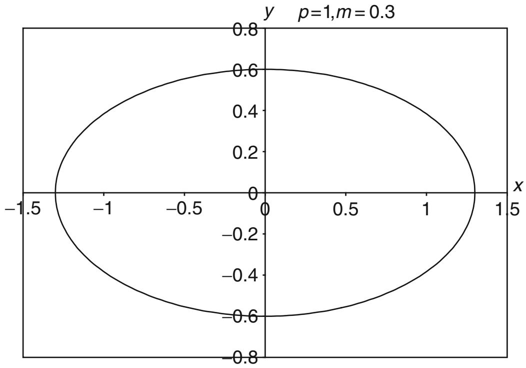
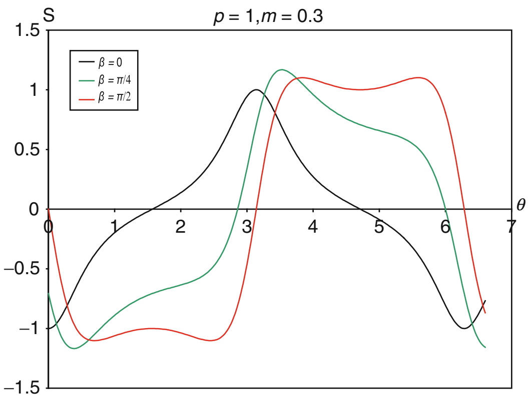


Goursat Functions of Thermoelastic Problem of an Infinite Plate with Hypitrochoidal Hole, Fig. 3 Elliptic hole



Goursat Functions of Thermoelastic Problem of an Infinite Plate with Hypitrochoidal Hole, Fig. 4 Variation of thermal stresses on the elliptic hole with different directions

where

$$\varepsilon(p) = p + (1 - 2p)\delta(p)$$

To get the thermoelastic stresses on the boundary, we have from (32) that

$$S = \frac{\sigma_{\vartheta\vartheta}|_{\rho=1}}{cqE\alpha} = \frac{1}{W(\vartheta)}[m\{p+\varepsilon(p)\}\cos(p\vartheta+\beta) - \{m^2p\varepsilon(p)+1\}\cos(\vartheta-\beta)] \quad (46)$$

where

$$W(\vartheta) = 1 + m^2p^2 - 2mp\cos(p+1)\vartheta$$

The formulae (44)–(46) agree with the formulae (VII.80)–(VII.82) for $p = 3$ and $m = -\frac{1}{6}$ and formulae (VII.83)–(VII.84) for $p = 2$ and $m = \frac{1}{3}$ and formulae (VII.85) and (VII.87) for $p = 1$, p.537 of [4].

In Fig. 1, we have the hypotrochoidal holes of the infinite plate at $p = 2, 3, 4$, while in Fig. 2, we have the relations between S and θ at the same holes of Fig. 1 for $\beta = 0$. Also the relation between the shear stress S and the angle θ for the infinite plate of the elliptic hole for different values of β shown in Figs. 3 and 4.

References

1. Sokolnikoff IS (1956) Mathematical theory of elasticity, 2nd edn. Graw-Hill Book Company, New York
2. Neumann F (1885) Vorlesungenuber die theorie der elasticity. Meyer, Brestau
3. Noda N, Hetnarski RB, Tanigawa Y (2003) Thermal stresses. Taylor and Francis, New York
4. Savin GN (1968) Stress distribution around holes. Inst. Mech. AH YCCP, Kiev (In Russian)
5. Muskhelishvili NI (1953) Some basic problems of the mathematical theory of elasticity. Noordroof, Holand
6. Parkus H (1976) Thermoelasticity. Springer, New York

Gradient Thermoplasticity

Samuel Forest
Mines ParisTech/CNRS, Centre des Matériaux/ CNRS UMR 7633, Evry Cedex, France

Overview

Strain gradient plasticity effects arise from the interplay between the typical wavelength of applied mechanical loading conditions and microstructural length scales. Grain and particle size effects are well known in mechanical metallurgy. They can be represented in the continuum thermomechanics of materials by means of additional contributions of the plastic strain or hardening variable gradients to the free energy density or viscoplastic dissipation potentials of nonlinear materials. A general micromorphic framework is presented here that is based on the introduction of additional degrees of freedom and their gradient into constitutive functionals. The theory is specialized so as to recover the well-known Aifantis strain gradient plasticity model which includes an extra hardening term associated with the Laplacian of cumulative plastic strain. The effect of the gradient of the plastic strain tensor leads to a similar Laplacian term in the expression of kinematical hardening. An analytical example shows the development of boundary layers in laminate microstructures under shear that depend on microstructure size and on the intrinsic length of the continuum model. Strain gradient plasticity models can also be used to predict strain localization bands with finite thickness, which is required for finite element simulations of regularized strain localization phenomena. In the particular case of single crystal plasticity, strain gradient plasticity models are used to account for the effect of the dislocation density tensor on material behavior. The temperature dependence of all material parameters and the coupling of mechanical dissipation with thermal effect are taken into account in the formulation. Finally, the question of the introduction of the temperature or entropy gradient into the internal or free energy densities is addressed, which may be significant in the presence of strong temperature gradients like in microsystems or during laser treatments.

Introduction

Gradient plasticity theories belong to the mechanics of generalized continua that incorporate intrinsic length effects into the usual material modeling setting. Size effects result from the interplay between the typical wavelength of the mechanical loading conditions and the characteristic sizes of the underlying microstructures. Within the context of plasticity in metals, dislocations move and interact with microstructural barriers like grain boundaries or interfaces, which give rise to a wide range of grain or precipitate/inclusion size effects on the plastic yielding and hardening of metals and alloys [5]. Microstructural lengths also appear in the finite width of strain localization bands in metals or soils and in the damage of materials.

The introduction of the gradient of plastic strain or of some other hardening variables into the constitutive framework makes it possible to incorporate a microstructural intrinsic length into the continuum mechanical framework. The strain gradient plasticity model [14, 23] is an extension to the nonlinear case of Mindlin's celebrated second gradient model [27] which contains a contribution of the second gradient of the displacement field, or equivalently of the strain gradient, in the material's free energy density. It can also be seen as a special case of elastoviscoplastic micromorphic continua based on Eringen's generalized continuum theory [12]. The usual thermoelastic–viscoplastic constitutive framework can be extended, while keeping the essential structure of continuum thermodynamics, especially the notion of local accompanying state and of local action [21]. Instead of formulating complex integrodifferential equations for nonlocal interactions in microstructured solids, it is assumed that local state functions can be formulated that depend on the values at a material point of an enlarged set of constitutive variables. The effective notion of internal variables driven by differential evolution equations is extended into the concept of internal degrees of freedom involved in generalized balance equations associated with new additional boundary conditions. The concept of standard generalized materials can be extended to gradient plasticity and damage according to the variational framework presented in [29].

This entry presents the most general micromorphic framework that incorporates such plasticity-related microstructural effects and its specialization to strain gradient plasticity models. Isotropic and kinematic hardening is shown to be affected by terms containing the Laplacian of plastic strain. Analytical solutions of archetypal examples are provided for the description of boundary layer development in heterogeneous hardening materials and for the modeling of strain localization bands with finite thickness. Regularization of strain localization phenomena is particularly important in the case of geomaterials [9]. The approach deals with viscoplasticity but is equally applicable to continuum damage mechanics [10, 16].

The presentation is limited to the context of small deformations for the sake of conciseness. The reader is referred to [6] and the references quoted therein for the corresponding formulation of micromorphic and strain gradient finite deformation plasticity.

The theory involves tensors of up to sixth order so that specific notations are required in this entry. Zeroth, first, second, third, fourth, and sixth order tensors are denoted by a, $\underline{\boldsymbol{a}}$, $\underset{\sim}{\boldsymbol{a}}$, $\underset{\approx}{\boldsymbol{a}}$, $\underset{\approx}{\boldsymbol{a}}$, and $\underset{\approx}{\boldsymbol{a}}$, respectively. The simple, double, and triple contractions are written $.$, $:$ and $\vdots$, respectively. In index form with respect to an orthonormal basis, these notations correspond to

$$\underline{\boldsymbol{a}} \,.\, \underline{\boldsymbol{b}} = a_i b_i, \quad \underset{\sim}{a} : \underset{\sim}{b} = a_{ij} b_{ij}, \quad \underset{\approx}{\boldsymbol{a}} \vdots \underset{\approx}{\boldsymbol{b}} = a_{ijk} b_{ijk} \tag{1}$$

where repeated indices are summed up. The tensor product is denoted by $\otimes$. The nabla operator with respect to spatial coordinates is denoted by ∇. For example, the Cartesian component ijk of $\nabla \underset{\sim}{\varepsilon}$ is $\varepsilon_{ij,k}$.

General Continuum Thermomechanical Formulation

The micromorphic theory has been proposed simultaneously by Eringen and Mindlin [13, 26]. It introduces a general noncompatible field of micro-deformation, in addition to the usual material deformation gradient, accounting for the deformation of a triad of microstructural directions. When the micro-deformation coincides with the deformation gradient, the micromorphic model reduces to Toupin and Mindlin's strain gradient theory. The micromorphic approach can, in fact, be applied to any macroscopic quantity in order to introduce an intrinsic length scale in the original standard continuum model in a systematic way, as done in [16]. The reference standard continuum plasticity model is presented before extending it to incorporate length scale effects. As usual, the total infinitesimal strain tensor is split into its elastic, thermal, and plastic parts:

$$\underset{\sim}{\varepsilon} = \underset{\sim}{\varepsilon}^e + \underset{\sim}{\varepsilon}^{th} + \underset{\sim}{\varepsilon}^p \tag{2}$$

The reference state space corresponding to a classical elastoplasticity model is

$$DOF0 = \{\underline{\boldsymbol{u}}\}, \quad STATE0 = \{\underset{\sim}{\varepsilon}^e, \quad \theta, \quad \alpha\} \tag{3}$$

The degrees of freedom (DOF) of the material point are the components of the displacement vector $\underline{\boldsymbol{u}}$. The state variables are the (infinitesimal) elastic strain tensor $\underset{\sim}{\varepsilon}^e$, temperature θ, and the internal variables α, see, for example, [25]. The latter include all hardening variables for which specific differential evolution equations are required. A standard yield function and a corresponding dissipation potential can be used to compute plastic flow and the evolution rules for the other internal variables.

In general, the set of degrees of freedom can be enlarged by introducing additional kinematic (micromorphic) or physical (phase field) degrees of freedom, ϕ. The notation ϕ denotes a list of scalar or tensor variables attached to each material point. The material behavior is then assumed to depend on the variable ϕ and its first gradient $\nabla\phi$. Accordingly, the sets of degrees of freedom and the state space are enhanced as follows:

$$\begin{aligned} DOF &= \{\underline{\boldsymbol{u}}, \quad \phi\} \\ STATE &= \{\underset{\sim}{\varepsilon}^e, \quad \theta, \quad \alpha, \quad \phi, \quad \nabla\phi\} \end{aligned} \tag{4}$$

The principle of virtual power is generalized to incorporate such additional microstructural effects. This represents a systematic use of the method of virtual power that Germain applied to Eringen's micromorphic medium in [20]. The powers of internal and contact forces for the body Ω are represented by corresponding power densities in the form:

$$\mathcal{P}^{(i)} = -\int_{\Omega} p^{(i)}\, dV, \quad \mathcal{P}^{(c)} = \int_{\partial\Omega} p^{(c)}\, dS \tag{5}$$

$$p^{(i)} = \underset{\sim}{\boldsymbol{\sigma}} : \dot{\underset{\sim}{\varepsilon}} + a\dot{\phi} + \underline{\boldsymbol{b}}.\nabla\dot{\phi}$$
$$p^{(c)} = \underline{\boldsymbol{t}}.\dot{\underline{\boldsymbol{u}}} + a^c\dot{\phi} \tag{6}$$

in which generalized stresses a and $\underline{\boldsymbol{b}}$ have been introduced in addition to the usual Cauchy stress tensor $\underset{\sim}{\boldsymbol{\sigma}}$. The traction vector $\underline{\boldsymbol{t}}$ and the generalized traction a^c act on the displacement velocity and micromorphic velocity, respectively. In the absence of volume or inertia forces for simplicity, the sum of the virtual power of internal and contact forces must vanish for all virtual fields $\underline{\boldsymbol{u}}$ and ϕ. This leads to the following local balance equations and natural boundary conditions:

$$\mathrm{div}\,\underset{\sim}{\boldsymbol{\sigma}} = 0, \quad \forall \underline{\boldsymbol{x}} \in \Omega,$$
$$\underline{\boldsymbol{t}} = \underset{\sim}{\boldsymbol{\sigma}} \cdot \underline{\boldsymbol{n}}, \quad \forall \underline{\boldsymbol{x}} \in \Omega \tag{7}$$

$$\mathrm{div}\,\underline{\boldsymbol{b}} - a = 0, \quad \forall \underline{\boldsymbol{x}} \in \Omega, \quad a^c = \underline{\boldsymbol{b}} \cdot \underline{\boldsymbol{n}}, \quad \forall \underline{\boldsymbol{x}} \in \partial\Omega \tag{8}$$

where $\underline{\boldsymbol{n}}$ is the unit outer normal vector at the boundary of the body. The microstructural effects therefore arise in the balance of energy in the form:

$$\rho\dot{\varepsilon} = p^{(i)} - \mathrm{div}\,\underline{\boldsymbol{q}} \tag{9}$$

where ε is the specific internal energy and $\boldsymbol{q}$ is the heat flux vector. The free energy density function ψ is assumed to be a function of elements of the previous set *STATE*. The entropy principle is formulated in the form

$$-\rho\left(\dot{\psi} + \eta\dot{T}\right) + p^{(i)} - \frac{\underline{\boldsymbol{q}}}{T} \cdot \nabla T \geq 0 \tag{10}$$

where η is the entropy density function and where the thermal contribution is included. Taking into account the functional dependence of all state functions, the Clausius–Duhem inequality is obtained as

$$\left(\underset{\sim}{\boldsymbol{\sigma}} - \rho\frac{\partial\psi}{\partial\underset{\sim}{\varepsilon}^e}\right) : \dot{\underset{\sim}{\varepsilon}}^e + \left(a - \rho\frac{\partial\psi}{\partial\phi}\right)\dot{\phi}$$
$$+ \left(\underline{\boldsymbol{b}} - \rho\frac{\partial\psi}{\partial\nabla\phi}\right) \cdot \nabla\dot{\phi} + \underset{\sim}{\boldsymbol{\sigma}} : \dot{\underset{\sim}{\varepsilon}}^p - X\dot{\alpha} \tag{11}$$
$$- \left(\eta + \rho\frac{\partial\psi}{\partial\theta} - \underset{\sim}{\boldsymbol{\sigma}} : \frac{\partial\underset{\sim}{\varepsilon}^{th}}{\partial\theta}\right)\dot{\theta} - \frac{\underline{\boldsymbol{q}}}{\theta} \cdot \nabla\theta \geq 0$$

where it was assumed that the thermal strain depends solely on temperature and the thermodynamic force associated with internal variables was defined as

$$X = \rho\frac{\partial\psi}{\partial\alpha} \tag{12}$$

Thermoelasticity does not induce dissipation, which provides two state laws:

$$\underset{\sim}{\boldsymbol{\sigma}} = \rho\frac{\partial\psi}{\partial\underset{\sim}{\varepsilon}^e}, \quad \eta = -\rho\frac{\partial\psi}{\partial\theta} + \underset{\sim}{\boldsymbol{\sigma}} : \frac{\partial\underset{\sim}{\varepsilon}^{th}}{\partial\theta} \tag{13}$$

The remaining state laws are then formulated in the following way:

$$a = \rho\frac{\partial\psi}{\partial\phi} + a^v, \quad \underline{\boldsymbol{b}} = \rho\frac{\partial\psi}{\partial\nabla\phi} \tag{14}$$

where a generalized viscous stress a^v was introduced that is responsible for additional dissipation following [22]. The residual dissipation rate finally is

$$\underset{\sim}{\boldsymbol{\sigma}} : \dot{\underset{\sim}{\varepsilon}}^p - X\dot{\alpha} + a^v\dot{\phi} - \frac{\boldsymbol{q}}{\theta}.\nabla\theta \geq 0 \tag{15}$$

At this place, the existence of a viscoplastic potential can be postulated which depends on all generalized stress driving forces, $\Omega(\underset{\sim}{\boldsymbol{\sigma}}, \mathrm{X}, a^v)$, from which evolution rules are derived:

$$\dot{\underset{\sim}{\varepsilon}}^p = \frac{\partial\Omega}{\partial\underset{\sim}{\boldsymbol{\sigma}}}, \quad \dot{\alpha} = -\frac{\partial\Omega}{\partial X}, \quad \dot{\phi} = \frac{\partial\Omega}{\partial a^v} \tag{16}$$

Convexity of the viscoplastic potential with respect to the driving forces $\underset{\sim}{\boldsymbol{\sigma}}, -X, a^v$ then ensures the positivity of the dissipation rate along any thermodynamic process. This terminates the general framework and more specific constitutive assumptions must now be made.

Gradient Plasticity with Size-Dependent Isotropic Hardening

A quadratic form is now proposed for the isothermal free energy density function which is assumed to depend on the cumulative plastic strain p in the form

$$\rho\psi(\underset{\sim}{\varepsilon}^e, p, \phi, \nabla\phi) = \frac{1}{2}\underset{\sim}{\varepsilon}^e : \underset{\approx}{\boldsymbol{C}} : \underset{\sim}{\varepsilon}^e + \frac{1}{2}Hp^2 + \frac{1}{2}H_\phi(p-\phi)^2 + \frac{1}{2}A\nabla\phi\cdot\nabla\phi \quad (17)$$

The corresponding classical model describes an elastoplastic material behavior with linear elasticity characterized by the tensor of elastic moduli $\underset{\approx}{\boldsymbol{C}}$ and the linear hardening modulus H. Isotropy has been assumed for the last term for the sake of brevity. Two additional material parameters are introduced in the micromorphic extension of this classical model, namely, the coupling modulus H_ϕ (unit MPa) and the micromorphic stiffness A (unit MPa.mm^2). The thermodynamic forces associated with the state variables are given by the relations (14)

$$\underset{\sim}{\boldsymbol{\sigma}} = \underset{\approx}{\boldsymbol{C}} : \underset{\sim}{\varepsilon}^e, \quad a = -H_\phi(p-\phi), \quad \underline{\boldsymbol{b}} = A\nabla\phi, \quad X = \left(H + H_\phi\right)p - H_\phi\phi \quad (18)$$

Note that when the relative plastic strain $e = p - \phi$ is close to zero, the linear hardening rule retrieves its classical form and the generalized stress a vanishes. Only the strain gradient ∇p remains in the enriched work of internal forces (6). This is the situation encountered in the strain gradient plasticity models developed in [15, 23]. When the value of H_ϕ is high enough, it acts as a penalty term forcing the micromorphic variable to follow the cumulative plastic strain as close as possible. When inserted in the additional balance (8), the previous state laws lead to the following partial differential equation:

$$\phi - \frac{A}{H_\phi}\Delta\phi = p \quad (19)$$

which is identical to the additional partial differential equation used in the so-called implicit gradient-enhanced elastoplasticity in [11]. The associated Neumann condition is used in the form:

$$\underline{\boldsymbol{b}}\cdot\underline{\boldsymbol{n}} = A\nabla\phi\cdot\underline{\boldsymbol{n}} = a^c \quad \text{on} \quad \partial\Omega \quad (20)$$

A material surface is free when $a^c = 0$.

The yield function is now chosen as

$$f\left(\underset{\sim}{\boldsymbol{\sigma}}, X\right) = \sigma_{eq} - \sigma_Y - X \quad (21)$$

where σ_{eq} is an equivalent stress measure and σ_Y the initial yield stress. Plastic flow is derived from the normality rule:

$$\dot{\underset{\sim}{\varepsilon}}^p = \lambda\frac{\partial f}{\partial\underset{\sim}{\boldsymbol{\sigma}}} \quad (22)$$

which defines the cumulative plastic strain as the result of time integration of the plastic multiplier $\dot{p} = \lambda$. After substituting the balance (19) into the hardening law, yielding takes place when

$$\sigma_{eq} = \sigma_Y + H\phi - A\left(1 + \frac{H}{H_\phi}\right)\Delta\phi \quad (23)$$

This expression coincides with the enhanced yield criterion originally proposed for gradient plasticity by Aifantis [1, 2] and used for strain localization simulations in [8], provided that the micromorphic variable remains as close as possible to the plastic strain, $\phi \equiv p$:

$$\sigma_{eq} = \sigma_Y + H_p - c\Delta p \quad (24)$$

where $c = A\left(1 + H/H_\phi\right)$. As a result, Aifantis' model has been retrieved from the micromorphic approach by choosing simple linear constitutive equations and introducing the internal constraint $\phi \equiv p$ stating that the micromorphic variable coincides with the plastic strain itself. This strain gradient plasticity model can be used with more general nonlinear hardening rules and with the introduction of dissipative strain gradient mechanisms [4].

Gradient Plasticity with Size-Dependent Kinematic Hardening

The approach is not restricted to scalar micromorphic variables. The additional degrees of freedom are now taken as the components of a symmetric second-order tensor $\underset{\sim}{\phi}$. The generalized stresses are symmetric second- and third-order tensors, respectively:

$$p^{(i)} = \underset{\sim}{\boldsymbol{\sigma}} : \dot{\underset{\sim}{\varepsilon}} + \underset{\sim}{a} : \dot{\underset{\sim}{\phi}} + \underset{\approx}{\boldsymbol{b}} \vdots \nabla \dot{\underset{\sim}{\phi}} \tag{25}$$

The symmetry condition applies only to the first two indices of b_{ijk}. The extended set of state variables is

$$STATE = \left\{ \underset{\sim}{\varepsilon}^e, \quad \underset{\sim}{\varepsilon}^p, \quad \underset{\sim}{\phi}, \quad \nabla \underset{\sim}{\phi} \right\} \tag{26}$$

When the micromorphic variable is constrained to remain as close as possible to the plastic strain tensor, the theory of gradient of plastic strain presented in [23] is recovered. As an illustration, the following quadratic form for the free energy potential is adopted:

$$\begin{aligned} p\psi\left(\underset{\sim}{\varepsilon}^e, \underset{\sim}{\varepsilon}^p, \underset{\sim}{\phi}, \nabla\underset{\sim}{\phi}\right) = & \frac{1}{2} \underset{\sim}{\varepsilon}^e : \underset{\approx}{\boldsymbol{C}} : \underset{\sim}{\varepsilon}^e + \frac{1}{3} C \underset{\sim}{\varepsilon}^p : \underset{\sim}{\varepsilon}^p \\ & + \frac{1}{3} \left(\underset{\sim}{\varepsilon}^p - \underset{\sim}{\phi}\right) : \underset{\approx}{\boldsymbol{C}}_{\phi} : \left(\underset{\sim}{\varepsilon}^p - \underset{\sim}{\phi}\right) \\ & + \frac{1}{2} \nabla \underset{\sim}{\phi} \vdots \underset{\approx}{\boldsymbol{A}} \vdots \nabla \underset{\sim}{\phi} \end{aligned} \tag{27}$$

from which the state laws are derived:

$$\begin{aligned} & \underset{\sim}{\boldsymbol{\sigma}} = \underset{\approx}{\boldsymbol{\Lambda}} : \underset{\sim}{\varepsilon}^e, \quad \underset{\sim}{\boldsymbol{X}} = \frac{2}{3} C \underset{\sim}{\varepsilon}^p + \frac{2}{3} \underset{\approx}{\boldsymbol{C}}_{\phi} : \left(\underset{\sim}{\varepsilon}^p - \underset{\sim}{\phi}\right), \\ & \underset{\sim}{\boldsymbol{a}} = -\frac{2}{3} \underset{\approx}{\boldsymbol{C}}_{\phi} : \left(\underset{\sim}{\varepsilon}^p - \underset{\sim}{\phi}\right), \quad \underset{\approx}{\boldsymbol{b}} = \underset{\approx}{\boldsymbol{A}} \vdots \nabla \underset{\sim}{\phi} \end{aligned} \tag{28}$$

In the simplified situation for which $\underset{\approx}{\boldsymbol{C}}_{\phi} = \underset{\approx}{C}_{\phi} \underset{\approx}{1}, \underset{\approx}{\boldsymbol{A}} = A \underset{\approx}{1}$ where $\underset{\approx}{1}$ and $\underset{\approx}{1}$ are the fourth rank and sixth rank identity tensors operating respectively on symmetric second-order tensors and symmetric (w.r.t. the first two indices) third-rank tensors; the combination of the additional balance equation and state laws leads to the following partial differential equation:

$$\underset{\sim}{\boldsymbol{a}} = \operatorname{div} \underset{\approx}{\boldsymbol{b}} = A \Delta \underset{\sim}{\varepsilon}^p = -\frac{2}{3} C_{\phi} \left(\underset{\sim}{\varepsilon}^p - \underset{\sim}{\phi}\right) \tag{29}$$

$$\underset{\sim}{\phi} - \frac{3A}{2C_{\phi}} \Delta \underset{\sim}{\phi} = \underset{\sim}{\varepsilon}^p \tag{30}$$

The differential operators act in the following way w.r.t. to a Cartesian frame $\left(\underline{e}_i\right)_{i=1,3}$:

$$\begin{aligned} \operatorname{div} \underset{\approx}{\boldsymbol{b}} &= b_{ijk,k} \underline{\boldsymbol{e}}_i \otimes \underline{\boldsymbol{e}}_j \\ \Delta \underset{\sim}{\phi} &= \left(\Delta \phi_{ij}\right) \underline{\boldsymbol{e}}_i \otimes \underline{\boldsymbol{e}}_j \end{aligned} \tag{31}$$

The associated boundary conditions on the frontier of the body are given by a set of 6 equations:

$$\underset{\approx}{\boldsymbol{b}} \cdot \underline{\boldsymbol{n}} = \underset{\sim}{\boldsymbol{a}}^c \tag{32}$$

The internal variable $\underset{\sim}{\alpha} = \underset{\sim}{\varepsilon}^p$ is the proper state variable for a plasticity theory incorporating linear kinematic hardening, $\underset{\sim}{\boldsymbol{X}}$ being the back-stress tensor. The retained isotropic yield function for extended J_2–plasticity is

$$\begin{aligned} f(\underset{\sim}{\sigma}, \underset{\sim}{\boldsymbol{X}}) &= J_2(\underset{\sim}{\sigma} - \underset{\sim}{\boldsymbol{X}}) - \sigma_Y \\ &= J_2\left(\underset{\sim}{\sigma} - \frac{2}{3}(C + C_{\phi}) \underset{\sim}{\varepsilon}^p + \frac{2}{3} C_{\phi} \underset{\sim}{\phi}\right) - \sigma_Y \\ &= J_2\left(\underset{\sim}{\sigma} - \frac{2}{3} C \underset{\sim}{\phi} + A\left(1 + \frac{C}{C_{\phi}}\right) \Delta \underset{\sim}{\phi}\right) - \sigma_Y \end{aligned}$$

where $J_2(\underset{\sim}{\boldsymbol{\sigma}}) = \sqrt{3\left(\underset{\sim}{\boldsymbol{\sigma}}^{\text{dev}} : \underset{\sim}{\boldsymbol{\sigma}}^{\text{dev}}\right)/2}$ is the von Mises second invariant for symmetric second-rank tensors. The normality rule is adopted:

$$\dot{\underset{\sim}{\varepsilon}}^p = \dot{\lambda} \frac{\partial f}{\partial \underset{\sim}{\boldsymbol{\sigma}}} = -\dot{\lambda} \frac{\partial f}{\partial \underset{\sim}{\boldsymbol{X}}} = \dot{p} \underset{\sim}{\boldsymbol{N}} \tag{33}$$

The intrinsic dissipation then takes its classical form:

$$\underset{\sim}{\boldsymbol{\sigma}} : \dot{\underset{\sim}{\varepsilon}}^p - \underset{\sim}{\boldsymbol{X}} : \dot{\underset{\sim}{\varepsilon}}^p = f \dot{p} + \sigma_Y \dot{p} \geq 0 \tag{34}$$

with energy storage associated with kinematic hardening. The plastic multiplier is deduced from the consistency condition:

$$\begin{aligned}\dot{f} &= \underset{\sim}{N} : \underset{\sim}{\dot{\sigma}} - \underset{\sim}{N} : \underset{\sim}{\dot{X}} = \underset{\sim}{N} : \underset{\approx}{C} : \left(\underset{\sim}{\dot{\varepsilon}} - \dot{p}\underset{\sim}{N}\right) \\ &- \underset{\sim}{N} : \left(\frac{2}{3}(C + C_\phi)\underset{\sim}{\dot{\varepsilon}}^p - \frac{2}{3}C_\phi \underset{\sim}{\dot{\phi}}\right) = 0\end{aligned} \quad (35)$$

$$\dot{p} = \frac{\underset{\sim}{N} : \left(\underset{\approx}{C} : \underset{\sim}{\dot{\varepsilon}} + \frac{2}{3}C_\phi \underset{\sim}{\dot{\phi}}\right)}{\underset{\sim}{N} : \underset{\approx}{C} : \underset{\sim}{N} + C + C_\phi} \quad (36)$$

where both $\underset{\sim}{\dot{\varepsilon}}^p$ and $\underset{\sim}{\dot{\phi}}$ are controllable independent variables at the material point. The consistency condition must be modified in the non-isothermal case due to the general dependence of all parameters on temperature, as shown in [16].

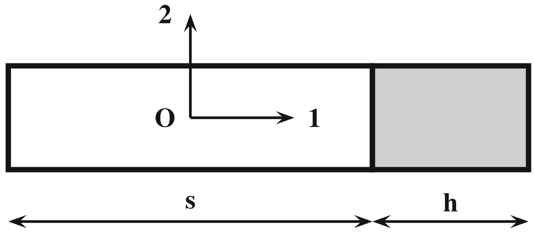
The fact that the gradient of the plastic strain tensor mainly impacts on the kinematic hardening of the material has been recognized in [23, 31].

Size Effects in Strain Gradient Plasticity

Aifantis isotropic strain gradient plasticity model is considered to investigate two distinct size-dependent behaviors, respectively associated with hardening and softening plasticity. The first example is taken from [3] and the second inspired from [19].

Boundary Layers in a Sheared Laminate ($H > 0$)

Laminate microstructures are prone to size effects especially in the case of metals for which the interfaces act as barriers for the motion of dislocations. The material response then strongly depends on the layer thickness. The considered laminate microstructure is a periodic arrangement of two phases including a purely elastic material and a plastic strain gradient layer. The unit cell corresponding to this arrangement is shown in Fig. 1. The thickness of the hard elastic layer is h, whereas the thickness of the soft plastic strain gradient layer is s. The unit cell of Fig. 1 is subjected to a mean simple



Gradient Thermoplasticity, Fig. 1 Unit cell of a periodic two-phase laminate

shear $\bar{\gamma}$ in direction 1. The origin of the coordinate system is the center of the soft phase.

The displacement field is of the form

$$u_1 = \bar{\gamma}x_2, \quad u_2(x_1) = u(x_1), \quad u_3 = 0 \quad (37)$$

where $u(x_1)$ is a periodic function which describes the fluctuation from the homogeneous shear $\bar{\gamma}$. This fluctuation is the main unknown of the boundary value problem. The gradient of the displacement field and the strain tensors are computed as

$$\begin{aligned}[\nabla \underline{u}] &= \begin{bmatrix} 0 & \bar{\gamma} & 0 \\ u_{,1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ [\underset{\sim}{\varepsilon}] &= \begin{bmatrix} 0 & \frac{1}{2}\left(\bar{\gamma} + u_{,1}\right) & 0 \\ \frac{1}{2}\left(\bar{\gamma} + u_{,1}\right) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\end{aligned} \quad (38)$$

where $u_{,1}$ denotes the derivative of the displacement u with respect to x_1. After Hooke's law, the only activated simple stress component is σ_{12}. Due to the balance of momentum equation and the continuity of the traction vector, this stress component is homogeneous throughout the laminate.

The elastic law in the elastic phase and the elastic–plastic response of the soft phase are then exploited to derive the partial differential equations for plastic strain and, finally, for the displacement fluctuation. The explicit solution is found after considering precise interface conditions regarding continuity of various variables. Note that the solution is known for conventional plasticity, that is, in the absence of strain gradient effect. The plastic strain is then expected to be homogeneous in the soft phase for any loading $\bar{\gamma}$. Plastic strain therefore exhibits the usual jump at the interface. The introduction of higher-order interface conditions, associated with strain gradient plasticity, will induce a nonhomogeneous plasticity field.

Assuming plastic loading in the soft phase, the von Mises criterion is fulfilled:

$$\sqrt{3}|\sigma_{12}| = R_0 + Hp - cp_{,11} \tag{39}$$

Since the stress component σ_{12} is uniform, the previous equation can be differentiated with respect to x_1, which gives

$$p_{,111} - \omega^2 p_{,1} = 0, \quad \omega^2 = \omega^2 \frac{H}{c} \tag{40}$$

The form of the plastic strain field for $H > 0$ therefore is

$$p = \alpha \cosh(\omega x_1) + \beta \tag{41}$$

where α and β are integration constants. In the elastic zone, the stress is given by

$$\sigma_{12} = \mu\left(\bar{\gamma} + u^h_{,1}\right) \quad \Rightarrow \quad u^h_{,1} = C \tag{42}$$

An additional integration constant C must be determined. The exponent ^{h} has been added to indicate the displacement fluctuation inside the elastic phase. The arbitrary translation for u^h will be set to zero. The field u^s can be determined from the elasticity law in the soft phase:

$$\sigma_{12} = \mu\left(\bar{\gamma} + u^s_{,1} - \sqrt{3}p\right) \tag{43}$$

An additional constant D arises from the integration of this equation that remains to be determined. The four unknown integration constants α, β, C, D will be determined from 4 conditions at the interface between both materials at $x_1 = \pm s/2$:

- Continuity of simple traction

$$\sqrt{3}\mu(\bar{\gamma} + C) = R_0 + H\beta \tag{44}$$

- Continuity of displacement $u(x_1)$ at $s/2$

$$u^s\left(\frac{s}{2}\right) = u^h\left(\frac{s}{2}\right) \tag{45}$$

$$\begin{aligned} u^h(x_1) &= Cx_1, \quad u^s(x_1) \\ &= \left(\frac{R_0}{\mu\sqrt{3}} + \left(\frac{H}{\mu\sqrt{3}} + \sqrt{3}\right)\beta - \bar{\gamma}\right)x_1 \\ &\quad + \frac{\sqrt{3}\alpha}{\omega}\sinh(\omega x_1) + D \end{aligned} \tag{46}$$

- Periodicity of displacement $u(x_1)$

$$u^s\left(-\frac{s}{2}\right) = u^h\left(\frac{s}{2} + h\right) \tag{47}$$

- Continuity of plastic strain p at the interface $x_1 = \frac{s}{2}$

$$p\left(\frac{s}{2}\right) = 0 \tag{48}$$

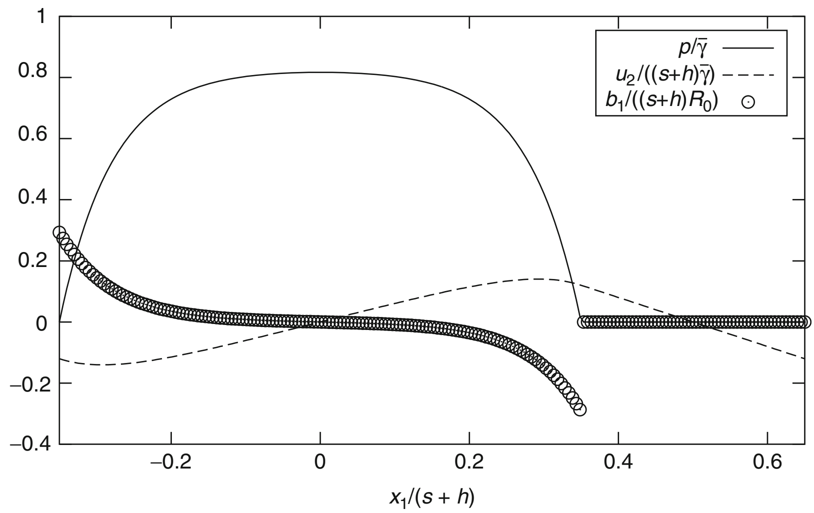
$$\alpha \cosh\left(\omega\frac{s}{2}\right) + \beta = 0 \tag{49}$$

The last condition is necessary to close the system. Differentiability and hence continuity of plastic strain p are required in strain gradient plasticity theory. In the elastic phase, $p = 0$ so that p should also vanish at the interface. The identification of the constants provides

$$\beta = \frac{\left(\bar{\gamma} - \frac{R_0}{\mu\sqrt{3}}\right)(s+h)}{\frac{H}{\mu\sqrt{3}}(s+h) + \sqrt{3}s - \tanh\left(\omega\frac{s}{2}\right)\frac{2\sqrt{3}}{\omega}} \tag{50}$$

$$\alpha = -\frac{\beta}{\cosh\left(\omega\frac{s}{2}\right)} \tag{51}$$

$$C = \frac{R_0}{\mu\sqrt{3}} - \bar{\gamma} + \frac{H}{\sqrt{3}\mu}\beta \tag{52}$$



Gradient Thermoplasticity, Fig. 2 Distributions of plastic strain, normalized displacement fluctuation, and normalized generalized stress vector component in the unit cell of the laminate microstructure

$$D = C\frac{s}{2} - \left(\frac{R_0}{\mu\sqrt{3}} + \left(\frac{H}{\mu\sqrt{3}} + \sqrt{3}\right)\beta - \bar{\gamma}\right)\frac{s}{2} - \frac{\sqrt{3}\alpha}{\omega}\sinh\left(\omega\frac{s}{2}\right) \tag{53}$$

where homogeneous elasticity has been assumed for simplicity, with μ being the shear modulus of both phases. As a result, we find that the double traction cannot vanish on the soft side of the interface, $x_1 = s^-/2$:

$$b_1(x_1) = c\alpha\sinh(\omega x_1),\quad b_1\left(\frac{s^-}{2}\right) = c\alpha\sinh\left(\omega\frac{s}{2}\right) \neq 0 \tag{54}$$

In the elastic phase, the generalized stress identically vanishes since no plastic strain occurs. It follows that the generalized traction b_1 exhibits a jump across the interface.

This solution is illustrated for a special choice of material parameters oriented towards plasticity of metals at the micron scale:

$$s = 0.007\ \text{mm},\ h = 0.003\ \text{mm},\quad \bar{\gamma} = \mu = 300\ \text{GPa},\ R_0 = 20\ \text{MPa},\quad H = 10\ \text{GPa},\quad c = 0.005\ \text{MPa}\cdot\text{mm}^2$$

The distribution of plastic slip, displacement, and generalized stress component b_1 are shown in Fig. 2. The plastic strain displays a typical *cosh* profile with boundary layer effects close to the interface, due to the continuity requirement. The displacement fluctuation is clearly periodic. The jump of the generalized traction at the interface is also visible.

Plastic Strain Localization in a Metal Foam ($H < 0$)

One of the most simple yield functions for compressible elastoplastic materials is the elliptic potential that depends on the first and second invariants of the stress tensor:

$$f(\underset{\sim}{\boldsymbol{\sigma}}) = \sigma_{eq} - \sigma_Y - R,\quad \sigma_{eq}^2 = \frac{3}{2}C\underset{\sim}{\boldsymbol{\sigma}}^{\text{dev}} : \underset{\sim}{\boldsymbol{\sigma}}^{\text{dev}} + F\left(\text{trace}\ \underset{\sim}{\boldsymbol{\sigma}}\right)^2 \tag{55}$$

where C and F are material parameters, possibly depending on material porosity, and R the hardening function. The tensor giving at each instant the direction of plastic flow is

$$\underset{\sim}{\boldsymbol{N}} = \frac{\partial f}{\partial \underset{\sim}{\boldsymbol{\sigma}}} = \frac{1}{\sigma_{eq}}\left(\frac{3}{2}C\underset{\sim}{\boldsymbol{\sigma}}^{\text{dev}} + F\left(\text{trace}\underset{\sim}{\boldsymbol{\sigma}}\right)\underset{\sim}{\mathbf{1}}\right)\quad \text{and}\quad \dot{\underset{\sim}{\varepsilon}}^p = \dot{p}\underset{\sim}{\boldsymbol{N}} \tag{56}$$

according to the normality rule adopted here for simplicity and valid for metallic foams. The second-order identity tensor is $\underset{\sim}{1}$. In the special case of tension/compression in direction 2, the direction of plastic flow becomes

$$[N] = \frac{\operatorname{sign}\sigma_{22}}{\sqrt{C+F}} \begin{bmatrix} F-\frac{C}{2} & 0 & 0 \\ 0 & C+F & 0 \\ 0 & 0 & F-\frac{C}{2} \end{bmatrix} \quad (57)$$

It can be seen that for $F = 0$ (von Mises plasticity), classical von Mises incompressible plasticity is retrieved. The special case $F = C/2$ is associated with no lateral plastic flow in tension/compression. This is a simplification often used for the modeling of the deformation of aluminum foams [19].

A softening modulus $H < 0$ (see (23) or (24)) is introduced to induce strain localization as observed in aluminum foams, for instance. The bifurcation analysis presented in [7] for general nonassociative and compressible elastoplasticity is applied here to the elliptic potential (55). The objective is to determine the orientation of possible strain localization bands that can be deduced from Rice's criterion of loss of ellipticity in elastoplastic solids. Under plane stress conditions, the orientation of the first possible localization band is given by

$$n_1^2 = \frac{2}{3C}\left(\frac{C}{2} - F\right), \quad n_2^2 = 1 - n_1^2, \quad n_3 = 0 \quad (58)$$

where n_i is the unit vector normal to the strain localization band. If $F = 0$ the classical orientation of shear bands at 55° from the loading axis is recovered. If $F = C/2$ it can be seen that $n_1 = 0, n_2 = 1$. This corresponds to a horizontal strain localization band with an opening mode [7]. This ideal orientation of strain localization bands is in accordance with the quasi-horizontal crushing bands frequently observed in aluminum foams under compression. In contrast, more inclined bands usually form in compressed samples of rocks and soils.

The choice $F = C/2$ (and also a vanishing Poisson ratio to simplify the analytical derivation) ensures that the problem becomes actually one dimensional so that an analytical solution of the localization problem can be worked out for tension/compression along direction 2:

$$\sigma_{eq} = \sqrt{C+F}|\sigma_{22}|, \quad \dot{\varepsilon}^p_{22} = \dot{p}\sqrt{C+F} \quad (59)$$

The tensile stress field is homogeneous due to the balance of momentum (7). The yield condition (23) can then be differentiated with respect to the coordinate 2, which provides the differential equation governing ϕ:

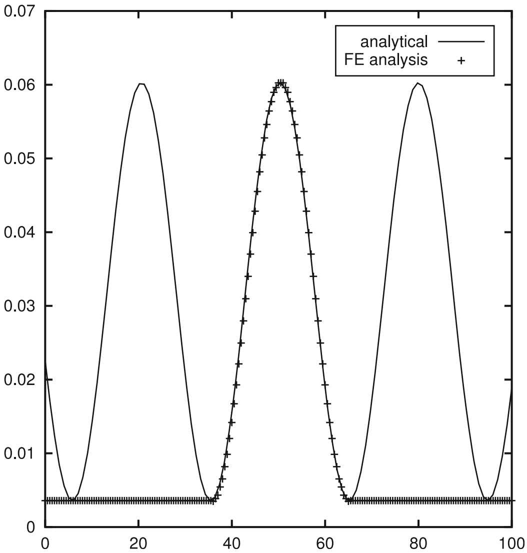
$$\phi_{,222} + \omega^2 \phi_{,2} = 0, \quad \text{with}, \quad \omega^2 = -\frac{H H_\phi}{A(H + H_\phi)} \quad (60)$$

The wave number ω is real when $H < 0$ and $H + H_\phi > 0$. The latter condition takes into account the fact that H_ϕ is a penalty factor that constrains the variable ϕ to remain close to p and should therefore be high enough.

If Aifantis model (24) is adopted instead, the plastic strain is solution of the differential equation:

$$p_{,222} + \omega^2 p_{,2} = 0, \quad \text{with} \quad \omega^2 = -\frac{H}{c} \quad (61)$$

In both cases, due to the fact that $H < 0$, a sinusoidal profile of plastic strain is expected. The strain localization band with indeterminate size predicted by the localization analysis of the classical model is therefore replaced by a strain localization zone of finite width directly related to $1/\omega$. The localization zone is an arc of sinus curve for this simple model. Additional boundary conditions are necessary to solve actually the differential equations. They concern the continuity of the displacement, ϕ or p, and of the dual forces $\underset{\sim}{\sigma} \cdot \underline{n}$ and $\underline{b} \cdot \underline{n}$ at the interface between the localized deformation band and the elastically unloading remaining part of the specimen. The micromorphic model was implemented in a finite element program. It can be checked that mesh



Gradient Thermoplasticity, Fig. 3 Finite element analysis of a homogeneous metal foam block with a central initial defect under uniaxial compression using the micromorphic foam model: Comparison of the strain localization band between the finite element and analytical solutions (horizontal axis, position in mm along the sample; vertical, axial strain component)

refinement leads to a converged deformation zone of finite size. It has the sinusoidal character predicted by the analytical model as shown in Fig. 3.

Gradient Crystal Plasticity

In the crystal plasticity theory at small deformation, the gradient of the velocity field can be decomposed into the elastic and plastic deformation rates:

$$\dot{\underset{\sim}{H}} = \dot{\underline{u}} \otimes \nabla = \dot{\underset{\sim}{H}}^e + \dot{\underset{\sim}{H}}^p \tag{62}$$

where

$$\dot{\underset{\sim}{H}}^p = \sum_\alpha \dot{\gamma}^\alpha \underline{l}^\alpha \otimes \underline{n}^\alpha \tag{63}$$

with α as the number of slip systems, $\dot{\gamma}^\alpha$ the slip rate for the slip system α, $\underline{l}$ the slip direction, and $\underline{n}$ the normal to the slip plane [33]. The elastic deformation $\underset{\sim}{H}^e$ bridges the gap between the compatible total deformation $\underset{\sim}{H}$ and the incompatible plastic deformation $\underset{\sim}{H}^p$. Applying the curl operator to a compatible field gives zero so that

$$\text{curl}\, \dot{\underset{\sim}{H}} = 0 = \text{curl}\, \dot{\underset{\sim}{H}}^e + \text{curl}\, \dot{\underset{\sim}{H}}^p \tag{64}$$

The incompatibility of plastic deformation is characterized by its curl part called dislocation density tensor $\underset{\sim}{\Gamma}$ [31] defined here as

$$\underset{\sim}{\Gamma} = \text{curl}\, \underset{\sim}{H}^p = -\text{curl}\, \underset{\sim}{H}^e \tag{65}$$

This prompts us to consider now a strain gradient plasticity theory which includes the curl of the plastic deformation tensor, $\underset{\sim}{H}^p$, instead of the full gradient with the following power densities of internal and contact forces [6]:

$$\begin{aligned} p^{(i)} &= \underset{\sim}{\sigma} : \dot{\underset{\sim}{H}} + \underset{\sim}{s} : \dot{\underset{\sim}{H}}^p + \underset{\sim}{M} : \text{curl}\, \dot{\underset{\sim}{H}}^p \\ p^{(c)} &= \underline{t} \cdot \dot{\underline{u}} + \underset{\sim}{m} : \dot{\underset{\sim}{H}}^p \end{aligned} \tag{66}$$

where $\underline{t}$, $\underset{\sim}{m}$ are, respectively, the surface simple and double tractions. Volume forces are not written for simplicity. For objectivity reasons, the stress tensor $\underset{\sim}{\sigma}$ is symmetric whereas the micro-stress tensor $\underset{\sim}{s}$ and the double-stress tensor $\underset{\sim}{M}$ are generally nonsymmetric. The method of virtual power can be used to derive the field equations governing this continuum:

$$\text{div}\, \underset{\sim}{\sigma} = 0, \quad \text{curl}\, \underset{\sim}{M} + \underset{\sim}{s} = 0, \quad \varepsilon_{jkl} M_{ik,l} + s_{ij} = 0 \tag{67}$$

for all regular points of the domain Ω. Furthermore, the following boundary conditions on $\partial\Omega$ can be derived:

$$\underline{t} = \underset{\sim}{\sigma} \cdot \underline{n}, \quad \underset{\sim}{m} = \underset{\sim}{M} \cdot \underset{\approx}{\varepsilon} \cdot \underline{n}, \quad m_{ij} = M_{ik} \varepsilon_{kjl} n_l \tag{68}$$

The free energy density is taken as a function of the elastic strain, $\underset{\sim}{\varepsilon}^e$; the dislocation density tensor, that is, curl $\underset{\sim}{H}^p$; and a generic internal hardening variable, α. As a result, the Clausius–Duhem inequality becomes

$$\left(\underset{\sim}{\boldsymbol{\sigma}} - \rho \frac{\partial \psi}{\partial \underset{\sim}{\varepsilon}^e}\right) : \dot{\underset{\sim}{\varepsilon}}^e + \left(\underset{\sim}{\boldsymbol{M}} - \rho \frac{\partial \psi}{\partial \text{curl}\, \underset{\sim}{\boldsymbol{H}}^p}\right) : \text{curl}\, \dot{\underset{\sim}{\boldsymbol{H}}}^p + \left(\underset{\sim}{\boldsymbol{\sigma}} + \underset{\sim}{\boldsymbol{s}}\right) : \dot{\underset{\sim}{\boldsymbol{H}}}^p - X\dot{\alpha} \geq 0 \tag{69}$$

Here, the constitutive assumption is made that the two first terms in the previous inequality are nondissipative and therefore should vanish. Then,

$$\underset{\sim}{\boldsymbol{\sigma}} = \rho \frac{\partial \psi}{\partial \underset{\sim}{\varepsilon}^e}, \quad \underset{\sim}{\boldsymbol{M}} = \rho \frac{\partial \psi}{\partial \text{curl}\, \underset{\sim}{\boldsymbol{H}}^p} \tag{70}$$

It follows that the residual dissipation rate is

$$\left(\underset{\sim}{\boldsymbol{\sigma}} + \underset{\sim}{\boldsymbol{s}}\right) : \dot{\underset{\sim}{\boldsymbol{H}}}^p - X\dot{\alpha} \geq 0 \tag{71}$$

The existence of a dissipation potential, namely, $\Omega\left(\underset{\sim}{\boldsymbol{\sigma}} + \underset{\sim}{\boldsymbol{s}}, X\right)$

$$\dot{\underset{\sim}{\boldsymbol{H}}}^p = \frac{\partial \Omega}{\partial \left(\underset{\sim}{\boldsymbol{\sigma}} + \underset{\sim}{\boldsymbol{s}}\right)}, \quad \dot{\alpha} = -\frac{\partial \Omega}{\partial X} \tag{72}$$

The viscoplastic potential is a function of the generalized Schmid criterion associated to each slip system:

$$f^s\left(\underset{\sim}{\boldsymbol{\sigma}} + \underset{\sim}{\boldsymbol{s}}\right) = \left|\left(\underset{\sim}{\boldsymbol{\sigma}} + \underset{\sim}{\boldsymbol{s}}\right) : \underline{\boldsymbol{l}}^s \otimes \underline{\boldsymbol{n}}^s\right| - \tau_c^s \tag{73}$$

where τ_c^s is the critical resolved shear stress for slip system s. The resolved shear stress is $\tau^s = \underset{\sim}{\boldsymbol{\sigma}} : \underline{\boldsymbol{l}}^s \otimes \underline{\boldsymbol{n}}^s$, and it can be seen that a back-stress can attributed to each slip system in the form:

$$\begin{aligned} x^s &= -\underset{\sim}{\boldsymbol{s}} : \underline{\boldsymbol{l}}^s \otimes \underline{\boldsymbol{n}}^s = \text{curl}\, \underset{\sim}{\boldsymbol{M}} : \underline{\boldsymbol{l}}^s \otimes \underline{\boldsymbol{n}}^s \\ &= A\, \text{curl}\, \text{curl}\, \underset{\sim}{\boldsymbol{H}}^p : \underline{\boldsymbol{l}}^s \otimes \underline{\boldsymbol{n}}^s \end{aligned} \tag{74}$$

where the balance (67) has been taken into account. A linear constitutive equation was finally assumed relating the double-stress tensor $\underset{\sim}{\boldsymbol{M}}$ to curl $\underset{\sim}{\boldsymbol{H}}^p$. As a result, this strain gradient plasticity theory contains a size-dependent kinematic hardening component for each slip system that is related to the second spatial derivative of plastic deformation. This theory can also be seen as a particular case of a more general micromorphic crystal plasticity model [6].

Temperature Gradient Effects

Strain gradient plasticity models are needed in situations where strong strain gradients are encountered due to strain localization or size effects. Such situations may well be associated with strong temperature gradients that can affect the material's behavior in a way different than in usual continuum thermomechanics. In particular, a dependence of the internal energy on the entropy gradient or on the free energy function on the temperature gradient can be envisaged in strongly out of equilibrium physical situations like material processing or assembling or laser treatment of materials and coatings. Several theories in that direction have been formulated recently [18, 24, 28, 30]. They lead to generalized heat equations coupled to mechanics [34].

The micromorphic approach can again be used to encompass a large number of such theories [17]. The presentation is now restricted to the thermal aspects only, the coupling with mechanics being a direct application of the previous sections. A micro-entropy variable $\dot{\phi}_\eta$ is introduced in addition to the usual entropy variable η:

$$DOF = \{\eta,\ \phi_\eta\},\ STATE = \left\{\eta,\ \phi_\eta,\ \nabla\phi_\eta\right\} \tag{75}$$

Within the micromorphic approach, there exists an additional independent power expenditure due to $\dot{\phi}_\eta$ and $\nabla\dot{\phi}_\eta$:

$$p^{(i)} = a\,\dot{\phi}_\eta + \underline{\boldsymbol{b}}.\nabla\dot{\phi}_\eta, \quad p^{(c)} = a^c\,\dot{\phi}_\eta \tag{76}$$

that leads to the additional balance (8), with the associated boundary conditions. Assuming the functional dependence $\varepsilon(\eta, \phi_\eta, \nabla\phi_\eta)$ for the internal energy density, the Clausius–Duhem inequality takes the form:

$$\rho\left(\theta - \frac{\partial \varepsilon}{\partial \eta}\right)\dot{\eta} + \left(a - \rho\frac{\partial \varepsilon}{\partial \phi_\eta}\right)\dot{\phi}_\eta + \left(\underline{\boldsymbol{b}} - \rho\frac{\partial \varepsilon}{\partial \nabla\phi_\eta}\right).\nabla\dot{\phi}_\eta - \underline{\boldsymbol{q}}.\frac{\nabla\theta}{\theta} \geq 0 \tag{77}$$

The following state equations are adopted:

$$\theta = \frac{\partial \varepsilon}{\partial \eta}, \quad a = \rho \frac{\partial \varepsilon}{\partial \phi_\eta}, \quad \underset{\sim}{\boldsymbol{b}} = \rho \frac{\partial \varepsilon}{\partial \nabla \phi_\eta} \tag{78}$$

Within a linear context, the following quadratic potential is proposed:

$$\begin{aligned} \rho\varepsilon\left(\eta, \phi_\eta, \nabla\phi_\eta\right) = \rho\eta\theta_0 + \frac{\rho^2(\eta-\eta_0)^2}{4\beta} \\ + \frac{1}{2}H_\phi\left(\eta - \phi_\eta\right)^2 \\ + \frac{1}{2}A\nabla\phi_\eta.\nabla\phi_\eta \end{aligned} \tag{79}$$

The parameter H_ϕ has been added in order to penalize the difference between micro- and macro-entropy. It follows that

$$\begin{aligned} \theta = \theta_0 + \frac{\rho}{2\beta}(\eta - \eta_0) + \frac{H_\phi}{\rho}(\eta - \phi_\eta) \\ a = -H_\phi(\eta - \phi_\eta), \quad \underset{\sim}{\boldsymbol{b}} = A\nabla\phi_\eta \end{aligned} \tag{80}$$

Positivity of dissipation is still ensured by a Fourier law of the form $\boldsymbol{q} = -\underset{\sim}{\boldsymbol{\kappa}} \cdot \nabla\eta$. The energy balance equation leads to the usual heat equation

$$\rho\eta\dot{\theta} = -\mathrm{div}\,\underset{\sim}{\boldsymbol{q}} \tag{81}$$

After combining the (81), the relevant state laws, and the balance (8), the following generalized heat equation is obtained:

$$\rho\theta_0\dot{\eta} = \kappa\left(\frac{\rho}{2\beta}\nabla^2\eta - \frac{A}{\rho}\nabla^4\phi_\eta\right) \tag{82}$$

An internal constraint can be enforced such that the micro-entropy coincides with the entropy itself. The micro-entropy theory then reduces to an entropy gradient model. Such a condition is almost exactly satisfied when the penalty factor H_ϕ becomes sufficiently high. For a strict respect of the internal constraint, a Lagrange multiplier must be introduced. The (82) then takes the form of a Cahn–Hilliard equation for entropy.

Modifications of the Fourier law can also been envisaged instead of enriching the internal energy function, thus introducing higher-order thermal effects in the dissipative part of material's behavior. A Cahn–Hilliard type of heat equation with respect to temperature is derived in that way in [32].

References

1. Aifantis E (1984) On the microstructural origin of certain inelastic models. J Eng Mater Technol 106:326–330
2. Aifantis E (1987) The physics of plastic deformation. Int J Plasticity 3:211–248
3. Altenbach H, Maugin GA, Erofeev V (2011) Mechanics of generalized continua, vol 7, Advanced structured materials. Springer, Berlin
4. Anand L, Aslan O, Chester S (2012) A large–deformation gradient theory for elastic–plastic materials: strain softening and regularization of shear bands. Int J Plasticity 30–31:116–143
5. Ashby M (1971) The deformation of plastically non-homogeneous alloys. In: Kelly A, Nicholson R (eds) Strengthening methods in crystals. Applied Science Publishers, London, pp 137–192
6. Aslan O, Cordero NM, Gaubert A, Forest S (2011) Micromorphic approach to single crystal plasticity and damage. Int J Eng Sci 49:1311–1325
7. Besson J (2004) Local approach to fracture. Ecole des Mines de Paris–Les Presses, Paris
8. Borst R, Sluys L, Mühlhaus H, Pamin J (1993) Fundamental issues in finite element analyses of localization of deformation. Eng Comput 10:99–121
9. Chambon R, Caillerie D, Matsuchima T (2001) Plastic continuum with microstructure, local second gradient theories for geomaterials. Int J Solids Struct 38:8503–8527
10. Dimitrijevica B, Aifantis K, Hackl K (2012) The influence of particle size and spacing on the fragmentation of nanocomposite anodes for Li batteries. J Power Sources 206:343–348
11. Engelen R, Geers M, Baaijens F (2003) Nonlocal implicit gradient-enhanced elasto-plasticity for the modelling of softening behaviour. Int J Plasticity 19:403–433
12. Eringen A (1999) Microcontinuum field theories. Springer, New York
13. Eringen A, Suhubi E (1964) Nonlinear theory of simple microelastic solids. Int J Eng Sci 2(189–203):389–404
14. Fleck N, Hutchinson J (1997) Strain gradient plasticity. Adv Appl Mech 33:295–361
15. Fleck N, Hutchinson J (2001) A reformulation of strain gradient plasticity. J Mech Phys Solids 49:2245–2271
16. Forest S (2009) The micromorphic approach for gradient elasticity, viscoplasticity and damage. ASCE J Eng Mech 135:117–131

17. Forest S, Aifantis EC (2010) Some links between recent gradient thermo-elasto-plasticity theories and the thermomechanics of generalized continua. Int J Solids Struct 47:3367–3376
18. Forest S, Amestoy M (2008) Hypertemperature in thermoelastic solids. C R Méc 336:347–353
19. Forest S, Blazy J, Chastel Y, Moussy F (2005) Continuum modelling of strain localization phenomena in metallic foams. J Mater Sci 40:5903–5910
20. Germain P (1973) The method of virtual power in continuum mechanics. Part 2: microstructure. SIAM J Appl Math 25:556–575
21. Germain P, Nguyen Q, Suquet P (1983) Continuum thermodynamics. J Appl Mech 50:1010–1020
22. Gurtin M (1996) Generalized Ginzburg–Landau and Cahn–Hilliard equations based on a microforce balance. Physica D 92:178–192
23. Gurtin M (2003) On a framework for small–deformation viscoplasticity: free energy, microforces, strain gradients. Int J Plasticity 19:47–90
24. Ireman P, Nguyen QS (2004) Using the gradients of temperature and internal parameters in continuum mechanics. C R Méc 332:249–255
25. Maugin GA (1992) Thermomechanics of plasticity and fracture. Cambridge University Press, Cambridge
26. Mindlin R (1964) Micro–structure in linear elasticity. Arch Rat Mech Anal 16:51–78
27. Mindlin R, Eshel N (1968) On first strain gradient theories in linear elasticity. Int J Solids Struct 4:109–124
28. Nguyen QS (2010) Gradient thermodynamics and heat equations. C R Méc 338:321–326
29. Nguyen QS (2011) Variational principles in the theory of gradient plasticity. C R Méc 339: 743–750
30. Nguyen QS, Andrieux S (2005) The non–local generalized standard approach: a consistent gradient theory. C R Méc 333:139–145
31. Steinmann P (1996) Views on multiplicative elastoplasticity and the continuum theory of dislocations. Int J Eng Sci 34:1717–1735
32. Temizer I, Wriggers P (2010) A micromechanically motivated higher–order continuum formulation of linear thermal conduction. ZAMM 90:768–782
33. Teodosiu C (1997) Large plastic deformation of crystalline aggregates, vol 376, CISM Courses and Lectures. Springer, Udine/Berlin
34. Voyiadjis G, Faghihi D (2012) Thermo-mechanical strain gradient plasticity with energetic and dissipative length scales. Int J Plasticity 30–31: 218–247

Granite

► Damage in Granite Under Temperature Variations

Green's Function

► Green's Function for Thermal and Mechanical Mixed Boundary Value Problem for an Elliptic Hole
► Green's Function of Heat Source for Mixed Boundary Value Problem
► Green's Function of Thermoelastic Mixed Boundary Value Problem for Elliptic Hole
► Magneto-Electro-Thermoelastic Problems: Fundamental Solutions and Green's Function
► Thermoelastostatics of Transversely Isotropic Materials: Fundamental Solutions and Green's Functions

Green's Function for Thermal and Mechanical Mixed Boundary Value Problem for an Elliptic Hole

Norio Hasebe
Department of Civil Engineering,
Nagoya Institute of Technology, Showa-ku,
Nagoya, Japan

Synonyms

Elliptical hole; Green's function; Mechanical mixed boundary value problem; Thermal mixed boundary value problem

Overview

Green's functions play an important role in the solution of problems in mechanics and physics of solids. They represent kernel functions or fundamental solutions for analytic or numerical methods, such as singular integral equation methods, boundary element methods, eigenstrain approach, and dislocation methods [1–3].

For thermoelastic problems, Green's function for a heat source has received considerable attention. Some fundamental thermoelastic problems were treated in [4–6]. Using these thermoelastic theories, Green's functions of a heat source near an