1 Linearized constitutive equations

A reference thermomechanical state ($\underline{\mathbf{E}}_0 = 0, \underline{\mathbf{K}}_0 = 0, T_0, (\nabla T)_0 = 0$) is considered and the kinematic, balance and constitutive equations are linearized with respect to this reference state. The small perturbation framework requires sufficiently small strains, strain gradients, temperature changes and temperature gradients. The relevant variables then are :

$$\underline{\tilde{\mathbf{E}}} = e(\underline{\mathbf{U}}), \quad \underline{\tilde{\mathbf{K}}} = \nabla \underline{\tilde{\mathbf{E}}}, \quad \Delta = T - T_0, \quad \nabla T \quad (27)$$

where the symmetric gradient operator e means $e(\underline{\mathbf{U}}) = (U_{i,j} + U_{j,i})/2 \underline{\mathbf{e}}_i \otimes \underline{\mathbf{e}}_j$. The free energy is then taken as a quadratic form in all these variables :

$$\begin{aligned} \rho \Psi &= \frac{1}{2} \underline{\tilde{\mathbf{E}}} : \underline{\tilde{\mathbf{C}}} : \underline{\tilde{\mathbf{E}}} - \Delta \underline{\tilde{\mathbf{E}}} : \underline{\tilde{\mathbf{C}}} : \underline{\boldsymbol{\alpha}} - \frac{1}{2} \beta \Delta^2 + \frac{1}{2} \underline{\tilde{\mathbf{K}}} : \underline{\tilde{\mathbf{A}}} : \underline{\tilde{\mathbf{K}}} + \underline{\tilde{\mathbf{E}}} : \underline{\tilde{\mathbf{M}}} : \underline{\tilde{\mathbf{K}}} + \Delta \underline{\tilde{\mathbf{K}}} : \underline{\tilde{\mathbf{H}}} \\ &+ \frac{1}{2} \nabla T : \underline{\tilde{\mathbf{B}}} : \nabla T + \Delta \underline{\tilde{\mathbf{F}}} : \nabla T - \underline{\tilde{\mathbf{K}}} : \left((\underline{\tilde{\mathbf{A}}} : \underline{\tilde{\mathbf{P}}}) : \nabla T \right) + \underline{\tilde{\mathbf{E}}} : \underline{\tilde{\mathbf{N}}} : \nabla T \end{aligned} \quad (28)$$

The state laws follow from (24) :

$$\underline{\tilde{\boldsymbol{\Sigma}}} = \underline{\tilde{\mathbf{C}}} : (\underline{\tilde{\mathbf{E}}} - \Delta \underline{\boldsymbol{\alpha}}) + \underline{\tilde{\mathbf{M}}} : \underline{\tilde{\mathbf{K}}} + \underline{\tilde{\mathbf{N}}} : \nabla T \quad (29)$$

$$\underline{\tilde{\mathbf{S}}} = \underline{\tilde{\mathbf{A}}} : (\underline{\tilde{\mathbf{K}}} - \underline{\tilde{\mathbf{P}}} \otimes \nabla T) + \underline{\tilde{\mathbf{M}}} : \underline{\tilde{\mathbf{E}}} + \Delta \underline{\tilde{\mathbf{H}}} \quad (30)$$

$$\underline{\mathbf{b}}^{(i)} = \underline{\tilde{\mathbf{N}}} : \underline{\tilde{\mathbf{E}}} - (\underline{\tilde{\mathbf{A}}} : \underline{\tilde{\mathbf{P}}}) : \underline{\tilde{\mathbf{K}}} + \Delta \underline{\tilde{\mathbf{F}}} + \underline{\tilde{\mathbf{B}}} : \nabla T \quad (31)$$

$$\rho s = (\underline{\tilde{\mathbf{C}}} : \underline{\boldsymbol{\alpha}}) : \underline{\tilde{\mathbf{E}}} - \underline{\tilde{\mathbf{H}}} : \underline{\tilde{\mathbf{K}}} + \beta \Delta - \underline{\tilde{\mathbf{F}}} : \nabla T + a^{(i)} \quad (32)$$

in which classical and additional terms can be recognized. In particular, there exist a thermal eigenstrain $\Delta\boldsymbol{\alpha}$ and, in the same way, an eigen-(strain gradient) $\underline{\mathbf{P}} \otimes \nabla T$, the interpretation of which is given in [Cardona et al., 1998]. In the linearized scheme, the introduced constitutive properties are independent from temperature. The constitutive equations can be written as a single relation linking the effective stress tensor $\boldsymbol{\tau} = \underline{\boldsymbol{\Sigma}} - \nabla \cdot \underline{\mathbf{S}}$ and strain and temperature gradients :

$$\boldsymbol{\tau} = \underline{\underline{\mathbf{C}}} : (\underline{\mathbf{E}} - \Delta\boldsymbol{\alpha}) + (\underline{\underline{\mathbf{M}}} - \underline{\underline{\mathbf{M}}^*}) : \underline{\underline{\mathbf{K}}} + (\underline{\underline{\mathbf{N}}} - \underline{\underline{\mathbf{H}}}) \cdot \nabla T - \underline{\underline{\mathbf{A}}} : (\nabla \underline{\underline{\mathbf{K}}} - \underline{\underline{\mathbf{P}}} \otimes (\nabla \nabla T)) \quad (33)$$

where $M_{ijppqr}^* = M_{pqijr}$. In this expression, second order strain gradients and second order temperature gradients necessarily appear.

3.2 Linearized heat equation

The heat equation is deduced from the energy equation (19) and takes the form :

$$\rho T \dot{s} = -\nabla \cdot \underline{\mathbf{Q}} + r \quad (34)$$

Substituting the linear state laws in (34) and taking the fact that $a^{(i)}$ is nothing but the divergence of $\underline{\mathbf{h}}^{(i)}$ (setting $\underline{\mathbf{h}}^{(e)}$ to zero without loss of generality) into account, the linearized heat equation is obtained :

$$T \beta \dot{T} = r - \underline{\mathbf{Q}} \cdot \underline{\nabla} - T \dot{\underline{\mathbf{E}}} : \underline{\underline{\mathbf{C}}} : \boldsymbol{\alpha} + T (\underline{\underline{\mathbf{A}}} : \underline{\underline{\mathbf{P}}}) :: \nabla \underline{\underline{\mathbf{K}}} - T \underline{\underline{\mathbf{B}}} : (\nabla \nabla \dot{T}) - T (\underline{\underline{\mathbf{N}}} - \underline{\underline{\mathbf{H}}}) : \underline{\underline{\mathbf{K}}} \quad (35)$$

In the isotropic case, all odd order constitutive tensors vanish so that the last term disappears. It turns out that the proposed approach leads to an additional thermo-mechanical coupling term $(\underline{\underline{\mathbf{A}}} : \underline{\underline{\mathbf{P}}}) :: \nabla \underline{\underline{\mathbf{K}}}$ in addition to the classical term $\dot{\underline{\mathbf{E}}} : \underline{\underline{\mathbf{C}}} : \boldsymbol{\alpha}$.

It can be noted that a generalized specific heat can be defined [Cardona et al., 1998], the positivity of which is preserved at least in the present linearized case. In the purely thermal case, the heat equation is also modified as indicated below. Introducing a linear relationship between the heat flux vector and the temperature gradient :

$$\underline{\mathbf{Q}} = -\boldsymbol{\kappa} \cdot \nabla T \quad (36)$$

that still identically fulfills the positive thermal dissipation requirement (25). The purely thermal part of the heat equation reduces, in the isotropic case, to :

$$\beta \dot{T} = \kappa \nabla \cdot \nabla T - B \nabla \cdot \nabla \dot{T} \quad (37)$$

where $\nabla \cdot \nabla$ simply is the Laplacian operator. It is interesting to notice that this equation is identical to the first Cattaneo equation presented for instance in [Müller and Ruggeri, 1993]. Cattaneo's argument is rather based on a modification of Fourier's law :

$$\underline{\mathbf{Q}} = -\kappa \nabla T + B \nabla \dot{T} \quad (38)$$

which may lead to up-hill heat diffusion, contrary to the present formulation. However Cattaneo proposed a second modification of Fourier's law aiming at restoring

the hyperbolic character of the heat equation. This modification falls into the framework of extended thermodynamics and is not contained in our formulation.

4. ASYMPTOTIC ANALYSIS OF HETEROGENEOUS MATERIALS

The aim of this section is to show that the additional constitutive properties associated with higher order strain and temperature gradients that have been introduced in a purely phenomenological manner in the previous sections, can be deduced from an analysis of the effective properties of a heterogeneous material subjected to slowly-varying mean fields. For simplicity, classical heterogeneous materials with periodic microstructure are considered, in the static case. In a first analysis, the local temperature field is assumed to be known and the fully coupled thermomechanical problem is treated at the end. The main features of the constitutive framework for thermoelastic second grade media proposed in section 2 and 3 are confirmed by the following analysis but additional ones are pointed out in the last section.

4.1 Field equations at the local scale

The linear thermomechanical properties of a heterogeneous classical material, adequately represented by a Cauchy continuum, are considered. The aim is to deduce the global properties of an homogeneous equivalent medium from the local properties. The local and global (effective) variables (free energy, deformation, stress, temperature, temperature change, specific entropy and heat flux) must be distinguished :

$$\text{local variables} : \quad \psi, \quad \underline{\underline{\boldsymbol{\varepsilon}}}, \quad \underline{\underline{\boldsymbol{\sigma}}}, \quad \theta, \quad \delta = \theta - \theta_{ref}, \quad \eta, \quad \underline{\mathbf{q}} \quad (39)$$

$$\text{global variables} : \quad \Psi, \quad \underline{\mathbf{E}}, \quad \underline{\underline{\boldsymbol{\Sigma}}}, \quad T, \quad \Delta = T - T_0, \quad s, \quad \underline{\mathbf{Q}} \quad (40)$$

The heterogeneous material is described locally by the following free energy with constitutive properties varying in space but independent from temperature (linearized formulation) :

$$\rho \psi(\underline{\underline{\boldsymbol{\varepsilon}}}, \theta) = \frac{1}{2} \underline{\underline{\boldsymbol{\varepsilon}}} : \underline{\underline{\mathbf{C}}} : \underline{\underline{\boldsymbol{\varepsilon}}} - \delta \underline{\underline{\boldsymbol{\varepsilon}}} : \underline{\underline{\boldsymbol{\alpha}}} : \underline{\underline{\boldsymbol{\alpha}}} - \frac{1}{2} \beta \delta^2 \quad (41)$$

The local specific heat for constant deformation is related to parameter β through :

$$\rho C_\varepsilon = \theta \rho \left(\frac{\partial \eta}{\partial \theta} \right)_{\underline{\underline{\boldsymbol{\varepsilon}}}} = \theta \beta = -\theta \frac{\partial^2 \psi}{\partial \theta^2} \quad (42)$$

The equations of the linear coupled thermomechanical problem \mathcal{P} to be solved on the heterogenous material are Hooke's and Fourier's laws and the balance of momentum and energy :

$$\underline{\underline{\boldsymbol{\sigma}}} = \underline{\underline{\boldsymbol{\alpha}}} : (\underline{\underline{\boldsymbol{\varepsilon}}} - \delta \underline{\underline{\boldsymbol{\alpha}}}); \quad \nabla \cdot \underline{\underline{\boldsymbol{\sigma}}} + \underline{\mathbf{f}} = 0 \quad (43)$$

$$\underline{\mathbf{q}} = -\underline{\underline{\boldsymbol{\lambda}}} \cdot \nabla T; \quad \beta \dot{\theta} = -\nabla \cdot \underline{\mathbf{q}} - \theta \underline{\underline{\boldsymbol{\alpha}}} : \underline{\underline{\boldsymbol{\alpha}}} : \underline{\underline{\boldsymbol{\varepsilon}}} + r \quad (44)$$

where $\underline{\mathbf{q}}$ stands in fact for $\underline{\mathbf{q}}/\theta_{ref}$, r for r/θ_{ref} and $\underline{\boldsymbol{\lambda}}$ for Fourier's heat conduction tensor divided by θ_{ref} . This convention holds for the remaining of this part. The unknowns fields are the local displacement $\underline{\mathbf{u}}(\underline{\mathbf{x}})$ and temperature $\theta(\underline{\mathbf{x}})$.

In this part, the problem is restricted to an infinite body, so that the additional problems associated with boundary conditions are not addressed. Initial conditions for the evolution problem close the formulation of \mathcal{P} .

4.2 Dimensional analysis and asymptotic developments

For simplicity, it is assumed that the heterogeneous material admits a periodic microstructure that can be described by a unit cell Y^l of characteristic size l . The macroscopic scale is characterized by a typical wave length L_ω of variation of the mean fields like overall stress and strain. In a finite body, L_ω may be of the order of magnitude of its size L . The dimension analysis performed below provides the small parameters involved in the thermomechanical problem, to be taken into account in a homogenization procedure.

Dimensionless space coordinates, displacements, time and temperature, several operators and constitutive properties can be defined :

$$\underline{\mathbf{x}}^* = \underline{\mathbf{x}}/L_\omega, \quad \underline{\mathbf{u}}^* = \underline{\mathbf{u}}/L_\omega, \quad t^* = t/\bar{t}, \quad \theta^* = \theta/\theta_{ref} \quad (45)$$

$$e(\underline{\mathbf{u}}) = e^*(\underline{\mathbf{u}}^*), \quad \nabla^* = L_\omega \nabla, \quad \nabla \theta = \frac{\theta_{ref}}{L_\omega} \nabla^* \theta^* \quad (46)$$

$$\underline{\mathbf{c}} = \bar{c} \underline{\mathbf{c}}^*, \quad \underline{\boldsymbol{\alpha}} = \bar{\alpha} \underline{\boldsymbol{\alpha}}^*, \quad \underline{\boldsymbol{\lambda}} = \frac{\bar{c} \bar{\alpha}}{\theta_{ref} \bar{t}} l_\lambda^2 \underline{\boldsymbol{\lambda}}^*, \quad \beta = \frac{\bar{c} \bar{\alpha}}{\theta_{ref}} \left(\frac{l_\lambda}{L_\omega} \right)^2 \frac{t_\beta}{\bar{t}} \beta^* \quad (47)$$

where reference time \bar{t} and characteristic time t_β and length l_λ have been introduced. A dimensionless formulation \mathcal{P}^* of the thermomechanical problem \mathcal{P} can then be expressed on a dimensionless unit cell Y^* :

$$\nabla^* \cdot \underline{\boldsymbol{\sigma}}^* + \underline{\mathbf{f}}^* = 0 \quad \text{with} \quad \underline{\boldsymbol{\sigma}}^* = \underline{\boldsymbol{\sigma}}/\bar{c} = \underline{\mathbf{c}}^* : (e^*(\underline{\mathbf{u}}^*) - \bar{\alpha} \theta_{ref} \delta^* \boldsymbol{\alpha}^*), \quad \underline{\mathbf{f}}^* = \frac{L_\omega}{\bar{c}} \underline{\mathbf{f}} \quad (48)$$

$$-\nabla^* \cdot \underline{\mathbf{q}}^* - \frac{\partial e^*(\underline{\mathbf{u}}^*)}{\partial t^*} : \underline{\mathbf{c}}^* : \underline{\boldsymbol{\alpha}}^* = \varepsilon_\beta \varepsilon_\lambda^2 \beta^* \frac{\partial \theta^*}{\partial t^*}, \quad \text{with} \quad \underline{\mathbf{q}}^* = \frac{\bar{c} \bar{\alpha} L_\omega}{\bar{t}} \underline{\mathbf{q}} \quad (49)$$

The characteristic numbers of the thermomechanical problem then are :

$$\varepsilon = \frac{l}{L_\omega}, \quad \varepsilon_\lambda = \frac{l_\lambda}{L_\omega}, \quad \varepsilon_\beta = \frac{t_\beta}{\bar{t}} \quad (50)$$

Note that another characteristic length appears if boundary conditions like heat convection are introduced but this is not treated here.

In the following a homogenization procedure is considered for which ε is the relevant small parameter whereas ε_λ and ε_β are regarded as constant and independent from ε . For that purpose and similarly to classical multiscale asymptotic methods used in periodic homogenization [Sanchez-Palencia, 1980], a series of problems

$(\mathcal{P}_\varepsilon)_{\varepsilon>0}$ is considered. Once the small parameter of the problem has been chosen after considering the dimensionless problem \mathcal{P}^* , it is possible to define each \mathcal{P}_ε on Y

$$Y = \{\underline{\mathbf{y}} \mid \underline{\mathbf{y}} = \underline{\mathbf{x}}/\varepsilon, \underline{\mathbf{x}} \in Y^l\} \quad (51)$$

and the equations of \mathcal{P}_ε are chosen to be the equations (43) and (44) in which displacements, temperature, stress and heat flux have to be replaced by $\underline{\mathbf{u}}^\varepsilon, \theta^\varepsilon, \underline{\boldsymbol{\sigma}}^\varepsilon$ and $\underline{\mathbf{q}}^\varepsilon$. Each constitutive tensor a appearing in these equations is to be replaced by a^ε such that

$$a^\varepsilon(\underline{\mathbf{y}}) = a(\varepsilon \underline{\mathbf{y}}) \quad (52)$$

The limiting case obtained for $\varepsilon \rightarrow 0$ gives the balance and constitutive equations of a homogeneous equivalent medium. This is recalled in the next section where the corrections for non-vanishing ε are also investigated since they are relevant when the macroscopic mean fields are not strictly constant but slowly varying. All fields are regarded as functions of the two variables $\underline{\mathbf{x}}$ and $\underline{\mathbf{y}}$, that can be expanded in a series of powers of small parameter ε :

$$\underline{\mathbf{u}}^\varepsilon(\underline{\mathbf{x}}) = \underline{\mathbf{u}}_0(\underline{\mathbf{x}}, \underline{\mathbf{y}}) + \varepsilon \underline{\mathbf{u}}_1(\underline{\mathbf{x}}, \underline{\mathbf{y}}) + \varepsilon^2 \underline{\mathbf{u}}_2(\underline{\mathbf{x}}, \underline{\mathbf{y}}) + \varepsilon^3 \underline{\mathbf{u}}_3(\underline{\mathbf{x}}, \underline{\mathbf{y}}) + \dots \quad (53)$$

$$\theta^\varepsilon(\underline{\mathbf{x}}) = \theta_0(\underline{\mathbf{x}}, \underline{\mathbf{y}}) + \varepsilon \theta_1(\underline{\mathbf{x}}, \underline{\mathbf{y}}) + \varepsilon^2 \theta_2(\underline{\mathbf{x}}, \underline{\mathbf{y}}) + \varepsilon^3 \theta_3(\underline{\mathbf{x}}, \underline{\mathbf{y}}) + \dots \quad (54)$$

$$\underline{\boldsymbol{\sigma}}^\varepsilon(\underline{\mathbf{x}}) = \underline{\boldsymbol{\sigma}}_0(\underline{\mathbf{x}}, \underline{\mathbf{y}}) + \varepsilon \underline{\boldsymbol{\sigma}}_1(\underline{\mathbf{x}}, \underline{\mathbf{y}}) + \varepsilon^2 \underline{\boldsymbol{\sigma}}_2(\underline{\mathbf{x}}, \underline{\mathbf{y}}) + \varepsilon^3 \underline{\boldsymbol{\sigma}}_3(\underline{\mathbf{x}}, \underline{\mathbf{y}}) + \dots \quad (55)$$

$$\underline{\mathbf{q}}^\varepsilon(\underline{\mathbf{x}}) = \underline{\mathbf{q}}_0(\underline{\mathbf{x}}, \underline{\mathbf{y}}) + \varepsilon \underline{\mathbf{q}}_1(\underline{\mathbf{x}}, \underline{\mathbf{y}}) + \varepsilon^2 \underline{\mathbf{q}}_2(\underline{\mathbf{x}}, \underline{\mathbf{y}}) + \varepsilon^3 \underline{\mathbf{q}}_3(\underline{\mathbf{x}}, \underline{\mathbf{y}}) + \dots \quad (56)$$

where the terms of the series are Y -periodic with respect to the second variable. In order to put these expressions into the balance and constitutive equations of \mathcal{P}_ε , the gradient operator can be split into partial derivatives with respect to $\underline{\mathbf{x}}$ and $\underline{\mathbf{y}}$:

$$\nabla = \nabla_x + \frac{1}{\varepsilon} \nabla_y, \quad e(\cdot) = e_x(\cdot) + \frac{1}{\varepsilon} e_y(\cdot) \quad (57)$$

4.3 Derivation of the effective balance and constitutive equations

The expansion of stress and heat flux are introduced into the equations of balance of momentum and energy. The terms can be ordered with respect to the powers of ε . Identifying the terms of same order, the different contributions in the expansions (53) to (56) can be shown to be solution of the following auxiliary problems on Y :

Problem \mathcal{A}_0 :

$$e_y(\underline{\mathbf{u}}_0) = 0; \quad \nabla_y \theta_0 = 0 \quad (58)$$

Problem \mathcal{A}_1 :

$$\underline{\boldsymbol{\sigma}}_0 = \underline{\underline{\mathbf{c}}}^\varepsilon : (e_x(\underline{\mathbf{u}}_0) + e_y(\underline{\mathbf{u}}_1) - \underline{\boldsymbol{\alpha}}^\varepsilon \delta_0); \quad \nabla_y \cdot \underline{\boldsymbol{\sigma}}_0 = 0 \quad (59)$$

$$\underline{\mathbf{q}}_0 = -\lambda^\varepsilon (\nabla_x \theta_0 + \nabla_y \theta_1); \quad \nabla_y \cdot \underline{\mathbf{q}}_0 = 0 \quad (60)$$

Problem \mathcal{A}_2 :

$$\underline{\boldsymbol{\sigma}}_1 = \underline{\underline{\mathbf{c}}}^\varepsilon : (e_x(\underline{\mathbf{u}}_1) + e_y(\underline{\mathbf{u}}_2) - \underline{\boldsymbol{\alpha}}^\varepsilon \delta_1); \quad \nabla_x \cdot \underline{\boldsymbol{\sigma}}_0 + \nabla_y \cdot \underline{\boldsymbol{\sigma}}_1 + \underline{\mathbf{f}}^\varepsilon = 0 \quad (61)$$

$$\underline{\mathbf{q}}_1 = -\lambda^\varepsilon (\nabla_x \theta_1 + \nabla_y \cdot \theta_2); \quad -\nabla_x \cdot \underline{\mathbf{q}}_0 - \nabla_y \cdot \underline{\mathbf{q}}_1 - \underline{\underline{\mathbf{c}}}^\varepsilon : \underline{\underline{\boldsymbol{\alpha}}}^\varepsilon : (e_x(\underline{\mathbf{u}}_0) + e_y(\underline{\mathbf{u}}_1)) + r^\varepsilon = \beta^\varepsilon \dot{\theta}_0 \quad (62)$$

Problem \mathcal{A}_3 :

$$\underline{\boldsymbol{\sigma}}_2 = \underline{\underline{\mathbf{c}}}^\varepsilon : (e_x(\underline{\mathbf{u}}_2) + e_y(\underline{\mathbf{u}}_3) - \underline{\boldsymbol{\alpha}}^\varepsilon \delta_2); \quad \nabla_x \cdot \underline{\boldsymbol{\sigma}}_1 + \nabla_y \cdot \underline{\boldsymbol{\sigma}}_2 = 0 \quad (63)$$

$$\underline{\mathbf{q}}_2 = -\lambda^\varepsilon (\nabla_x \theta_2 + \nabla_y \cdot \theta_3); \quad -\nabla_x \cdot \underline{\mathbf{q}}_1 - \nabla_y \cdot \underline{\mathbf{q}}_2 - \underline{\underline{\mathbf{c}}}^\varepsilon : \underline{\underline{\boldsymbol{\alpha}}}^\varepsilon : (e_x(\underline{\mathbf{u}}_1) + e_y(\underline{\mathbf{u}}_2)) = \beta^\varepsilon \dot{\theta}_1 \quad (64)$$

The solutions of problem \mathcal{A}_i are $(\underline{\mathbf{u}}_i, \underline{\boldsymbol{\sigma}}_{i-1}, \theta_i, \underline{\mathbf{q}}_{i-1})$, for $i > 0$. Problem \mathcal{A}_0 gives $\underline{\mathbf{u}}$ and θ_0 . The resolution of \mathcal{A}_0 and \mathcal{A}_1 corresponds to classical homogenization theory applied to coupled thermoelasticity. The reader is referred to [Francfort, 1983] and [Brahim-Otsmane et al., 1992] for a detailed solution. We simply give the form of the solution :

$$\underline{\mathbf{u}}_0(\underline{\mathbf{x}}, \underline{\mathbf{y}}) = \underline{\mathbf{U}}_0(\underline{\mathbf{x}}); \quad \theta_0(\underline{\mathbf{x}}, \underline{\mathbf{y}}) = \Theta_0(\underline{\mathbf{x}}) \quad (65)$$

$$\underline{\mathbf{u}}_1(\underline{\mathbf{x}}, \underline{\mathbf{y}}) = \underline{\mathbf{U}}_1(\underline{\mathbf{x}}) + \underline{\underline{\mathbf{X}}}(\underline{\mathbf{y}}) : e_x(\underline{\mathbf{U}}_0) + \underline{\underline{\mathbf{X}}}(\underline{\mathbf{y}}) \delta_0; \quad \theta_1(\underline{\mathbf{x}}, \underline{\mathbf{y}}) = \Theta_1(\underline{\mathbf{x}}) + \underline{\underline{\mathbf{X}}}''(\underline{\mathbf{y}}) \cdot \nabla_x \theta_0 \quad (66)$$

where $\underline{\underline{\mathbf{X}}}$, $\underline{\underline{\mathbf{X}}}'$, $\underline{\underline{\mathbf{X}}}''$ are concentration tensors, the existence of which is ensured by the linearity of the problem.

4.4 Links with second gradient theory

Following the technique used for instance in [Boutin, 1996], the problems \mathcal{A}_2 and \mathcal{A}_3 dealing with the correcting terms $\underline{\mathbf{u}}_2$ and $\underline{\mathbf{u}}_3$ can be solved in order to evidence some links with second gradient theory. In this section and for simplicity, the analysis is restricted to the thermomechanical problem for a given temperature field $\theta^\varepsilon(\underline{\mathbf{x}}) = \Theta_0(\underline{\mathbf{x}})$. The problem associated with the two first equations of \mathcal{A}_2 (resp. \mathcal{A}_3) can then be interpreted as an elasticity problem with fictitious body forces that are linear in $e_x(\underline{\mathbf{U}}_1)$, $\nabla_x e_x(\underline{\mathbf{U}}_0)$ and $\nabla_x \theta_0$ (resp. $e_x(\underline{\mathbf{U}}_2)$, $\nabla_x e_x(\underline{\mathbf{U}}_1)$, $\nabla_x \nabla_x e_x(\underline{\mathbf{U}}_0)$ and $\nabla_x \nabla_x \theta_0$). Accordingly, there exist 4 additional concentration tensors such that :

$$\underline{\mathbf{u}}_2(\underline{\mathbf{x}}, \underline{\mathbf{y}}) = \underline{\mathbf{U}}_2(\underline{\mathbf{x}}) + \underline{\underline{\mathbf{X}}}(\underline{\mathbf{y}}) : e_x(\underline{\mathbf{U}}_1) + \underline{\underline{\mathbf{Y}}}(\underline{\mathbf{y}}) : (\nabla_x e_x(\underline{\mathbf{U}}_0)) + \underline{\underline{\mathbf{Y}}}(\underline{\mathbf{y}}) \cdot \nabla_x \theta_0 \quad (67)$$

$$\begin{aligned} \underline{\mathbf{u}}_3(\underline{\mathbf{x}}, \underline{\mathbf{y}}) &= \underline{\mathbf{U}}_3(\underline{\mathbf{x}}) + \underline{\underline{\mathbf{X}}}(\underline{\mathbf{y}}) : e_x(\underline{\mathbf{U}}_2) + \underline{\underline{\mathbf{Y}}}(\underline{\mathbf{y}}) : (\nabla_x e_x(\underline{\mathbf{U}}_1)) \\ &+ \underline{\underline{\mathbf{Z}}}(\underline{\mathbf{y}}) : (\nabla_x \nabla_x e_x(\underline{\mathbf{U}}_0)) + \underline{\underline{\mathbf{Z}}}(\underline{\mathbf{y}}) : (\nabla_x \nabla_x \theta_0) \end{aligned} \quad (68)$$

where the $\underline{\mathbf{U}}_i$ are constant translation terms. The concentration tensors can be taken such that their average over the unit cell vanishes. These expressions are now used to compute the local strain fields with the correctors :

$$\begin{aligned} e(\underline{\mathbf{u}}^\varepsilon) &= e_x(\underline{\mathbf{U}}_0) + \underline{\underline{\hat{\mathbf{X}}}}(\underline{\mathbf{y}}) : e_x(\underline{\mathbf{U}}_0) + \underline{\underline{\hat{\mathbf{X}}}}(\underline{\mathbf{y}}) \delta_0 - \underline{\boldsymbol{\alpha}}^\varepsilon \delta_0 \\ &+ \varepsilon \left(\underline{\underline{\hat{\mathbf{X}}}}(\underline{\mathbf{y}}) : (\nabla_x e_x(\underline{\mathbf{U}}_0)) + \underline{\underline{\hat{\mathbf{Y}}}}(\underline{\mathbf{y}}) : (\nabla_x e_x(\underline{\mathbf{U}}_0)) + \underline{\underline{\hat{\mathbf{X}}}}(\underline{\mathbf{y}}) \cdot \nabla_x \theta_0 + \underline{\underline{\hat{\mathbf{Y}}}}(\underline{\mathbf{y}}) \cdot \nabla_x \theta_0 \right) \\ &+ \varepsilon^2 \left(\underline{\underline{\hat{\mathbf{Y}}}}(\underline{\mathbf{y}}) : (\nabla_x \nabla_x e_x(\underline{\mathbf{U}}_0)) + \underline{\underline{\hat{\mathbf{Z}}}}(\underline{\mathbf{y}}) : (\nabla_x \nabla_x e_x(\underline{\mathbf{U}}_0)) \right) \\ &+ \varepsilon^2 \left(\underline{\underline{\hat{\mathbf{Y}}}}(\underline{\mathbf{y}}) \cdot (\nabla_x \nabla_x \theta_0) + \underline{\underline{\hat{\mathbf{Z}}}}(\underline{\mathbf{y}}) : \nabla_x \nabla_x \theta_0 \right) \end{aligned} \quad (69)$$

where concentration tensors modified by application of the symmetric gradient operator have been introduced and the translation terms have been put into the first term $\underline{\mathbf{U}}_0$ for simplicity. The mean stress is obtained by averaging the local stress over the unit cell :

$$\begin{aligned} \langle \underline{\boldsymbol{\sigma}}^\varepsilon \rangle &= \langle \underline{\underline{\mathbf{c}}}^\varepsilon : (\underline{\mathbf{e}}(\underline{\mathbf{u}}^\varepsilon) - \delta^\varepsilon \underline{\boldsymbol{\alpha}}^\varepsilon) \rangle = \underline{\underline{\mathbf{C}}}^0 : (e_x(\underline{\mathbf{U}}_0) - \delta_0 \underline{\boldsymbol{\alpha}}^0) \\ &+ \varepsilon \underline{\underline{\mathbf{C}}}^1 : (\nabla e_x(\underline{\mathbf{U}}_0) - \underline{\underline{\boldsymbol{\alpha}}}^1 \cdot \nabla \theta_0) + \varepsilon^2 \underline{\underline{\mathbf{C}}}^2 : (\nabla \nabla e_x(\underline{\mathbf{U}}_0) - \underline{\underline{\boldsymbol{\alpha}}}^2 : \nabla \nabla \theta_0) \end{aligned} \quad (70)$$

where $\langle . \rangle$ denotes the averaging process. This relation can be interpreted as the effective constitutive equation for the homogenization problem up to order 3. The involved constitutive tensors are related to the concentration tensors as follows :

$$\underline{\underline{\mathbf{C}}}^0 = \langle \underline{\underline{\mathbf{c}}}^\varepsilon : \hat{\underline{\mathbf{X}}} \rangle; \quad \underline{\underline{\mathbf{C}}}^0 : \underline{\boldsymbol{\alpha}}^0 = \langle \underline{\underline{\mathbf{c}}}^\varepsilon : (\hat{\underline{\mathbf{X}}} - \underline{\boldsymbol{\alpha}}^\varepsilon) \rangle; \quad \underline{\underline{\mathbf{C}}}^1 = \langle \underline{\underline{\mathbf{c}}}^\varepsilon : (\underline{\mathbf{X}} + \hat{\underline{\mathbf{Y}}}) \rangle; \quad \dots \quad (71)$$

It appears that the overall constitutive equations involve higher order gradients of the overall strain and temperature fields. They are found to be identical to the constitutive equations (33) of a linear thermoelastic medium, providing that the mean stress is interpreted not as an overall Cauchy stress tensor but rather as the “effective” stress $\underline{\boldsymbol{\tau}}$ of a second grade medium defined by (33). The mean stress can be shown to fulfill the following effective balance equation :

$$\langle \nabla \cdot \underline{\boldsymbol{\sigma}}^\varepsilon \rangle + \langle \underline{\mathbf{f}}^\varepsilon \rangle = 0. \quad (72)$$

which actually is the balance of momentum equation (14) satisfied by the “effective” stress $\underline{\boldsymbol{\tau}}$ of a second grade medium. The identification of the homogeneous equivalent medium with a second grade thermoelastic material should be closed by the statement of the boundary conditions of the boundary value problem for the body endowed with the properties of the HEM.

4.5 Coupled thermomechanical problem

The resolution of the fully coupled thermomechanical problem of heterogeneous elasticity goes through the resolution in cascade of problems \mathcal{A}_0 to \mathcal{A}_3 for the displacement and temperature fields $\underline{\mathbf{u}}_i$ and θ_i . Note that, in each case, the thermal and mechanical problems are decoupled, since the coupling term in the heat equation of problem \mathcal{A}_{i+1} involves the solution of problem \mathcal{A}_i only. The mechanical part of the problem has been solved in the previous section. The partial differential equation of each thermal problem in \mathcal{A}_i can be interpreted as a heat equation with fictitious heat source terms linear in $\dot{\theta}_0$, $e_x(\dot{\underline{\mathbf{U}}}_0)$, $\nabla \dot{\theta}_0$, $\nabla e_x(\dot{\underline{\mathbf{U}}}_0)$, so that the solutions for temperature can be proved to have the form :

$$\theta_2(\underline{\mathbf{x}}, \underline{\mathbf{y}}) = \underline{\underline{\mathbf{Y}}}''(\underline{\mathbf{y}}) : (\nabla_x \nabla_x \theta_0) + \underline{\underline{\mathbf{Y}}}^{(3)}(\underline{\mathbf{y}}) : e_x(\dot{\underline{\mathbf{U}}}_0) + Y^{(4)}(\underline{\mathbf{y}}) \dot{\theta}_0 \quad (73)$$

$$\theta_3(\underline{\mathbf{x}}, \underline{\mathbf{y}}) = \underline{\underline{\mathbf{Z}}}^{(4)}(\underline{\mathbf{y}}) : (\nabla_x \nabla_x \nabla_x \theta_0) + \underline{\underline{\mathbf{Z}}}^{(5)}(\underline{\mathbf{y}}) : (\nabla e_x(\dot{\underline{\mathbf{U}}}_0)) + \underline{\underline{\mathbf{Z}}}^{(6)}(\underline{\mathbf{y}}) \cdot \nabla \dot{\theta}_0 \quad (74)$$

The thermomechanical coupling terms in problem \mathcal{A}_3 leads to the following additional contributions to the previous result (68) :

$$\underline{\mathbf{u}}_3(\underline{\mathbf{x}}, \underline{\mathbf{y}}) = \underline{\underline{\mathbf{Z}}}(\underline{\mathbf{y}}) :: (\nabla_x \nabla_x e_x(\underline{\mathbf{U}}_0)) + \underline{\underline{\mathbf{Z}}}(\underline{\mathbf{y}}) : (\nabla_x \nabla_x \theta_0) + \underline{\underline{\mathbf{Z}}}''(\underline{\mathbf{y}}) : e_x(\underline{\dot{\mathbf{U}}}_0) + \underline{\underline{\mathbf{Z}}}^{(3)} \dot{\theta}_0 \quad (75)$$

As a result, the expression of the overall constitutive equations for the effective stress (70) must be complemented by additional constitutive tensors of the form $\varepsilon^2(\underline{\underline{\mathbf{C}}}^{(3)} : e_x(\underline{\dot{\mathbf{U}}}_0) + \underline{\underline{\mathbf{C}}}^{(4)} \dot{\theta}_0)$, that are active only in the transient regime. Similarly, the expression of the effective heat flux can be derived :

$$\begin{aligned} \underline{\mathbf{Q}} &= \langle \underline{\mathbf{q}}^\varepsilon \rangle = -\underline{\underline{\mathbf{\Lambda}}}^{(0)} \cdot \nabla \theta_0 + \varepsilon \left(\underline{\underline{\mathbf{\Lambda}}}^{(1)} : \nabla \nabla \theta_0 + \underline{\underline{\mathbf{\Lambda}}}^{(2)} : e_x(\underline{\dot{\mathbf{U}}}_0) + \underline{\underline{\mathbf{\Lambda}}}^{(3)} \dot{\theta}_0 \right) \\ &+ \varepsilon^2 \left(\underline{\underline{\mathbf{\Lambda}}}^{(4)} : (\nabla \nabla \nabla \theta_0) + \underline{\underline{\mathbf{\Lambda}}}^{(5)} : \nabla e_x(\underline{\dot{\mathbf{U}}}_0) + \underline{\underline{\mathbf{\Lambda}}}^{(6)} \cdot \nabla \dot{\theta}_0 \right) + \dots \end{aligned} \quad (76)$$

The expressions relating the introduced effective constitutive tensors and the concentration tensors appearing in (73) and (74) are not given here for conciseness. The first term shows that, at the first order, Fourier's law is retrieved at the macroscopic level. Taking higher order terms into account leads to a generalized Fourier law involving higher order temperature gradients and transient terms. Note that the dissipation inequality remains fulfilled per construction at the macroscopic level, at least up to the considered precision in ε . New coupling terms arise then in the effective heat equation : In addition to terms in $\nabla \nabla \dot{T}$ and $\nabla \underline{\underline{\mathbf{K}}}$ that are expected according to the thermoelastic framework depicted in part 3, terms involving $\underline{\underline{\mathbf{K}}}$, $\nabla \dot{T}$, $\underline{\underline{\mathbf{E}}}$ and $\underline{\underline{\mathbf{T}}}$ arise necessarily. This seems to indicate that an even more general phenomenological framework that the one proposed in part 2 should be considered for which ∇T and $\underline{\underline{\mathbf{T}}}$ should be introduced together. In particular the relations between the $\underline{\underline{\mathbf{T}}}$ term appearing then in the heat equation, and the second Cattaneo equation (see [Müller, Ruggieri, 1993]) remain to be investigated.

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