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Finite-deformation second-order micromorphic theory and its relations to strain and stress gradient models

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Abstract
Germain's general micromorphic theory of order n is extended to fully non-symmetric higher-order tensor degrees of freedom. An interpretation of the microdeformation kinematic variables as relaxed higher-order gradients of the displacement field is proposed. Dynamical balance laws and hyperelastic constitutive equations are derived within the finite deformation framework. Internal constraints are enforced to recover strain gradient theories of grade n. An extension to finite deformations of a recently developed stress gradient continuum theory is then presented, together with its relation to the second-order micromorphic model. The linearization of the combination of stress and strain gradient models is then shown to deliver formulations related to Eringen's and Aifantis's well-known gradient models involving the Laplacians of stress and strain tensors. Finally, the structures of the dynamical equations are given for strain and stress gradient media, showing fundamental differences in the dynamical behaviour of these two classes of generalized continua.

Keywords
Micromorphic continuum, strain gradient theory, stress gradient theory, finite deformation, method of virtual power, generalized continua, dispersion relations

1. Introduction

The micromorphic theory is currently arousing strong interest in the mechanics of materials community, due to its ability to account for size effects in the continuum modelling of many physical phenomena like strain and damage localization or the dispersion of elastic waves; see for example [1] and [2]. Eringen and Mindlin's original model goes back to the early 1960s; see [3] and [4] for the presentation at finite deformations. The generally incompatible microdeformation field variable $\underset{\sim}{\boldsymbol{\chi}}$ was introduced by these authors to represent the deformation of a triad of directors attached to the material's microstructure, like lattice vectors in crystalline solids or fibre directions in composite materials. This represents a drastic enhancement of the continuum theory by nine additional degrees of freedom complementing the displacement vector of the material point.

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The most ambitious extension of the classical Cauchy continuum model is probably Germain's general micromorphic theory which introduces higher-order microdeformations, $\chi_{ij}, \chi_{ijk}, \chi_{ijkl\ldots}$, of increasing tensor order up to order n [5]. Germain's vision of this hierarchy of additional degrees of freedom is related to a Taylor expansion of the description around the material point. Due to their definition in terms of a Taylor expansion, the general microdeformations are symmetric with respect to all the indices except the first one. Germain's paper provides the hierarchy of balance of momentum equations. However, it does not provide constitutive equations for this class of media. It should be noted that Eringen himself also extended the original micromorphic model to higher-order microdeformation tensors based on averaging procedures [6, 7]. However, his theory was shown to be incomplete by Germain, leaving indeterminate higher-order micromorphic variables.

The multipolar continuum mechanics proposed by Green and Rivlin [8] also represents one of the most general continuum models involving both higher-order field variables and higher-order gradients. Green and Rivlin's multipolar theory was compared to the general micromorphic one by Germain [5]. Such a comparison will also be drawn for the theory proposed in the present work.

Suitable internal constraints on the microdeformation can be introduced so that Eringen's micromorphic model reduces to the strain gradient theory proposed by Mindlin [9], which is identical to the second gradient of the displacement model. Such a reduction has been proposed by Bleustein [10] and recently discussed by Madeo et al.[11] and Broese et al. [12]. This reduction can be applied to Germain's general micromorphic media. For instance, the second-order micromorphic theory ($n = 2$) can be shown to degenerate into Mindlin's second strain gradient model by constraining the first and second microdeformations to coincide with the deformation first and second gradients, respectively; see [9,13,14]. In gradient theories, the micromorphic degrees of freedom are therefore eliminated and a single balance equation of higher order remains to solve for the displacement field. Finite element simulations of strain gradient materials very often rely on the use of such a constrained micromorphic theory; see [15,16,17,18,19] and the discussion in [20]. Mindlin's second strain gradient theory [9] was recently applied to the elasticity of nano-objects with surface effects [14] and to the numerical analysis of singularities at edges and corners [21]. The associated computational analysis was based on a constrained second-order micromorphic theory where the first- and second-order microdeformations are constrained to coincide with the first and second gradients of the displacement field [22,14]. These internal constraints were only considered for micromorphic media at small strains. Conversely, the micromorphic continua can be seen as relaxed higher-grade materials.

A third-rank tensor as additional independent kinematic degrees of freedom is also present in the stress gradient theory recently proposed by Forest and Sab [23] and Sab et al. [24]. The stress gradient model is a completely new continuum theory which was shown to be fundamentally different from the strain gradient approach. It was inspired by the bending gradient theory for thick plates according to Lebée and Sab [25,26]. The third-order degrees of freedom are conjugate to the gradient of the classical stress tensor in the work of internal forces. They therefore have a physical unit different to Germain's second-order microdeformation. An alternative stress gradient theory was designed by Polizzotto [27,28] where third-order kinematic test functions also arise but are not treated as independent degrees of freedom of the theory. The original stress gradient elasticity model was then shown to lead to a well-posed boundary value problem with the new boundary conditions proposed by Forest and Sab [23] and Sab et al. [24]. In particular, in a static stress gradient medium, the full stress tensor can be prescribed at the boundary, in contrast to Cauchy's model for which only the traction vector is controlled. The stress gradient theories were originally formulated in the context of linear elasticity and a complete stress gradient theory at finite deformation is still missing.

The theory of elastoviscoplasticity for first-order micromorphic media at finite deformation has been well established since the works by Forest and Sievert [29,30], Regueiro [31] and Sansour et al. [32], and so on. First strain gradient theory was also explored at finite elastoviscoplastic deformations in [29,33,34]. Second strain gradient hyperelasticity at finite deformation was then considered by Javili et al. [13]. However, the reduction of the hyperelastic laws based on invariance requirements was not performed in the latter reference, so suitable Lagrangian strain measures remain to be defined. The treatment of internal constraints in gradient continua at finite strains was presented recently by Bertram and Glüge [35].

The objective of the present work is to formulate a phenomenological theory of higher-order micromorphic media generalizing Germain's one, and to establish the links to strain and stress gradient models. The formulation is presented in the finite deformation setting. For that purpose, a constitutive framework is proposed for hyperelasticity, taking into account possible internal constraints. Its linearization will be shown to provide general equations for the statics and dynamics of second strain gradient media and of stress gradient media. In particular, the dynamical stress gradient theory will be formulated at finite strain for the first time, providing the balance, boundary and Lagrangian constitutive equations.

The proposed extension of Germain's theory consists in abandoning the symmetry requirements for the microdeformation tensors, thus departing from the Taylor expansion approach. The theory is presented for the order $n = 2$ (first- and second-order microdeformation tensors of orders two and three respectively) and limited to the grade $p = 1$, considering only the first gradient of all degrees of freedom. Extensions to order n and grade p are possible but not considered in the present work for the sake of simplicity.

The higher-order kinematics is presented in Section 2. Following [5,36], the method of virtual power is used to derive balance equations and the associated boundary conditions. Lagrangian strain measures are proposed to formulate hyperelastic constitutive equations in Section 3, based on the Helmholtz or Gibbs free energy potentials. This section ends with the consideration of general internal constraints and the consequences of the formulation of hyperelastic laws based on the exploitation of the second principle of thermodynamics. Specific internal constraints are then discussed in Section 4 in order to relate the general theory to strain gradient and stress gradient models. Section 4.2 is devoted to the formulation of the stress gradient theory at finite deformation. This theory is then linearized and shown to coincide with the already existing linear elastic stress gradient model. The applications presented in Section 5 deal with the statics and dynamics of combined linear stress and strain gradient media.

1.1. Notation

The material points of the body are labelled according to their position vectors $\underline{\boldsymbol{X}}$ with respect to a reference configuration Ω_0. They occupy the positions $\underline{\boldsymbol{x}} = \Phi(\underline{\boldsymbol{X}}, t)$ in the current configuration Ω of the body at time t. Their coordinates are expressed in two distinct Cartesian orthonormal bases:

$$\underline{\boldsymbol{X}} = X_I\, \underline{\boldsymbol{E}}_I, \quad \underline{\boldsymbol{x}} = x_i\, \underline{\boldsymbol{e}}_i \tag{1}$$

where upper-case (respectively, lower-case) letters are used for indices referring to the reference (respectively, current) configuration of the body. The components X_I (respectively, x_i) are called Lagrangian (respectively, Eulerian) coordinates. Einstein's convention on summation of repeated indices is enforced.

The Lagrangian (respectively, Eulerian) volume and surface elements are dV (respectively, dv) and dS (respectively, ds), respectively. The gradient operators are written as

$$\nabla^0 = \frac{\partial}{\partial X_I}\underline{\boldsymbol{E}}_I, \quad \nabla = \frac{\partial}{\partial x_i}\underline{\boldsymbol{e}}_i \tag{2}$$

In index notation, we write

$$u_{i,J} = \frac{\partial u_i}{\partial X_J} \tag{3}$$

An intrinsic type of notation is used whereby tensors of orders one, two, three, four and six are respectively denoted by $\underline{\boldsymbol{a}}, \underset{\sim}{\boldsymbol{A}}, \underset{\approx}{\boldsymbol{A}}$ (or $\underset{\simeq}{\boldsymbol{A}}$), $\underset{\approx}{\boldsymbol{A}}$, and $\underset{\approx\atop\sim}{\boldsymbol{A}}$. To avoid any ambiguity, the corresponding index notation is often provided together with the intrinsic one.

2. Kinematics and balance laws for higher-order micromorphic media

2.1. Higher-order micromorphic degrees of freedom

Each material point is endowed with the following set of generalized degrees of freedom:

$$\begin{aligned} \text{DOF} &= \{\underline{\boldsymbol{u}}, \quad \underset{\sim}{\chi}, \quad \underset{\simeq}{\chi}, \quad \underset{\approx}{\chi}, \quad \dots\} \\ \text{DOF} &= \{u_i, \quad \chi_{iJ}, \quad \chi_{iJK}, \quad \chi_{iJKL}, \quad \dots\} \end{aligned} \tag{4}$$

where $\underline{\boldsymbol{u}} = \Phi(\underline{\boldsymbol{X}}, t) - \underline{\boldsymbol{X}}$ is the displacement vector and the micromorphic degrees of freedom are independent tensors of increasing orders. Following notation similar to that introduced by Mindlin [9], the same letter χ is used for the independent micromorphic degrees of freedom of various orders. These variables can be distinguished by the indication of respective tensor rank in the tensor notation or from the number of indices.

The first- and higher-order microdeformation tensor variables are generally non-compatible fields. They do not possess any symmetry property a priori. In particular, the second-order microvariable χ_{iJK} does not exhibit any symmetry with respect to the last two indices. This is in contrast to Germain's general micromorphic medium where the higher-order microvariables are conceived as the coefficients in a Taylor expansion of the relative motion of the material particle with respect to its centre of mass. Such symmetry properties also hold in Eringen's higher-grade micromorphic theory where higher-order moments of the microfields are introduced [7]. The physical meaning and physical dimension of the higher-order micromorphic variables are left unspecified since they will depend on the specific application. However, if

$$\underset{\sim}{\boldsymbol{F}} = \underset{\sim}{\boldsymbol{1}} + \underline{\boldsymbol{u}} \otimes \nabla^0 = F_{iJ}\, \underline{\boldsymbol{e}}_i \otimes \underline{\boldsymbol{E}}_J, \quad J = \det \underset{\sim}{\boldsymbol{F}} > 0 \tag{5}$$

is the deformation gradient of the continuum, the first interpretation of the micromorphic degrees of freedom proposed in this work is the relaxation of $\underset{\sim}{\boldsymbol{F}}$ and its gradients of increasing order. The microdeformation tensor $\underset{\sim}{\boldsymbol{\chi}}$ is viewed as the relaxed counterpart of $\underset{\sim}{\boldsymbol{F}}$ meaning that it is a generally incompatible deformation field, in contrast to $\underset{\sim}{\boldsymbol{F}}$. The reference state of the microdeformation is $\underset{\sim}{\boldsymbol{\chi}} = \underset{\sim}{\boldsymbol{1}}$ and it is assumed that $\det \underset{\sim}{\boldsymbol{\chi}} > 0$. Similarly, the second-order microdeformation χ_{iJK} represents the relaxed microdeformation gradient $\chi_{iJ,K}$, and, consequently, the relaxation of the second gradient $F_{iJ,K} = u_{i,JK}$ which is symmetric with respect to the last two indices. The reference states of the second- and higher-order microdeformation tensors are zero. According to this definition, the physical dimension of each higher-order microdeformation tensor is that of the corresponding higher gradient of $\underset{\sim}{\boldsymbol{F}}$, so $\underset{\sim}{\boldsymbol{\chi}}$ is dimensionless, $\underset{\approx}{\boldsymbol{\chi}}$ is in m^{-1}, and so on.

The transformation of the deformation gradient by a change of observer represented by the time-dependent rotation $\underset{\sim}{\boldsymbol{Q}}(t)$ is as follows:

$$\underset{\sim}{\boldsymbol{F}} \longrightarrow \underset{\sim}{\boldsymbol{Q}} \cdot \underset{\sim}{\boldsymbol{F}} \tag{6}$$

The following transformation rules are assumed for the micromorphic degrees of freedom:

$$\underset{\sim}{\boldsymbol{\chi}} \longrightarrow \underset{\sim}{\boldsymbol{Q}} \cdot \underset{\sim}{\boldsymbol{\chi}} \qquad F_{iJ} \longrightarrow Q_{ik}F_{kJ} \tag{7}$$

$$\underset{\approx}{\boldsymbol{\chi}} \longrightarrow \underset{\sim}{\boldsymbol{Q}} \cdot \underset{\approx}{\boldsymbol{\chi}} \qquad \chi_{iJK} \longrightarrow Q_{il}\chi_{lJK} \tag{8}$$

$$\ldots$$

The generalized degrees of freedom $\chi_{iJK\ldots}$ have the same structure and transformation rules as the multipolar displacements introduced by [8], later called *multipolar deformation* fields in [37]. The fundamental difference with the present theory lies in the generalization of the balance of momentum equation which is absent in Green and Rivlin's theory, as recognized by Germain [5]. Green and Rivlin's interpretation of the multipolar fields, presented in the appendix of [8], involves a collection of particles and is based on a Taylor expansion whereby the multipolar tensors must be completely symmetric with respect to all indices excepted the first. Their multipolar displacements therefore represent a generalization of this concept to multiparticle systems.

The interpretation of the first microdeformation tensor as the linear transformation of a triad of directors attached to the microstructure is particularly illustrative. Extension to higher-order microdeformation is possible based on higher-order tensor products of directors as proposed by Green et al. [38].

2.2. Lagrangian generalized strain measures

The proposed theory is a continuum model of order n and of grade 1, meaning that only the first gradient of all degrees of freedom is considered. The set GRAD contains all the available constitutive variables of the model:

$$\mathrm{GRAD} = \{\underline{\boldsymbol{u}}, \quad \underline{\boldsymbol{u}} \otimes \nabla^0, \quad \underset{\sim}{\boldsymbol{\chi}}, \quad \underset{\sim}{\boldsymbol{\chi}} \otimes \nabla^0, \quad \underset{\approx}{\boldsymbol{\chi}}, \quad \underset{\approx}{\boldsymbol{\chi}} \otimes \nabla^0, \quad \ldots\} \tag{9}$$

$$\mathrm{GRAD} = \{u_i, \quad u_{i,J}, \quad \chi_{iJ}, \quad \chi_{iJ,K}, \quad \chi_{iJK}, \quad \chi_{iJK,L}, \quad \ldots\}$$

Appropriate strain measures are obtained by considering the invariance properties of the constitutive functions. In the case of hyperelasticity, the constitutive function is the Helmholtz free energy potential ψ per unit volume. It is a priori a function depending on all the variables contained in GRAD. According to the requirement of Galilean invariance, it must be invariant with respect to all Galilean transformations. The translation invariance excludes the presence of the displacement in the arguments of the constitutive function. The invariance with respect to all constant rotations $\underset{\sim}{\boldsymbol{Q}}_0$ is

$$
\begin{aligned}
&\psi(\underset{\sim}{\boldsymbol{F}},\quad \underset{\sim}{\boldsymbol{\chi}},\quad \underset{\sim}{\boldsymbol{\chi}}\otimes\nabla^0,\quad \underset{\approx}{\boldsymbol{\chi}},\quad \underset{\approx}{\boldsymbol{\chi}}\otimes\nabla^0,\quad \ldots)\\
&=\psi(\underset{\sim}{\boldsymbol{Q}}_0\cdot\underset{\sim}{\boldsymbol{F}},\quad \underset{\sim}{\boldsymbol{Q}}_0\cdot\underset{\sim}{\boldsymbol{\chi}},\quad \underset{\sim}{\boldsymbol{Q}}_0\cdot\underset{\sim}{\boldsymbol{\chi}}\otimes\nabla^0,\quad \underset{\sim}{\boldsymbol{Q}}_0\cdot\underset{\approx}{\boldsymbol{\chi}},\quad \underset{\sim}{\boldsymbol{Q}}_0\cdot\underset{\approx}{\boldsymbol{\chi}}\otimes\nabla^0,\quad \ldots)\\
&=\psi(\underset{\sim}{\boldsymbol{U}},\quad \underset{\sim}{\boldsymbol{R}}^{\mathrm{T}}\cdot\underset{\sim}{\boldsymbol{\chi}},\quad \underset{\sim}{\boldsymbol{R}}^{\mathrm{T}}\cdot\underset{\sim}{\boldsymbol{\chi}}\otimes\nabla^0,\quad \underset{\sim}{\boldsymbol{R}}^{\mathrm{T}}\cdot\underset{\approx}{\boldsymbol{\chi}},\quad \underset{\sim}{\boldsymbol{R}}^{\mathrm{T}}\cdot\underset{\approx}{\boldsymbol{\chi}}\otimes\nabla^0,\quad \ldots)\\
&=\psi(U_{IJ},\quad R^{\mathrm{T}}_{Ik}\chi_{kJ},\quad R^{\mathrm{T}}_{Ij}\chi_{jK,L},\quad R^{\mathrm{T}}_{Ij}\chi_{jKL},\quad R^{\mathrm{T}}_{Ij}\chi_{jKL,M},\quad \ldots)
\end{aligned}
\tag{10}
$$

where the specific choice $\underset{\sim}{\boldsymbol{Q}}_0 = \underset{\sim}{\boldsymbol{R}}^{\mathrm{T}}$ was made among all possible rotations, $\underset{\sim}{\boldsymbol{R}}$ being the rotation part in the polar decomposition of the deformation gradient $\underset{\sim}{\boldsymbol{F}} = \underset{\sim}{\boldsymbol{R}}\cdot\underset{\sim}{\boldsymbol{U}}$, with $\underset{\sim}{\boldsymbol{U}}$ the Lagrangian stretch tensor. The last expression shows that the arguments of the free energy potential are Lagrangian strain measures appropriate for constitutive modelling. In the present work, the following set of Lagrangian strain measures is adopted, without loss of generality:

$$
\psi(\underset{\sim}{\boldsymbol{C}},\quad \underset{\sim}{\boldsymbol{\Upsilon}}:=\underset{\sim}{\boldsymbol{\chi}}^{-1}\cdot\underset{\sim}{\boldsymbol{F}},\quad \underset{\approx}{\boldsymbol{K}}:=\underset{\sim}{\boldsymbol{\chi}}^{-1}\cdot(\underset{\sim}{\boldsymbol{\chi}}\otimes\nabla^0),\quad \underset{\approx}{\boldsymbol{\Upsilon}}:=\underset{\sim}{\boldsymbol{\chi}}^{-1}\cdot\underset{\approx}{\boldsymbol{\chi}},\quad \underset{\approx}{\boldsymbol{K}}:=\underset{\sim}{\boldsymbol{\chi}}^{-1}\cdot(\underset{\approx}{\boldsymbol{\chi}}\otimes\nabla^0),\quad \ldots)
\tag{11}
$$

$$
\psi(C_{IJ},\quad \Upsilon_{IJ}:=\chi^{-1}_{Ik}F_{kJ},\quad K_{IJK}:=\chi^{-1}_{Il}\chi_{lJ,K},\quad \Upsilon_{IJK}:=\chi^{-1}_{Il}\chi_{lJK},\quad K_{IJKL}:=\chi^{-1}_{Im}\chi_{mJK,L},\quad \ldots)
$$

The pull-back of all quantities by $\underset{\sim}{\boldsymbol{\chi}}^{-1}$ provides Lagrangian strain measures. The right Cauchy–Green strain is $\underset{\sim}{\boldsymbol{C}}=\underset{\sim}{\boldsymbol{F}}^{\mathrm{T}}\cdot\underset{\sim}{\boldsymbol{F}}$. The generalized strain measures $\underset{\sim}{\boldsymbol{\Upsilon}}$ and $\underset{\approx}{\boldsymbol{K}}$ as defined in equation (11) are the ones used in Eringen's original micromorphic theory [39,29]. All the arguments of the constitutive function (11) can be expressed in terms of the arguments found in the reduced constitutive law (10), as they should. The Lagrangian generalized strain measure $\underset{\approx}{\boldsymbol{K}}$ possesses the remarkable property that its time derivative can be directly and simply related to the Eulerian gradient of the microdeformation rate:

$$
(\dot{\underset{\sim}{\boldsymbol{\chi}}}\cdot\underset{\sim}{\boldsymbol{\chi}}^{-1})\otimes\nabla = \underset{\sim}{\boldsymbol{\chi}}\cdot\dot{\underset{\approx}{\boldsymbol{K}}}:(\underset{\sim}{\boldsymbol{\chi}}^{-1}\boxtimes\underset{\sim}{\boldsymbol{F}}^{-1}),\quad (\dot{\chi}_{iL}\chi^{-1}_{Lj})_{,k}=\chi_{iP}\dot{K}_{PQR}\chi^{-1}_{Qj}F^{-1}_{Rk}
\tag{12}
$$

Other pull-backs are possible and lead to different strain measures, as usual in finite deformation theories. However, they lead to more complicated expressions as discussed by [39,29] and [33] for the special case of strain gradient theory. The new tensors $\underset{\approx}{\boldsymbol{\Upsilon}}$ and $\underset{\approx}{\boldsymbol{K}}$ represent direct extensions to second-order micromorphic media.

Note that Green and Rivlin [8] define similar Lagrangian generalized strain measures pulled back by means of $\underset{\sim}{\boldsymbol{F}}^{\mathrm{T}}$ instead of $\underset{\sim}{\boldsymbol{\chi}}^{-1}$.

2.3. Generalized principle of virtual power

The method of virtual power is used to introduce the generalized stress tensors of the theory. The virtual power density of internal forces is computed with respect to any subdomain $\mathcal{D}_0$ of the reference configuration Ω_0:

$$
\mathcal{P}^{(i)}(\delta\underline{\boldsymbol{u}},\delta\underset{\sim}{\boldsymbol{\chi}},\delta\underset{\approx}{\boldsymbol{\chi}},\ldots) = -\int_{\mathcal{D}_0} Jp^{(i)}(\delta\underline{\boldsymbol{u}},\delta\underset{\sim}{\boldsymbol{\chi}},\delta\underset{\approx}{\boldsymbol{\chi}},\ldots)\,dV
\tag{13}
$$

where $p^{(i)}$ is the virtual power density of internal forces per unit volume of the current configuration, and J is the Jacobian. It is introduced as a function of the virtual fields represented by variations of displacements and microdeformations of any order. The virtual power density of internal forces is taken as a linear form with respect to the variations of the Lagrangian strain measures (11):

$$
\begin{aligned}
Jp^{(i)} &= \frac{1}{2}\underset{\sim}{\boldsymbol{\Pi}}:\delta\underset{\sim}{\boldsymbol{C}}+\underset{\sim}{\boldsymbol{T}}:\delta\underset{\sim}{\boldsymbol{\Upsilon}}+\underset{\approx}{\boldsymbol{M}}\,\vdots\,\delta\underset{\approx}{\boldsymbol{K}}+\underset{\approx}{\boldsymbol{T}}\,\vdots\,\delta\underset{\approx}{\boldsymbol{\Upsilon}}+\underset{\approx}{\boldsymbol{M}}::\delta\underset{\approx}{\boldsymbol{K}}\\
&= \frac{1}{2}\Pi_{IJ}\,\delta C_{IJ}+T_{IJ}\,\delta\Upsilon_{IJ}+M_{IJK}\,\delta K_{IJK}+T_{IJK}\,\delta\Upsilon_{IJK}+M_{IJKL}\,\delta K_{IJKL}
\end{aligned}
\tag{14}
$$

where $\underset{\sim}{\boldsymbol{\Pi}}$ is the usual Piola stress tensor, and $\underset{\sim}{\boldsymbol{T}}$ and $\underset{\approx}{\boldsymbol{T}}$ are the stress tensors conjugate to the microdeformations, whereas $\underset{\approx}{\boldsymbol{M}}$ and $\underset{\approx}{\underset{\sim}{\boldsymbol{M}}}$ are generalized couple stress tensors. The variations of the strains can be expressed in terms of the virtual displacements and microdeformations:

$$\delta\underset{\sim}{\boldsymbol{C}} = \underset{\sim}{\boldsymbol{F}}^{\mathrm{T}}\cdot\delta\underset{\sim}{\boldsymbol{F}} + \delta\underset{\sim}{\boldsymbol{F}}^{\mathrm{T}}\cdot\underset{\sim}{\boldsymbol{F}} \tag{15}$$

$$\delta\underset{\sim}{\boldsymbol{\Upsilon}} = \underset{\sim}{\boldsymbol{\chi}}^{-1}\cdot\delta\underset{\sim}{\boldsymbol{F}} - \underset{\sim}{\boldsymbol{\chi}}^{-1}\cdot\delta\underset{\sim}{\boldsymbol{\chi}}\cdot\underset{\sim}{\boldsymbol{\Upsilon}} \tag{16}$$

$$\delta\underset{\approx}{\boldsymbol{K}} = \underset{\sim}{\boldsymbol{\chi}}^{-1}\cdot(\delta\underset{\sim}{\boldsymbol{\chi}}\otimes\nabla^0) - \underset{\sim}{\boldsymbol{\chi}}^{-1}\cdot\delta\underset{\sim}{\boldsymbol{\chi}}\cdot\underset{\approx}{\boldsymbol{K}} \tag{17}$$

$$\delta\underset{\approx}{\boldsymbol{\Upsilon}} = \underset{\sim}{\boldsymbol{\chi}}^{-1}\cdot\delta\underset{\approx}{\boldsymbol{\chi}} - \underset{\sim}{\boldsymbol{\chi}}^{-1}\cdot\delta\underset{\sim}{\boldsymbol{\chi}}\cdot\underset{\approx}{\boldsymbol{\Upsilon}} \tag{18}$$

$$\delta\underset{\approx}{\underset{\sim}{\boldsymbol{K}}} = \underset{\sim}{\boldsymbol{\chi}}^{-1}\cdot(\delta\underset{\approx}{\boldsymbol{\chi}}\otimes\nabla^0) - \underset{\sim}{\boldsymbol{\chi}}^{-1}\cdot\delta\underset{\sim}{\boldsymbol{\chi}}\cdot\underset{\approx}{\underset{\sim}{\boldsymbol{K}}} \tag{19}$$

The virtual power of contact forces acting on the boundary $\partial\mathcal{D}_0$ takes the form

$$\begin{aligned}\mathcal{P}^{(c)}(\delta\underline{\boldsymbol{u}},\delta\underset{\sim}{\boldsymbol{\chi}},\delta\underset{\approx}{\boldsymbol{\chi}},\ldots) &= \int_{\partial\mathcal{D}_0} p^{(c)}(\delta\underline{\boldsymbol{u}},\delta\underset{\sim}{\boldsymbol{\chi}},\delta\underset{\approx}{\boldsymbol{\chi}},\ldots)\,dS\\ &= \int_{\partial\mathcal{D}_0} \underline{\boldsymbol{t}}\cdot\delta\underline{\boldsymbol{u}} + \underset{\sim}{\boldsymbol{t}}:\delta\underset{\sim}{\boldsymbol{\chi}} + \underset{\approx}{\boldsymbol{t}}\,\vdots\,\delta\underset{\approx}{\boldsymbol{\chi}} + \ldots\; dS\end{aligned} \tag{20}$$

The virtual power of external forces acting at a distance is of the form

$$\begin{aligned}\mathcal{P}^{(e)}(\delta\underline{\boldsymbol{u}},\delta\underset{\sim}{\boldsymbol{\chi}},\delta\underset{\approx}{\boldsymbol{\chi}},\ldots) &= \int_{\mathcal{D}_0} Jp^{(e)}(\delta\underline{\boldsymbol{u}},\delta\underset{\sim}{\boldsymbol{\chi}},\delta\underset{\approx}{\boldsymbol{\chi}},\ldots)\,dV\\ &= \int_{\mathcal{D}_0} \underline{\boldsymbol{f}}\cdot\delta\underline{\boldsymbol{u}} + \underset{\sim}{\boldsymbol{f}}:\delta\underset{\sim}{\boldsymbol{\chi}} + \underset{\approx}{\boldsymbol{f}}\,\vdots\,\delta\underset{\approx}{\boldsymbol{\chi}} + \ldots\; dV\end{aligned} \tag{21}$$

Higher-order volume forces working with the gradient of the microdeformation tensors could be introduced in the latter expression in the spirit of [40].

The virtual power of acceleration forces is taken in the form

$$\begin{aligned}\mathcal{P}^{(a)}(\delta\underline{\boldsymbol{u}},\delta\underset{\sim}{\boldsymbol{\chi}},\delta\underset{\approx}{\boldsymbol{\chi}},\ldots) &= -\int_{\mathcal{D}_0} Jp^{(a)}(\delta\underline{\boldsymbol{u}},\delta\underset{\sim}{\boldsymbol{\chi}},\delta\underset{\approx}{\boldsymbol{\chi}},\ldots)\,dV\\ &= -\int_{\mathcal{D}_0} \rho_0\left(\underline{\boldsymbol{a}}\cdot\delta\underline{\boldsymbol{u}} + (\ddot{\underset{\sim}{\boldsymbol{\chi}}}\cdot\underset{\sim}{\boldsymbol{I}}) : \delta\underset{\sim}{\boldsymbol{\chi}} + (\ddot{\underset{\approx}{\boldsymbol{\chi}}} : \underset{\approx}{\underset{\sim}{\boldsymbol{I}}})\,\vdots\,\delta\underset{\approx}{\boldsymbol{\chi}} + \ldots\right) dV\\ &= -\int_{\mathcal{D}_0} \rho_0\left(a_i\delta u_i + \ddot{\chi}_{iL}I_{LJ}\delta\chi_{iJ} + \ddot{\chi}_{iPQ}I_{PQJK}\delta\chi_{iJK} + \ldots\right) dV\end{aligned} \tag{22}$$

where $\underset{\sim}{\boldsymbol{I}}$ and $\underset{\approx}{\underset{\sim}{\boldsymbol{I}}}$ are Lagrangian generalized microinertia tensors. The mass densities per unit reference or current volume are called ρ_0 and ρ, respectively. They are such that $J = \det\underset{\sim}{\boldsymbol{F}} = \rho_0/\rho$ to comply with mass conservation. The acceleration vector is $\underline{\boldsymbol{a}}$. This Lagrangian formulation is in contrast to Germain's general micromorphic dynamics which is primarily introduced in the Eulerian framework; see [5]. Note that the acceleration of the multipolar deformations was not considered by Green and Rivlin [8], even though these authors take the acceleration of directors into account in subsequent works [38,37].

The principle of virtual power for general micromorphic media stipulates that

$$\mathcal{P}^{(i)} + \mathcal{P}^{(c)} + \mathcal{P}^{(e)} + \mathcal{P}^{(a)} = 0, \quad \forall\delta\underline{\boldsymbol{u}},\delta\underset{\sim}{\boldsymbol{\chi}},\delta\underset{\approx}{\boldsymbol{\chi}} \quad \text{and} \quad \forall\mathcal{D}_0\subset\Omega_0 \tag{23}$$

2.4. Balance laws of generalized moments of momentum

The exploitation of the principle of virtual power stated in the previous section leads to the derivation of the balance equations valid for all $\underline{\boldsymbol{X}} \in \Omega_0$ in the form

$$\mathrm{Div}\,(\underset{\sim}{\boldsymbol{S}} + \underset{\sim}{\boldsymbol{T}}^B) + \underline{\boldsymbol{f}} = \rho_0 \underline{\boldsymbol{a}}, \qquad S_{iJ,J} + T^B_{iJ,J} + f_i = \rho_0 a_i \tag{24}$$

$$\mathrm{Div}\,\underset{\eqsim}{\boldsymbol{M}}^B - \underset{\sim}{\boldsymbol{T}}^C + \underset{\sim}{\boldsymbol{f}} = \rho_0 \ddot{\underset{\sim}{\boldsymbol{\chi}}} \cdot \underset{\sim}{\boldsymbol{I}}, \qquad M^B_{iJK,K} - T^C_{iJ} + f_{iJ} = \rho_0 \ddot{\chi}_{iK} I_{KJ} \tag{25}$$

$$\mathrm{Div}\,\underset{\approx}{\boldsymbol{M}}^B - \underset{\eqsim}{\boldsymbol{T}}^B + \underset{\eqsim}{\boldsymbol{f}} = \rho_0 \ddot{\underset{\eqsim}{\boldsymbol{\chi}}} : \underset{\approx}{\boldsymbol{I}}, \qquad M^B_{iJKL,L} - T^B_{iJK} + f_{iJK} = \rho_0 \ddot{\chi}_{iPQ} I_{PQJK} \tag{26}$$

and of the Neumann boundary conditions valid for all $\underline{\boldsymbol{X}} \in \partial\Omega_0$:

$$(\underset{\sim}{\boldsymbol{S}} + \underset{\sim}{\boldsymbol{T}}^B) \cdot \underline{\boldsymbol{N}} = \underline{\boldsymbol{t}}, \quad (S_{iJ} + T^B_{iJ}) N_J = t_i \tag{27}$$

$$\underset{\eqsim}{\boldsymbol{M}}^B \cdot \underline{\boldsymbol{N}} = \underset{\sim}{\boldsymbol{t}}, \quad M^B_{iJK} N_K = t_{iJ} \tag{28}$$

$$\underset{\approx}{\boldsymbol{M}}^B \cdot \underline{\boldsymbol{N}} = \underset{\eqsim}{\boldsymbol{t}}, \quad M^B_{iJKL} N_L = t_{iJK} \tag{29}$$

Tensor $\underset{\sim}{\boldsymbol{S}} = \underset{\sim}{\boldsymbol{F}} \cdot \underset{\sim}{\boldsymbol{\Pi}}$ is the usual Boussinesq stress tensor, also called first Piola–Kirchhoff stress tensor. The label ^{B} refers to generalized Boussinesq tensors:

$$\underset{\sim}{\boldsymbol{T}}^B = \underset{\sim}{\boldsymbol{\chi}}^{-T} \cdot \underset{\sim}{\boldsymbol{T}}, \quad T^B_{iK} = \chi^{-T}_{iJ} T_{JK} \tag{30}$$

$$\underset{\eqsim}{\boldsymbol{T}}^B = \underset{\sim}{\boldsymbol{\chi}}^{-T} \cdot \underset{\eqsim}{\boldsymbol{T}}, \quad T^B_{iJK} = \chi^{-T}_{iI} T_{IJK} \tag{31}$$

$$\underset{\eqsim}{\boldsymbol{M}}^B = \underset{\sim}{\boldsymbol{\chi}}^{-T} \cdot \underset{\eqsim}{\boldsymbol{M}}, \quad M^B_{iJK} = \chi^{-T}_{iI} M_{IJK} \tag{32}$$

$$\underset{\approx}{\boldsymbol{M}}^B = \underset{\sim}{\boldsymbol{\chi}}^{-T} \cdot \underset{\approx}{\boldsymbol{M}}, \quad M^B_{iJKL} = \chi^{-T}_{iI} M_{IJKL} \tag{33}$$

The generalized stress tensor $\underset{\sim}{\boldsymbol{T}}^C$ arising in equation (25) couples all other micromorphic stress tensors in the following way:

$$\underset{\sim}{\boldsymbol{T}}^C = \underset{\sim}{\boldsymbol{T}}^B \cdot \underset{\sim}{\boldsymbol{F}}^{\mathrm{T}} \cdot \underset{\sim}{\boldsymbol{\chi}}^{-T} + \underset{\eqsim}{\boldsymbol{M}}^B : \underset{\eqsim}{\boldsymbol{K}}^{\mathrm{T}} \cdot \underset{\sim}{\boldsymbol{\chi}}^{-T} + \underset{\eqsim}{\boldsymbol{T}}^B : \underset{\eqsim}{\boldsymbol{\chi}}^{\mathrm{T}} \cdot \underset{\sim}{\boldsymbol{\chi}}^{-T} + \underset{\approx}{\boldsymbol{M}}^B \vdots \underset{\approx}{\boldsymbol{K}}^{\mathrm{T}} \cdot \underset{\sim}{\boldsymbol{\chi}}^{-T} \tag{34}$$

$$T^C_{pQ} = T^B_{pJ} F^{\mathrm{T}}_{Ji} \chi^{-T}_{iQ} + M^B_{pJK} K_{iJ,K} \chi^{-T}_{iQ} + T^B_{pJK} \chi_{iJ,K} \chi^{-T}_{iQ} + M^B_{pJKL} K_{iJK,L} \chi^{-T}_{iQ}$$

In contrast to the derived generalized balance of momentum equations, Green, Rivlin and Naghdi considered a single balance of momentum equation [8,38,37]. Relations are introduced involving the divergence of multipolar stresses defining auxiliary higher-order stress tensors akin to the present tensors $T_{iJK\ldots}$. These relations (see the equations (13.2) in [8]) are regarded as constitutive, not as balance equations. There are substituted into a single additional boundary condition involving the heat flux through the surface. Germain noted that this is due to their derivation of all equations from the energy conservation law which, in the classical case, leads to the same results as the method of virtual power, but to fewer equations in the case of micromorphic media; see [5].

2.5. Linearized balance laws

The previous field equations are now linearized to obtain a more simple form and recover some existing balance laws for micromorphic continua. Deformations are small when

$$\|\underset{\sim}{\boldsymbol{F}} - \underset{\sim}{\boldsymbol{1}}\| \ll 1 \tag{35}$$

Considering that the microdeformations are relaxed strain-gradient-like variables, the generalized microdeformations are small when

$$\|\underset{\sim}{\boldsymbol{\chi}} - \underset{\sim}{\boldsymbol{1}}\| \ll 1, \quad L\|\underset{\eqsim}{\boldsymbol{K}}\| \ll 1, \quad L\|\underset{\eqsim}{\boldsymbol{\chi}}\| \ll 1, \quad L^2\|\underset{\approx}{\boldsymbol{K}}\| \ll 1 \tag{36}$$

where L is a characteristic length related to the structure or to the wavelength of the applied loading conditions. In the following, $\underset{\sim}{\boldsymbol{\chi}} - \underset{\sim}{\boldsymbol{1}}$ is replaced by the same notation $\underset{\sim}{\boldsymbol{\chi}}$. The acceleration terms involve constitutive inertia tensors of orders two and four.

Within the context of small deformations and microdeformations, the generalized strain measures are linearized as follows:

$$\underset{\sim}{\boldsymbol{C}} \simeq \underset{\sim}{\boldsymbol{1}} + 2\underset{\sim}{\boldsymbol{\varepsilon}}, \quad \text{with} \quad 2\underset{\sim}{\boldsymbol{\varepsilon}} = \underline{\boldsymbol{u}} \otimes \nabla + \nabla \otimes \underline{\boldsymbol{u}} \tag{37}$$

$$\underset{\sim}{\boldsymbol{\Upsilon}} \simeq \underline{\boldsymbol{u}} \otimes \nabla - \underset{\sim}{\boldsymbol{\chi}}, \quad \underset{\sim}{\underline{\boldsymbol{\Upsilon}}} \simeq \underset{\sim}{\underline{\boldsymbol{\chi}}} \tag{38}$$

The linearized stress measures are then

$$\underset{\sim}{\boldsymbol{S}} = \underset{\sim}{\boldsymbol{F}} \cdot \underset{\sim}{\boldsymbol{\Pi}} = J\underset{\sim}{\boldsymbol{\sigma}} \cdot \underset{\sim}{\boldsymbol{F}}^{-T} \simeq \underset{\sim}{\boldsymbol{\sigma}} \tag{39}$$

$$\underset{\sim}{\boldsymbol{T}}^B \simeq \underset{\sim}{\boldsymbol{T}}, \quad \underset{\sim}{\underline{\boldsymbol{T}}}^B \simeq \underset{\sim}{\underline{\boldsymbol{T}}}, \quad \underset{\sim}{\boldsymbol{T}}^C \simeq \underset{\sim}{\boldsymbol{T}} \tag{40}$$

$$\underset{\sim}{\underline{\boldsymbol{M}}}^B \simeq \underset{\sim}{M}, \quad \underset{\approx}{\boldsymbol{M}}^B \simeq \underset{\approx}{T} \tag{41}$$

where $\underset{\sim}{\boldsymbol{\sigma}}$ is the usual Cauchy stress tensor.
As a result, the balance laws (24) to (26) reduce to

$$\operatorname{div}(\underset{\sim}{\boldsymbol{\sigma}} + \underset{\sim}{\boldsymbol{T}}) + \underline{\boldsymbol{f}} = \rho\underline{\boldsymbol{a}} \tag{42}$$

$$\operatorname{div}\underset{\sim}{\underline{\boldsymbol{M}}} - \underset{\sim}{\boldsymbol{T}} + \underset{\sim}{\boldsymbol{f}} = \rho\ddot{\underset{\sim}{\boldsymbol{\chi}}} \cdot \underset{\sim}{\boldsymbol{I}} \tag{43}$$

$$\operatorname{div}\underset{\approx}{\boldsymbol{M}} - \underset{\sim}{\underline{\boldsymbol{T}}} + \underset{\sim}{\underline{\boldsymbol{f}}} = \rho\ddot{\underset{\sim}{\underline{\boldsymbol{\chi}}}} : \underset{\approx}{\boldsymbol{I}} \tag{44}$$

These equations have the same form as the Eulerian balance laws according to Germain's general micromorphic theory, except that the present stress tensors $\underset{\sim}{\boldsymbol{T}}, \underset{\sim}{\underline{\boldsymbol{T}}}, \underset{\sim}{\underline{\boldsymbol{M}}}, \underset{\approx}{\boldsymbol{M}}$ do not exhibit any symmetry property, in contrast to Germain's ones.

3. Hyperelasticity of second-order micromorphic media

The constitutive theory of hyperelastic second-order micromorphic media is based on the objective Helmholtz free energy potential per unit volume:

$$\psi(\underset{\sim}{\boldsymbol{C}}, \quad \underset{\sim}{\boldsymbol{\Upsilon}}, \quad \underset{\sim}{\underline{\boldsymbol{K}}}, \quad \underset{\sim}{\underline{\boldsymbol{\Upsilon}}}, \quad \underset{\approx}{\boldsymbol{K}}) \tag{45}$$

3.1. Derivation of the hyperelastic state laws

The local form of the Clausius–Duhem inequality for second-order micromorphic media is

$$Jp^{(i)} - \dot{\psi} \geq 0 \tag{46}$$

Taking the expression (14) of the power of internal forces and the dependence (45) into account, it gives

$$\left(\frac{\underset{\sim}{\boldsymbol{\Pi}}}{2} - \frac{\partial\psi}{\partial\underset{\sim}{\boldsymbol{C}}}\right) : \dot{\underset{\sim}{\boldsymbol{C}}} + \left(\underset{\sim}{\boldsymbol{T}} - \frac{\partial\psi}{\partial\underset{\sim}{\boldsymbol{\Upsilon}}}\right) : \dot{\underset{\sim}{\boldsymbol{\Upsilon}}} + \left(\underset{\sim}{\underline{\boldsymbol{M}}} - \frac{\partial\psi}{\partial\underset{\sim}{\underline{\boldsymbol{K}}}}\right) \vdots \dot{\underset{\sim}{\underline{\boldsymbol{K}}}} + \left(\underset{\sim}{\underline{\boldsymbol{T}}} - \frac{\partial\psi}{\partial\underset{\sim}{\underline{\boldsymbol{\chi}}}}\right) \vdots \dot{\underset{\sim}{\underline{\boldsymbol{\Upsilon}}}} + \left(\underset{\approx}{\boldsymbol{M}} - \frac{\partial\psi}{\partial\underset{\approx}{\boldsymbol{K}}}\right) :: \dot{\underset{\approx}{\boldsymbol{K}}} \geq 0 \tag{47}$$

The positiveness of this linear form with respect to the strain rates requires that the conjugate quantities vanish. The hyperelastic laws follow:

$$\underset{\sim}{\boldsymbol{\Pi}} = 2\frac{\partial\psi}{\partial\underset{\sim}{\boldsymbol{C}}}, \quad \underset{\sim}{\boldsymbol{T}} = \frac{\partial\psi}{\partial\underset{\sim}{\boldsymbol{\Upsilon}}}, \quad \underset{\sim}{\underline{\boldsymbol{M}}} = \frac{\partial\psi}{\partial\underset{\sim}{\underline{\boldsymbol{K}}}}, \quad \underset{\sim}{\underline{\boldsymbol{T}}} = \frac{\partial\psi}{\partial\underset{\sim}{\underline{\boldsymbol{\chi}}}}, \quad \underset{\approx}{\boldsymbol{M}} = \frac{\partial\psi}{\partial\underset{\approx}{\boldsymbol{K}}} \tag{48}$$

3.2. Consideration of internal constraints

Internal constraints are envisaged in the form of relations that must be fulfilled by kinematic quantities. As an example, two constraints are introduced:

$$f(\underset{\sim}{\boldsymbol{\Upsilon}}) = 0, \quad g(\underset{\approx}{\boldsymbol{\Upsilon}}, \underset{\approx}{\boldsymbol{K}}) = 0 \tag{49}$$

where f and g are differentiable scalar functions. This specific example is motivated by the applications in Section 4, without loss of generality. Following the usual procedure, see for instance [41], extended for gradient materials by [35], the only evolutions of the strain variables compatible with the constraints are

$$\frac{\partial f}{\partial \underset{\sim}{\boldsymbol{\Upsilon}}} : \dot{\underset{\sim}{\boldsymbol{\Upsilon}}} = 0, \quad \frac{\partial g}{\partial \underset{\approx}{\boldsymbol{\Upsilon}}} : \dot{\underset{\approx}{\boldsymbol{\Upsilon}}} + \frac{\partial g}{\partial \underset{\approx}{\boldsymbol{K}}} : \dot{\underset{\approx}{\boldsymbol{K}}} = 0 \tag{50}$$

Some state laws (48) must be amended as follows to enforce the internal constraints:

$$\underset{\sim}{\boldsymbol{T}} = \frac{\partial \psi}{\partial \underset{\sim}{\boldsymbol{\Upsilon}}} + \underset{\sim}{\boldsymbol{T}}^R, \quad \underset{\approx}{\boldsymbol{M}} = \frac{\partial \psi}{\partial \underset{\approx}{\boldsymbol{K}}} + \underset{\approx}{\boldsymbol{M}}^R, \quad \underset{\approx}{\boldsymbol{T}} = \frac{\partial \psi}{\partial \underset{\approx}{\boldsymbol{\chi}}} + \underset{\approx}{\boldsymbol{T}}^R \tag{51}$$

where the quantities with the label ^{R} are reaction stresses of the form

$$\underset{\sim}{\boldsymbol{T}}^R = \lambda_f \frac{\partial f}{\partial \underset{\sim}{\boldsymbol{\Upsilon}}}, \quad \underset{\approx}{\boldsymbol{M}}^R = \lambda_g \frac{\partial g}{\partial \underset{\approx}{\boldsymbol{K}}}, \quad \underset{\approx}{\boldsymbol{T}}^R = \lambda_g \frac{\partial g}{\partial \underset{\approx}{\boldsymbol{\Upsilon}}} \tag{52}$$

The reaction stresses involve the two Lagrange multipliers λ_f and λ_g. Their values must be determined from the resolution of the boundary value problem.

3.3. Hyperelasticity based on the Gibbs free energy potential

The Gibbs free energy potential is obtained by the Legendre transform of the Helmholtz potential:

$$\psi^\star(\underset{\sim}{\boldsymbol{\Pi}}, \underset{\sim}{\boldsymbol{T}}, \underset{\approx}{\boldsymbol{M}}, \underset{\approx}{\boldsymbol{T}}, \underset{\underset{\sim}{\approx}}{\boldsymbol{M}}) = \underset{\left\{\underset{\sim}{\boldsymbol{C}}, \underset{\sim}{\boldsymbol{\Upsilon}}, \underset{\approx}{\boldsymbol{K}}, \underset{\approx}{\boldsymbol{\Upsilon}}, \underset{\underset{\sim}{\approx}}{\boldsymbol{K}}\right\}}{\text{Max}} \left(\psi(\underset{\sim}{\boldsymbol{C}}, \underset{\sim}{\boldsymbol{\Upsilon}}, \underset{\approx}{\boldsymbol{K}}, \underset{\approx}{\boldsymbol{\Upsilon}}, \underset{\underset{\sim}{\approx}}{\boldsymbol{K}}) - \underset{\sim}{\boldsymbol{\Pi}} : \frac{\underset{\sim}{\boldsymbol{C}}}{2} - \underset{\sim}{\boldsymbol{T}} : \underset{\sim}{\boldsymbol{\Upsilon}} - \underset{\approx}{\boldsymbol{M}} \vdots \underset{\approx}{\boldsymbol{K}} - \underset{\approx}{\boldsymbol{T}} \vdots \underset{\approx}{\boldsymbol{\Upsilon}} - \underset{\underset{\sim}{\approx}}{\boldsymbol{M}} :: \underset{\underset{\sim}{\approx}}{\boldsymbol{K}} \right) \tag{53}$$

The arguments of this constitutive function are the generalized stress tensors ($\underset{\sim}{\boldsymbol{\Pi}}, \quad \underset{\sim}{\boldsymbol{T}}, \quad \underset{\approx}{\boldsymbol{M}}, \quad \underset{\approx}{\boldsymbol{T}}, \quad \underset{\underset{\sim}{\approx}}{\boldsymbol{M}}$).

The dissipation inequality (46) transforms into

$$\begin{aligned} Jp^{(i)} - \dot{\psi}^\star - \underset{\sim}{\boldsymbol{\Pi}} : \frac{\dot{\underset{\sim}{\boldsymbol{C}}}}{2} - \underset{\sim}{\boldsymbol{T}} : \dot{\underset{\sim}{\boldsymbol{\Upsilon}}} - \underset{\approx}{\boldsymbol{M}} \vdots \dot{\underset{\approx}{\boldsymbol{C}}} - \underset{\approx}{\boldsymbol{T}} \vdots \dot{\underset{\approx}{\boldsymbol{\Upsilon}}} - \underset{\underset{\sim}{\approx}}{\boldsymbol{M}} :: \dot{\underset{\underset{\sim}{\approx}}{\boldsymbol{K}}} \\ - \dot{\underset{\sim}{\boldsymbol{\Pi}}} : \frac{\underset{\sim}{\boldsymbol{C}}}{2} - \dot{\underset{\sim}{\boldsymbol{T}}} : \underset{\sim}{\boldsymbol{\Upsilon}} - \dot{\underset{\approx}{\boldsymbol{M}}} \vdots \underset{\approx}{\boldsymbol{K}} - \dot{\underset{\approx}{T}} \vdots \underset{\approx}{\boldsymbol{\Upsilon}} - \dot{\underset{\underset{\sim}{\approx}}{\boldsymbol{M}}} :: \underset{\underset{\sim}{\approx}}{\boldsymbol{K}} \geq 0 \end{aligned} \tag{54}$$

and consequently,

$$-\dot{\psi}^\star - \dot{\underset{\sim}{\boldsymbol{\Pi}}} : \frac{\underset{\sim}{\boldsymbol{C}}}{2} - \dot{\underset{\sim}{\boldsymbol{T}}} : \underset{\sim}{\boldsymbol{\Upsilon}} - \dot{\underset{\approx}{\boldsymbol{M}}} \vdots \underset{\approx}{\boldsymbol{K}} - \dot{\underset{\approx}{\boldsymbol{T}}} \vdots \underset{\approx}{\boldsymbol{\Upsilon}} - \dot{\underset{\underset{\sim}{\approx}}{\boldsymbol{M}}} :: \underset{\underset{\sim}{\approx}}{\boldsymbol{K}} \geq 0$$

$$\left(\frac{\partial \psi^\star}{\partial \underset{\sim}{\boldsymbol{\Pi}}} + \frac{\underset{\sim}{\boldsymbol{C}}}{2} \right) : \dot{\underset{\sim}{\boldsymbol{\Pi}}} + \left(\frac{\partial \psi^\star}{\partial \underset{\sim}{\boldsymbol{T}}} + \underset{\sim}{\boldsymbol{\Upsilon}} \right) : \dot{\underset{\sim}{\boldsymbol{T}}} + \left(\frac{\partial \psi^\star}{\partial \underset{\approx}{\boldsymbol{M}}} + \underset{\approx}{\boldsymbol{K}} \right) \vdots \dot{\underset{\approx}{M}} + \left(\frac{\partial \psi^\star}{\partial \underset{\approx}{\boldsymbol{T}}} + \underset{\approx}{\boldsymbol{\Upsilon}} \right) \vdots \dot{\underset{\approx}{\boldsymbol{T}}} + \left(\frac{\partial \psi^\star}{\partial \underset{\underset{\sim}{\approx}}{\boldsymbol{M}}} + \underset{\underset{\sim}{\approx}}{\boldsymbol{K}} \right) :: \dot{\underset{\underset{\sim}{\approx}}{\boldsymbol{M}}} \leq 0 \tag{55}$$

from which the dual state laws are derived:

$$\underset{\sim}{\boldsymbol{C}} = -2 \frac{\partial \psi^\star}{\partial \underset{\sim}{\boldsymbol{\Pi}}}, \quad \underset{\sim}{\boldsymbol{\Upsilon}} = -\frac{\partial \psi^\star}{\partial \underset{\sim}{\boldsymbol{T}}}, \quad \underset{\approx}{\boldsymbol{K}} = -\frac{\partial \psi^\star}{\partial \underset{\approx}{\boldsymbol{M}}}, \quad \underset{\approx}{\boldsymbol{\Upsilon}} = -\frac{\partial \psi^\star}{\partial \underset{\approx}{\boldsymbol{T}}}, \quad \underset{\underset{\sim}{\approx}}{\boldsymbol{K}} = -\frac{\partial \psi^\star}{\partial \underset{\underset{\sim}{\approx}}{\boldsymbol{M}}} \tag{56}$$

These constitutive laws must be amended in the presence of internal constraints. The consideration of internal constraints is usually limited to kinematic constraints; see [35]. Static constraints can also be envisaged. As an example, let us consider a constraint linking two stress tensors:

$$f(\underset{\sim}{\boldsymbol{\Pi}}, \underset{\approx}{\boldsymbol{M}}) = 0 \tag{57}$$

The stress increments must then be such that

$$\frac{\partial f}{\partial \underset{\sim}{\boldsymbol{\Pi}}} : \dot{\underset{\sim}{\boldsymbol{\Pi}}} + \frac{\partial f}{\partial \underset{\approx}{\boldsymbol{M}}} : \dot{\underset{\approx}{\boldsymbol{M}}} = 0 \tag{58}$$

The state laws $(56)_1$ and $(56)_5$ must then be amended as follows:

$$\underset{\sim}{\boldsymbol{C}} = -2\frac{\partial \psi^\star}{\partial \underset{\sim}{\boldsymbol{\Pi}}} + 2\lambda_f \frac{\partial f}{\partial \underset{\sim}{\boldsymbol{\Pi}}}, \quad \underset{\approx}{\boldsymbol{K}} = -\frac{\partial \psi^\star}{\partial \underset{\approx}{\boldsymbol{M}}} + \lambda_f \frac{\partial f}{\partial \underset{\approx}{\boldsymbol{M}}} \tag{59}$$

where the Lagrange multiplier λ_f must be determined from the resolution of the boundary value problem.

4. Relation to strain and stress gradient theories

Specific internal constraints are now presented to retrieve the strain gradient and stress gradient models as limit cases.

4.1. Strain gradient theory

The following constraints are considered:

$$f(\underset{\sim}{\boldsymbol{\Upsilon}}) = \underset{\sim}{\boldsymbol{\Upsilon}} - \underset{\sim}{\mathbf{1}} = 0, \quad g(\underset{\approx}{\boldsymbol{K}}, \underset{\approx}{\boldsymbol{\Upsilon}}) = \underset{\approx}{\boldsymbol{\Upsilon}} - \underset{\approx}{\boldsymbol{K}} = 0 \tag{60}$$

The relative deformation tensor is assumed to remain the identity during any motion, which means that the microdeformation $\underset{\sim}{\boldsymbol{\chi}}$ coincides with the deformation gradient itself. As a result,

$$\underset{\sim}{\boldsymbol{\chi}} \equiv \underset{\sim}{\boldsymbol{F}}, \quad \underset{\approx}{\boldsymbol{K}} \equiv \underset{\sim}{\boldsymbol{F}} \otimes \nabla^0 \tag{61}$$

so that the theory includes the second gradient of displacement. The second-order microdeformation is forced to coincide with the gradient of the microdeformation:

$$\underset{\approx}{\boldsymbol{\Upsilon}} \equiv \underset{\sim}{\boldsymbol{\chi}} \otimes \nabla^0 \equiv \underset{\sim}{\boldsymbol{F}} \otimes \nabla^0, \quad \underset{\approx}{\boldsymbol{K}} \equiv \underset{\sim}{\boldsymbol{F}} \otimes \nabla^0 \otimes \nabla^0 \tag{62}$$

The constrained second-order micromorphic theory therefore reduces to the third displacement gradient model, or equivalently the second strain gradient theory of Mindlin [9].

In this case, the reaction stresses (52) become

$$\underset{\sim}{\boldsymbol{T}}^R = \lambda_f \underset{\sim}{\mathbf{1}}, \quad \underset{\approx}{\boldsymbol{M}}^R = \lambda_g \underset{\approx}{\mathbf{1}}, \quad \underset{\approx}{\boldsymbol{T}}^R = -\lambda_g \underset{\approx}{\mathbf{1}} \tag{63}$$

The constitutive equations therefore leave the spherical part of the coupling stress tensor $\underset{\sim}{\boldsymbol{T}}$ indeterminate. It will be shown explicitly in the linearized case how the three balance equations (24) to (26) can be reduced to a single set of higher-order partial differential equations for the unknown displacement components; see Section 5.3.

As noted by [42], enforcing internal constraints makes it possible to derive the balance and constitutive equations in the bulk for gradient theories but it does not provide the corresponding boundary conditions. The latter are rather sophisticated and must be derived from the direct formulation of the grade n model; see [43, 42, 44, 45]. A further discussion of the reduction of the first-order micromorphic model to Mindlin's strain gradient elasticity can be found in [12].

4.2. Stress gradient theory at finite deformation

The static stress gradient theory was proposed for the first time in [23] and shown to be fundamentally distinct from the strain gradient approach. It was presented within the small strain framework and the objective of this subsection and of the next one is to extend it to the general finite deformation and dynamical cases. The theory is first introduced within the formalism of a constrained second-order micromorphic model and then a direct construction is presented enforcing the constraint. However, it will become apparent that the mechanical meaning of the additional degrees of freedom arising in the stress gradient theory differs from that of the microdeformation tensors regarded as *relaxed deformation gradients* introduced in Section 2.1. As a result, alternative strain measures, different from (11), are proposed in what follows.

In [23], the stress gradient model was interpreted as a constrained and truncated second-order micromorphic medium where each material point is endowed with first- and third-rank tensors of degrees of freedom u_i, Φ_{iJK}, thus leaving aside first-order additional degrees of freedom:

$$\text{DOF} = \{\underline{\boldsymbol{u}}, \quad \underset{\approx}{\boldsymbol{\Phi}}\}, \quad \text{DOF} = \{u_i, \quad \Phi_{iJK}\}$$

$$\text{GRAD} = \{\underline{\boldsymbol{u}}, \quad \underline{\boldsymbol{u}} \otimes \nabla^0, \quad \underset{\approx}{\boldsymbol{\Phi}}, \quad \underset{\approx}{\boldsymbol{\Phi}} \otimes \nabla^0\}, \quad \text{GRAD} = \{u_i, \quad u_{i,L}, \quad \Phi_{iJK}, \quad \Phi_{iJK,L}\}$$

The first-order microdeformation tensor being absent from this special theory, the Lagrangian strain measures (11) must be reconsidered. The three Lagrangian constitutive variables proposed for the stress gradient model are

$$\underset{\sim}{\boldsymbol{C}} := \underset{\sim}{\boldsymbol{F}}^{\mathrm{T}} \cdot \underset{\sim}{\boldsymbol{F}}, \quad \underset{\approx}{\boldsymbol{\Upsilon}} := \underset{\sim}{\boldsymbol{F}}^{\mathrm{T}} \cdot \underset{\approx}{\boldsymbol{\Phi}}, \quad \underset{\approx}{\boldsymbol{K}} := \underset{\sim}{\boldsymbol{F}}^{\mathrm{T}} \cdot \underset{\approx}{\boldsymbol{\Phi}} \otimes \nabla^0 \tag{64}$$

$$C_{IJ} := F^{\mathrm{T}}_{Ik} F_{kJ}, \quad \Upsilon_{IJK} := F^{\mathrm{T}}_{Ii} \Phi_{iJK}, \quad K_{IJKL} := F^{\mathrm{T}}_{Ii} \Phi_{iJK,L}$$

The motivation for the pull-back by means of the $\underset{\sim}{\boldsymbol{F}}^{\mathrm{T}}$ variable, instead of $\underset{\sim}{\boldsymbol{F}}^{-1}$, is the relation to the Boussinesq stress tensor, which will become apparent in the following. The conjugate stress tensors are introduced in the virtual power density of internal forces which is formally a truncation of (14):

$$Jp^{(i)} = \underset{\sim}{\boldsymbol{\Pi}} : \underset{\sim}{\boldsymbol{F}}^{\mathrm{T}} \cdot \delta \underset{\sim}{\boldsymbol{F}} + \underset{\approx}{\boldsymbol{T}} \vdots \delta \underset{\approx}{\boldsymbol{\Upsilon}} + \underset{\approx}{\boldsymbol{M}} :: \delta \underset{\approx}{\boldsymbol{K}} \tag{65}$$

The elastic dual potential is then $\psi^\star(\underset{\sim}{\boldsymbol{\Pi}}, \underset{\approx}{\boldsymbol{T}}, \underset{\approx}{\boldsymbol{M}})$.

An internal constraint linking the fourth-rank stress tensor $\underset{\approx}{\boldsymbol{M}}$ to the Piola stress tensor is now introduced:

$$f(\underset{\sim}{\boldsymbol{\Pi}}, \underset{\approx}{\boldsymbol{M}}) = \underset{\approx}{\boldsymbol{M}} - \underset{\sim}{\boldsymbol{\Pi}} \otimes \underset{\sim}{\boldsymbol{1}} = 0 \tag{66}$$

A consequence of this constraint is that $\underset{\approx}{\boldsymbol{M}}$ becomes symmetric with respect to the first two indices and last two indices. It is equivalent to the constraint that the fourth-rank Boussinesq stress tensor is directly related to the Boussinesq second-rank stress tensor:

$$\underset{\approx}{\boldsymbol{M}}^B = \underset{\sim}{\boldsymbol{F}} \cdot \underset{\approx}{\boldsymbol{M}} = \underset{\sim}{\boldsymbol{S}} \otimes \underset{\sim}{\boldsymbol{1}}, \quad M^B_{iJKL} = S_{iJ} \delta_{KL} \tag{67}$$

It must be noted that the physical dimension implied for the stress tensor components M^B_{iJKL} by the constraint (67) is different from the one proposed in the previous section in the case of micromorphic theory regarded as a relaxed strain gradient model: Pa instead of $Pa.m^2 = N$. An internal length could be introduced to restore the dimensional consistency but it is preferable to keep the proposed form (67) which will be related to a stress gradient theory. Accordingly, the physical dimension of the additional degrees of freedom, Φ_{iJK}, is that of *microdisplacements*, in m, instead of *microdeformations* in the original theory of Section 2.1.

The presence of the internal constraint leads to the modification (59) of the constitutive equations in the form

$$\underset{\sim}{\boldsymbol{C}} = -2\frac{\partial \psi^\star}{\partial \underset{\sim}{\boldsymbol{\Pi}}} - 2\lambda_f \underset{\sim}{\boldsymbol{1}}, \quad \underset{\approx}{\boldsymbol{T}} = -\frac{\partial \psi^\star}{\partial \underset{\sim}{\boldsymbol{\Upsilon}}}, \quad \underset{\approx}{\boldsymbol{K}} = -\frac{\partial \psi^\star}{\partial \underset{\approx}{\boldsymbol{M}}} + \lambda_f \underset{\approx}{\boldsymbol{1}} \tag{68}$$

where λ_f is a Lagrange multiplier to be determined from the boundary conditions.

Due to this specific definition of the higher-order stress tensors, and, as a consequence, to an alternative definition of the Lagrangian strain measures different from (11), the generalized balance of momentum equations must be derived anew. For that purpose, the constraint (67) is directly implemented in the virtual power density of internal forces (65):

$$
\begin{aligned}
Jp^{(i)} &= \underset{\sim}{\boldsymbol{\Pi}} : \underset{\sim}{\boldsymbol{F}}^{\mathrm{T}} \cdot \delta \underset{\sim}{\boldsymbol{F}} + \underset{\approx}{\boldsymbol{T}} \,\vdots\, (\delta \underset{\sim}{\boldsymbol{F}}^{\mathrm{T}} \cdot \underset{\approx}{\boldsymbol{\Phi}} + \underset{\sim}{\boldsymbol{F}}^{\mathrm{T}} \cdot \delta \underset{\approx}{\boldsymbol{\Phi}}) + \underset{\approx}{\boldsymbol{M}} :: (\delta \underset{\sim}{\boldsymbol{F}}^{\mathrm{T}} \cdot \underset{\approx}{\boldsymbol{\Phi}} \otimes \nabla^0 + \underset{\sim}{\boldsymbol{F}}^{\mathrm{T}} \cdot \delta \underset{\approx}{\boldsymbol{\Phi}} \otimes \nabla^0) \\
&= \Pi_{IJ} F^{\mathrm{T}}_{Ik} \delta F_{kJ} + T_{IJK} \left(\delta F^{\mathrm{T}}_{Ik} \Phi_{kJK} + F^{\mathrm{T}}_{Ik} \delta \Phi_{kJK} \right) + \Pi_{IJ} \delta_{KL} \left(\delta F^{\mathrm{T}}_{Ik} \Phi_{kJK,L} + F^{\mathrm{T}}_{Ik} \delta \Phi_{kJK,L} \right) \\
&= \underset{\sim}{\boldsymbol{\Pi}} : \left(\frac{\delta \underset{\sim}{\boldsymbol{C}}}{2} + \delta (\underset{\sim}{\boldsymbol{F}}^{\mathrm{T}} \cdot \underset{\approx}{\boldsymbol{\Phi}} \cdot \nabla^0) \right) + \underset{\approx}{\boldsymbol{T}} \,\vdots\, \delta (\underset{\sim}{\boldsymbol{F}}^{\mathrm{T}} \cdot \underset{\approx}{\boldsymbol{\Phi}}) \qquad (69) \\
&= \left(\Pi_{IJ} F^{\mathrm{T}}_{Jk} + \Pi_{IJ} \Phi_{kJL,L} + T_{IJK} \Phi_{kJK} \right) \delta u_{k,I} + T_{IJK} F^{\mathrm{T}}_{Ik} \delta \Phi_{kJK} + \Pi_{IJ} F^{\mathrm{T}}_{Ik} \delta \Phi_{kJL,L} \\
&= \left(S_{kI} \delta u_k + \Pi_{IJ} \Phi_{kJL,L} \delta u_k + T_{IJK} \Phi_{kJK} \delta u_k \right)_{,I} - \left(S_{kI} + \Pi_{IJ} \Phi_{kJL,L} + T_{IJK} \Phi_{kJK} \right)_{,I} \delta u_k \\
&\quad + T^B_{kJK} \delta \Phi_{kJK} + (S_{kJ} \delta \Phi_{kJL})_{,L} - S_{kJ,L} \delta \Phi_{kJK} \qquad (70)
\end{aligned}
$$

where the generalized Boussinesq tensor, $\underset{\approx}{\boldsymbol{T}}^B := \underset{\sim}{\boldsymbol{F}} \cdot \underset{\approx}{\boldsymbol{T}}$, has been introduced. As a result, the virtual power of internal forces of a subdomain $\mathcal{D}_0 \subset \Omega_0$ can be written as the sum of a volume and a surface contribution:

$$
\begin{aligned}
\mathcal{P}^{(i)} = & \int_{\mathcal{D}_0} \delta \underline{\boldsymbol{u}} \cdot \left(\underset{\sim}{\boldsymbol{S}} + (\underset{\approx}{\boldsymbol{\Phi}} \cdot \nabla^0) \cdot \underset{\sim}{\boldsymbol{\Pi}} + \underset{\approx}{\boldsymbol{\Phi}} : \underset{\approx}{\boldsymbol{T}}^{\mathrm{T}} \right) \cdot \nabla^0 \, dV \\
& - \int_{\partial \mathcal{D}_0} \delta \underline{\boldsymbol{u}} \cdot \left(\underset{\sim}{\boldsymbol{S}} + (\underset{\approx}{\boldsymbol{\Phi}} \cdot \nabla^0) \cdot \underset{\sim}{\boldsymbol{\Pi}} + \underset{\approx}{\boldsymbol{\Phi}} : \underset{\approx}{\boldsymbol{T}}^{\mathrm{T}} \right) \cdot \underline{\boldsymbol{N}} \, dS \\
& - \int_{\mathcal{D}_0} \left(\underset{\approx}{\boldsymbol{T}}^B - \underset{\sim}{\boldsymbol{S}} \otimes \nabla^0 \right) \vdots\, \delta \underset{\approx}{\boldsymbol{\Phi}} \, dV - \int_{\partial \mathcal{D}_0} (\underset{\sim}{\boldsymbol{S}} \otimes \underline{\boldsymbol{N}}) \,\vdots\, \delta \underset{\approx}{\boldsymbol{\Phi}} \, dS \qquad (71)
\end{aligned}
$$

where the transposition for third-order tensors is taken as $T^{\mathrm{T}}_{JKI} = T_{IJK}$. The latter expression of the power of internal forces dictates the form of the power of contact forces to be

$$
\mathcal{P}^{(c)}(\delta \underline{\boldsymbol{u}}, \delta \underset{\approx}{\boldsymbol{\Phi}}) = \int_{\mathcal{D}_0} \left(\underset{\sim}{\boldsymbol{t}} \cdot \delta \underline{\boldsymbol{u}} + \underset{\approx}{\boldsymbol{t}} \,\vdots\, \delta \underset{\approx}{\boldsymbol{\Phi}} \right) dS \qquad (72)
$$

where $\underset{\sim}{\boldsymbol{t}}$ and $\underset{\approx}{\boldsymbol{t}}$ are generalized surface traction tensors.

The virtual power of external and acceleration forces is taken as

$$
\mathcal{P}^{(e)}(\delta \underline{\boldsymbol{u}}, \delta \underset{\approx}{\boldsymbol{\Phi}}) = \int_{\mathcal{D}_0} \underline{\boldsymbol{f}} \cdot \delta \underline{\boldsymbol{u}} + \underset{\approx}{\boldsymbol{f}} \,\vdots\, \delta \underset{\approx}{\boldsymbol{\Phi}} \, dV \qquad (73)
$$

$$
\mathcal{P}^{(a)}(\delta \underline{\boldsymbol{u}}, \delta \underset{\approx}{\boldsymbol{\Phi}}) = - \int_{\mathcal{D}_0} \rho_0 \left(\underline{\boldsymbol{a}} \cdot \delta \underline{\boldsymbol{u}} + (\ddot{\underset{\approx}{\boldsymbol{\Phi}}} : \underset{\approx}{\boldsymbol{I}}) \,\vdots\, \delta \underset{\approx}{\boldsymbol{\Phi}} \right) dV \qquad (74)
$$

The resulting field equations are

$$
\operatorname{Div} \left(\underset{\sim}{\boldsymbol{S}} + (\operatorname{Div} \underset{\approx}{\boldsymbol{\Phi}}) \cdot \underset{\sim}{\boldsymbol{\Pi}} + \underset{\approx}{\boldsymbol{\Phi}} : \underset{\approx}{\boldsymbol{T}}^{\mathrm{T}} \right) + \underline{\boldsymbol{f}} = \rho_0 \ddot{\underline{\boldsymbol{u}}} \qquad (75)
$$

$$
\underset{\approx}{\boldsymbol{T}}^B - \underset{\sim}{\boldsymbol{S}} \otimes \nabla^0 + \underset{\approx}{\boldsymbol{f}} = \rho_0 \ddot{\underset{\approx}{\boldsymbol{\Phi}}} : \underset{\approx}{\boldsymbol{I}} \qquad (76)
$$

The gradient of the classical Boussinesq stress tensor $\underset{\sim}{\boldsymbol{S}}$ arises in the latter equation. In the static case in particular and in the absence of third-rank body forces, the generalized stress tensor $\underset{\approx}{\boldsymbol{T}}^B$ is nothing but the stress gradient, hence the name *stress gradient theory*.

The exploitation of the principle of virtual power at the boundary provides the expression of the generalized traction tensors:

$$\underline{t} = \left(\left(\underset{\sim}{\boldsymbol{F}} + \underset{\approx}{\boldsymbol{\Phi}} \cdot \nabla^0\right) \cdot \underset{\sim}{\boldsymbol{\Pi}} + \underset{\approx}{\boldsymbol{\Phi}} : \underset{\approx}{\boldsymbol{T}}^{\mathrm{T}}\right) \cdot \underline{\boldsymbol{N}}, \quad t_i = \left((F_{iJ} + \Phi_{iJL,L})\Pi_{JK} + \Phi_{iJL}T_{JLK}\right) N_K \tag{77}$$

$$\underset{\sim}{\boldsymbol{t}} = \underset{\sim}{\boldsymbol{S}} \otimes \underline{\boldsymbol{N}}, \quad t_{iJK} = S_{iJ}N_K \tag{78}$$

The consideration of equations (75) and (77) leads to the definition of a new Boussinesq stress tensor for the stress gradient theory:

$$\underset{\sim}{\boldsymbol{S}}^B := \underset{\sim}{\boldsymbol{S}} + (\mathrm{Div}\, \underset{\approx}{\boldsymbol{\Phi}}) \cdot \underset{\sim}{\boldsymbol{\Pi}} + \underset{\approx}{\boldsymbol{\Phi}} : \underset{\approx}{\boldsymbol{T}}^{\mathrm{T}} = \left(\underset{\sim}{\boldsymbol{F}} + \mathrm{Div}\, \underset{\approx}{\boldsymbol{\Phi}}\right) \cdot \underset{\sim}{\boldsymbol{\Pi}} + \underset{\approx}{\boldsymbol{\Phi}} : \underset{\approx}{\boldsymbol{T}}^{\mathrm{T}} \tag{79}$$

whose divergence balances volume and inertia forces and whose action on the normal vector at the boundary delivers the generalized traction.

4.3. Hyperelastic stress gradient model and its linearization

The reduced hyperelastic laws of the stress gradient medium can be obtained by eliminating the Lagrange multiplier in equation (68). For that purpose, note that

$$\underset{\approx}{\boldsymbol{K}} : \underset{\sim}{\boldsymbol{1}} = \underset{\sim}{\boldsymbol{F}}^{\mathrm{T}} \cdot \mathrm{Div}\, \underset{\approx}{\boldsymbol{\Phi}} \tag{80}$$

so that

$$\lambda_f \underset{\sim}{\boldsymbol{1}} = \underset{\sim}{\boldsymbol{F}}^{\mathrm{T}} \cdot \mathrm{Div}\, \underset{\approx}{\boldsymbol{\Phi}} + \frac{\partial \psi^\star}{\partial \underset{\approx}{\boldsymbol{M}}} : \underset{\sim}{\boldsymbol{1}} \tag{81}$$

Finally, it is found that

$$\frac{\underset{\sim}{\boldsymbol{C}}}{2} + \underset{\sim}{\boldsymbol{F}}^{\mathrm{T}} \cdot \mathrm{Div}\, \underset{\approx}{\boldsymbol{\Phi}} = -\left(\frac{\partial \psi^\star}{\partial \underset{\sim}{\boldsymbol{\Pi}}} + \frac{\partial \psi^\star}{\partial \underset{\approx}{\boldsymbol{M}}} : \underset{\sim}{\boldsymbol{1}}\right) = -\frac{\partial \Psi^\star}{\partial \underset{\sim}{\boldsymbol{\Pi}}} \tag{82}$$

after recalling that $\underset{\approx}{\boldsymbol{M}} = \underset{\sim}{\boldsymbol{\Pi}} \otimes \underset{\sim}{\boldsymbol{1}}$ and introducing a reduced stress potential $\Psi^\star(\underset{\sim}{\boldsymbol{\Pi}}, \underset{\approx}{\boldsymbol{T}})$ such that

$$\underset{\sim}{\boldsymbol{E}} = \frac{\underset{\sim}{\boldsymbol{C}}}{2} + \underset{\sim}{\boldsymbol{F}}^{\mathrm{T}} \cdot \underset{\approx}{\boldsymbol{\Phi}} \cdot \nabla^0 = -\frac{\partial \Psi^\star}{\partial \underset{\sim}{\boldsymbol{\Pi}}}, \quad \underset{\approx}{\boldsymbol{\Upsilon}} = \underset{\sim}{\boldsymbol{F}}^{\mathrm{T}} \cdot \underset{\approx}{\boldsymbol{\Phi}} = -\frac{\partial \Psi^\star}{\partial \underset{\approx}{\boldsymbol{T}}} \tag{83}$$

Accordingly, the dual potential $\Psi(\underset{\sim}{\boldsymbol{E}}, \underset{\approx}{\boldsymbol{\Upsilon}})$ can be derived based on the reduced strain tensor $\underset{\sim}{\boldsymbol{E}}$ and the generalized strain measure $\underset{\approx}{\boldsymbol{\Upsilon}}$:

$$\delta\Psi = \frac{\partial \Psi}{\partial \underset{\sim}{\boldsymbol{E}}} : \delta \underset{\sim}{\boldsymbol{E}} + \frac{\partial \Psi}{\partial \underset{\approx}{\boldsymbol{\Upsilon}}} : \delta \underset{\approx}{\boldsymbol{\Upsilon}} = \underset{\sim}{\boldsymbol{\Pi}} : \delta \underset{\sim}{\boldsymbol{E}} + \underset{\approx}{\boldsymbol{T}} \,\vdots\, \delta \underset{\approx}{\boldsymbol{\Upsilon}} \tag{84}$$

after identification of the conjugate stress tensors as they appear in the increment of virtual work of internal forces; see equation (69).

The linearization of the proposed theory is consistent with the linear stress gradient model presented in the static case by Forest and Sab [23], as will be shown. The small deformation framework for the stress gradient theory is characterized by

$$\|\underset{\sim}{\boldsymbol{F}} - \underset{\sim}{\boldsymbol{1}}\| \ll 1, \quad L^{-1}\|\underset{\approx}{\boldsymbol{\Phi}}\| \ll 1, \quad \|\underset{\approx}{\boldsymbol{\Phi}} \otimes \nabla\| \ll 1 \tag{85}$$

The generalized strain measures are linearized as

$$\underset{\sim}{\boldsymbol{E}} - \underset{\sim}{\boldsymbol{1}}/2 = (\underset{\sim}{\boldsymbol{C}} - \underset{\sim}{\boldsymbol{1}})/2 + \underset{\sim}{\boldsymbol{F}}^{\mathrm{T}} \cdot \underset{\approx}{\boldsymbol{\Phi}} \simeq \underset{\sim}{\boldsymbol{\varepsilon}} + \underset{\approx}{\boldsymbol{\Phi}} \cdot \nabla \equiv \underset{\sim}{\boldsymbol{e}}, \quad \underset{\approx}{\boldsymbol{\Upsilon}} = \underset{\sim}{\boldsymbol{F}}^{\mathrm{T}} \cdot \underset{\approx}{\boldsymbol{\Phi}} \simeq \underset{\approx}{\boldsymbol{\Phi}} \tag{86}$$

where $\underset{\sim}{\boldsymbol{\varepsilon}}$ is the infinitesimal strain tensor and $\underset{\sim}{\boldsymbol{e}}$ is the generalized strain measure found in [23]. Regarding stresses, we obtain

$$\underset{\sim}{\boldsymbol{\Pi}} \simeq \underset{\sim}{\boldsymbol{\sigma}}, \quad \underset{\approx}{\boldsymbol{T}} \simeq \underset{\sim}{\boldsymbol{\sigma}} \otimes \nabla \equiv \underset{\approx}{\boldsymbol{R}} \tag{87}$$

where $\underset{\sim}{\boldsymbol{\sigma}}$ is the Cauchy stress tensor and $\underset{\approx}{\boldsymbol{R}}$ was the name given to the infinitesimal stress gradient in [23], in the static case. The linearization of the balance laws (75) and (76) proceeds as follows:

$$\underset{\sim}{\boldsymbol{S}} + (\operatorname{Div} \underset{\approx}{\boldsymbol{\Phi}}) \cdot \underset{\sim}{\boldsymbol{\Pi}} + \underset{\approx}{\boldsymbol{\Phi}} : \underset{\approx}{\boldsymbol{T}}^{\mathrm{T}} \simeq \underset{\sim}{\boldsymbol{\sigma}}, \quad \text{since} \quad \|\underset{\sim}{\boldsymbol{S}}\| \gg \|(\operatorname{Div} \underset{\approx}{\boldsymbol{\Phi}}) \cdot \underset{\sim}{\boldsymbol{\Pi}}\|, \quad \|\underset{\sim}{\boldsymbol{S}}\| \gg \|\underset{\approx}{\boldsymbol{\Phi}} : \underset{\approx}{\boldsymbol{T}}^{\mathrm{T}}\| \tag{88}$$

As a consequence, the balance laws of the linear and static stress gradient theory by Forest and Sab [23] are recovered:

$$\operatorname{div} \underset{\sim}{\boldsymbol{\sigma}} + \underline{\boldsymbol{f}} = 0, \quad \underset{\approx}{\boldsymbol{R}} - \underset{\sim}{\boldsymbol{\sigma}} \otimes \nabla + \underset{\approx}{\boldsymbol{f}} = 0 \tag{89}$$

After linearization of the stress and strain measures and their substitution into the hyperelastic laws (84), the elastic laws proposed in [23] are also retrieved:

$$\underset{\sim}{\boldsymbol{e}} = \frac{\partial w^{\star}(\underset{\sim}{\boldsymbol{\sigma}}, \underset{\approx}{\boldsymbol{R}})}{\partial \underset{\sim}{\boldsymbol{\sigma}}}, \quad \underset{\approx}{\boldsymbol{\Phi}} = \frac{\partial w^{\star}(\underset{\sim}{\boldsymbol{\sigma}}, \underset{\approx}{\boldsymbol{R}})}{\partial \underset{\approx}{\boldsymbol{R}}}, \quad \underset{\sim}{\boldsymbol{\sigma}} = \frac{\partial w(\underset{\sim}{\boldsymbol{e}}, \underset{\approx}{\boldsymbol{\Phi}})}{\partial \underset{\sim}{\boldsymbol{e}}}, \quad \underset{\approx}{\boldsymbol{R}} = \frac{\partial w(\underset{\sim}{\boldsymbol{e}}, \underset{\approx}{\boldsymbol{\Phi}})}{\partial \underset{\approx}{\boldsymbol{\Phi}}} \tag{90}$$

It is worth considering now the linearization of the virtual power of contact forces taken from equation (71):

$$\delta \underline{\boldsymbol{u}} \cdot \left(\underset{\sim}{\boldsymbol{S}} + (\underset{\approx}{\boldsymbol{\Phi}} \cdot \nabla^0) \cdot \underset{\sim}{\boldsymbol{\Pi}} + \underset{\approx}{\boldsymbol{\Phi}} : \underset{\approx}{\boldsymbol{T}}^{\mathrm{T}} \right) \cdot \underline{\boldsymbol{N}} + (\underset{\sim}{\boldsymbol{S}} \otimes \underline{\boldsymbol{N}}) \,\vdots\, \delta \underset{\approx}{\boldsymbol{\Phi}} \simeq \underset{\sim}{\boldsymbol{\sigma}} : (\delta \underline{\boldsymbol{u}} \otimes \underline{\boldsymbol{N}} + \delta \underset{\approx}{\boldsymbol{\Phi}} \cdot \underline{\boldsymbol{N}}) \tag{91}$$

which is the expression found in [23]. This structure for the virtual work of contact forces allows us to prescribe the full stress tensor along a boundary, which is a peculiar feature of the linear stress gradient theory. Existence and uniqueness theorems were established by Sab et al. [24] for such stress-based boundary conditions. Corresponding existence theorems remain to be derived for the finite deformation stress gradient model proposed in the present work.

5. Applications

The proposed applications deal first with a combination of strain and stress gradient theories in the static case and then with the one-dimensional dynamics of second-order micromorphic and stress gradient media. They are limited to the small strain case.

5.1. Combination of stress and strain gradient models

A second-order micromorphic medium is considered that involves the following degrees of freedom:

$$\mathrm{DOF} = \{\underline{\boldsymbol{u}}, \quad \underset{\sim}{\boldsymbol{\chi}}, \quad \underset{\approx}{\boldsymbol{\Phi}}\} \tag{92}$$

The following expression of the work of internal forces, combining the features of first-order micromorphic and stress gradient theories, is adopted:

$$p^{(i)} = \underset{\sim}{\boldsymbol{\sigma}} : (\underset{\sim}{\boldsymbol{\varepsilon}} + \underset{\approx}{\boldsymbol{\Phi}} \cdot \nabla) + \underset{\approx}{\boldsymbol{R}} \,\vdots\, \underset{\approx}{\boldsymbol{\Phi}} + \underset{\sim}{\boldsymbol{s}} : (\underset{\sim}{\boldsymbol{\chi}} - \underset{\sim}{\boldsymbol{\varepsilon}}) + \underset{\approx}{\boldsymbol{M}} \,\vdots\, \underset{\sim}{\boldsymbol{\chi}} \otimes \nabla \tag{93}$$

within the infinitesimal deformation framework. The field balance equations associated with such a medium are threefold, as deduced from the earlier analyses:

$$\operatorname{div} (\underset{\sim}{\boldsymbol{\sigma}} + \underset{\sim}{\boldsymbol{s}}) = 0, \quad \underset{\approx}{\boldsymbol{R}} = \underset{\sim}{\boldsymbol{\sigma}} \otimes \nabla, \quad \operatorname{div} \underset{\approx}{\boldsymbol{M}} + \underset{\sim}{\boldsymbol{s}} = 0 \tag{94}$$

in the static case and in the absence of body forces. Implementing the internal constraint $\underset{\sim}{\boldsymbol{\chi}} \equiv \underset{\sim}{\boldsymbol{\varepsilon}}$ corresponding to the strain gradient part of the model, the three previous equations reduce to two:

$$\operatorname{div} (\underset{\sim}{\boldsymbol{\sigma}} - \operatorname{div} \underset{\approx}{\boldsymbol{M}}) = 0, \quad \underset{\approx}{\boldsymbol{R}} = \underset{\sim}{\boldsymbol{\sigma}} \otimes \nabla \tag{95}$$

The linear constitutive laws for the stress gradient part of the model are as in [23]:

$$\underset{\sim}{\boldsymbol{\sigma}} = \underset{\approx}{\boldsymbol{C}} : \underset{\sim}{\boldsymbol{e}} = \underset{\approx}{\boldsymbol{C}} : (\underset{\sim}{\boldsymbol{\varepsilon}} + \underset{\approx}{\boldsymbol{\Phi}} \cdot \nabla), \quad \underset{\approx}{\boldsymbol{R}} = \underset{\underset{\sim}{\approx}}{\boldsymbol{D}} \,\vdots\, \underset{\approx}{\boldsymbol{\Phi}} \tag{96}$$

where $\underset{\approx}{\boldsymbol{C}}$ and $\underset{\approx}{\underset{\sim}{\boldsymbol{D}}}$ are fourth- and sixth-rank tensors of generalized elastic moduli. In fact, for the purpose of comparison with Eringen's model, it is sufficient to consider the simplified law

$$\underset{\approx}{\boldsymbol{\Phi}} = \ell_\sigma^2 \underset{\approx}{\boldsymbol{C}}^{-1} : \underset{\approx}{\boldsymbol{R}} \tag{97}$$

so that

$$\underset{\sim}{\boldsymbol{\sigma}} - \ell_\sigma^2 \nabla^2 \underset{\sim}{\boldsymbol{\sigma}} = \underset{\approx}{\boldsymbol{C}} : \underset{\sim}{\boldsymbol{\varepsilon}} \tag{98}$$

which is identical to Eringen's model involving the Laplacian of the stress tensor; see [46]. The fact that Eringen's model can be retrieved from a stress gradient theory was recognized by Polizzotto [28].

On the other hand, following [47], a simplified higher-order elasticity law linking the hyperstress tensor to the strain gradient is

$$\underset{\approx}{\boldsymbol{M}} = \ell_\varepsilon^2 \underset{\approx}{\boldsymbol{C}} : (\underset{\sim}{\boldsymbol{\varepsilon}} \otimes \nabla) \tag{99}$$

which is similar to equation (97). This leads to a generalized elasticity law for the effective stress tensor, $\underset{\sim}{\boldsymbol{\tau}}$, whose divergence vanishes under static conditions, according to [40]:

$$\underset{\sim}{\boldsymbol{\tau}} = \underset{\sim}{\boldsymbol{\sigma}} - \operatorname{div} \underset{\approx}{\boldsymbol{M}} = \underset{\sim}{\boldsymbol{\sigma}} - \ell_\varepsilon^2 \underset{\approx}{\boldsymbol{C}} : \nabla^2 \underset{\sim}{\boldsymbol{\varepsilon}} \tag{100}$$

which coincides with Aifantis's celebrated gradient elasticity model [48]. Combining the constitutive laws (98) and (100), the following partial differential equation for stress and strain tensors is obtained:

$$\underset{\sim}{\boldsymbol{\tau}} - \ell_\sigma^2 \nabla^2 \underset{\sim}{\boldsymbol{\sigma}} = \underset{\approx}{\boldsymbol{C}} : (\underset{\sim}{\boldsymbol{\varepsilon}} - \ell_\varepsilon^2 \nabla^2 \underset{\sim}{\boldsymbol{\varepsilon}}) \tag{101}$$

This equation involves the Laplacians of both stress and strain tensors. Note that both stress tensors $\underset{\sim}{\boldsymbol{\sigma}}$ and $\underset{\sim}{\boldsymbol{\tau}}$ are present.

In the context of a combined stress and strain gradient theory, the question arises of the choice of the proper stress tensor whose gradient should be incorporated in the model. Here, we have considered $\underset{\sim}{\boldsymbol{\sigma}} \otimes \nabla$, but the theory could instead involve $\underset{\sim}{\boldsymbol{\tau}} \otimes \nabla$. For that purpose, an alternative combination of stress and strain gradient theories can be proposed from the following modification of the power of internal forces (93):

$$p^{(i)} = \underset{\sim}{\boldsymbol{\sigma}} : \underset{\sim}{\dot{\boldsymbol{\varepsilon}}} + (\underset{\sim}{\boldsymbol{\sigma}} + \underset{\sim}{\boldsymbol{s}}) : \underset{\approx}{\dot{\boldsymbol{\Phi}}} \cdot \nabla + \underset{\approx}{\boldsymbol{R}} \,\vdots\, \underset{\approx}{\dot{\boldsymbol{\Phi}}} + \underset{\sim}{\boldsymbol{s}} : (\underset{\sim}{\dot{\boldsymbol{\chi}}} - \underset{\sim}{\dot{\boldsymbol{\varepsilon}}}) + \underset{\approx}{\boldsymbol{M}} \,\vdots\, \underset{\sim}{\dot{\boldsymbol{\chi}}} \otimes \nabla \tag{102}$$

which leads to the following static balance equations:

$$\operatorname{div}(\underset{\sim}{\boldsymbol{\sigma}} + \underset{\sim}{\boldsymbol{s}}) = 0, \quad \underset{\approx}{\boldsymbol{R}} = (\underset{\sim}{\boldsymbol{\sigma}} + \underset{\sim}{\boldsymbol{s}}) \otimes \nabla, \quad \operatorname{div} \underset{\approx}{\boldsymbol{M}} + \underset{\sim}{\boldsymbol{s}} = 0 \tag{103}$$

Eliminating the relative stress tensor $\underset{\sim}{\boldsymbol{s}}$ provides the balance equations with respect to the effective stress $\underset{\sim}{\boldsymbol{\tau}} = \underset{\sim}{\boldsymbol{\sigma}} - \operatorname{div} \underset{\approx}{\boldsymbol{M}}$:

$$\operatorname{div} \underset{\sim}{\boldsymbol{\tau}} = 0, \quad \underset{\approx}{\boldsymbol{R}} = \underset{\sim}{\boldsymbol{\tau}} \otimes \nabla \tag{104}$$

which shows that it is a theory for the effective stress gradient. Implementing again the internal constraint $\underset{\sim}{\boldsymbol{\chi}} \equiv \underset{\sim}{\boldsymbol{\varepsilon}}$ corresponding to the strain gradient part of the model, the following constitutive equations are retained:

$$\underset{\sim}{\boldsymbol{\tau}} = \underset{\approx}{\boldsymbol{C}} : \underset{\sim}{\boldsymbol{e}} = \underset{\approx}{\boldsymbol{C}} : (\underset{\sim}{\boldsymbol{\varepsilon}} + \operatorname{div} \underset{\approx}{\boldsymbol{\Phi}}), \quad \underset{\approx}{\boldsymbol{\Phi}} = \ell_\sigma^2 \underset{\approx}{\boldsymbol{C}}^{-1} : \underset{\approx}{\boldsymbol{R}} \tag{105}$$

The following relation between stress and strain gradients is deduced:

$$\underset{\sim}{\boldsymbol{\sigma}} - \ell_\sigma^2 \nabla^2 \underset{\sim}{\boldsymbol{\tau}} = \underset{\approx}{\boldsymbol{C}} : (\underset{\sim}{\boldsymbol{\varepsilon}} + \ell_\varepsilon^2 \nabla^2 \underset{\sim}{\boldsymbol{\varepsilon}}) \tag{106}$$

This equation differs from (101) by the interchange of $\underset{\sim}{\boldsymbol{\sigma}}$ and $\underset{\sim}{\boldsymbol{\tau}}$ on the left-hand side and by the plus sign on the right-hand side.

Within the small strain framework, Gutkin and Aifantis [49] and Aifantis [50] proposed a gradient elasticity constitutive law involving both strain and stress gradient variables in the form

$$(1 - l_\sigma^2 \nabla^2)\underset{\sim}{\boldsymbol{\sigma}} = (1 - l_\varepsilon^2 \nabla^2)(\lambda(\text{trace}\, \underset{\sim}{\boldsymbol{\varepsilon}})\underset{\sim}{\mathbf{1}} + 2\mu \underset{\sim}{\boldsymbol{\varepsilon}}) \tag{107}$$

Equations (101) and (106) also involve Laplacians of both stress and strain tensors. They are reminiscent of Aifantis's generalized gradient elasticity model (107). In this equation, $\underset{\sim}{\boldsymbol{\sigma}}$ is presented as the usual stress tensor. Forest and Aifantis [47] suggested however that it should be interpreted as the effective stress $\underset{\sim}{\boldsymbol{\tau}}$ from the strain gradient model. Equation (107) is presented as a constitutive law. In fact, it is a combination of balance and constitutive equations as in the derivation of equations (101) and (106). The ambiguity on the nature of the stress tensor remains as long as the suitable balance, boundary and constitutive equations are not provided.

The merits of a combined stress and strain gradient model are suggested by the recent work of Tran [51] where the stress gradient model was shown to predict softening effects at small scales. In contrast, stiffening effects are generally obtained with a strain gradient model. Depending on boundary or interface conditions related to either stress gradient or strain gradient types, such stiffening and softening effects may be competing in actual materials.

5.2. Higher-order dynamics of linear micromorphic and strain gradient media

The dynamics of micromorphic media is derived in the case of linear elasticity. The obtained field equations are then specialized in the presence of internal constraints linking the micromorphic deformations to strain gradients.

The generalized Hooke laws are given in the case of a centro-symmetrical second-order micromorphic material for the sake of brevity. The retained form is that proposed in [14]:

$$\underset{\sim}{\boldsymbol{\sigma}} = \underset{\approx}{\boldsymbol{c}} : \underset{\sim}{\boldsymbol{\varepsilon}} \tag{108}$$

$$\underset{\sim}{\boldsymbol{T}} = \underset{\approx}{\boldsymbol{a}} : (\underline{\boldsymbol{u}} \otimes \nabla - \underset{\sim}{\boldsymbol{\chi}}) \tag{109}$$

$$\underset{\approx}{\boldsymbol{M}} = \underset{\underset{\sim}{\approx}}{\boldsymbol{b}} \,\vdots\, \underset{\approx}{\boldsymbol{K}} \tag{110}$$

$$\underset{\approx}{\boldsymbol{T}} = \underset{\underset{\sim}{\approx}}{\boldsymbol{a}} \,\vdots\, (\underset{\approx}{\boldsymbol{K}} - \underset{\approx}{\boldsymbol{\chi}}) \tag{111}$$

$$\underset{\approx}{\boldsymbol{M}} = \underset{\underset{\approx}{\approx}}{\boldsymbol{b}} :: \underset{\approx}{\boldsymbol{K}} \tag{112}$$

The relations (109) and (111) penalize the difference between the first- (respectively, second-) order microdeformation and the first (respectively, second) gradient of the displacement fields. Very large values of the higher-order moduli $\underset{\approx}{\boldsymbol{a}}$ and $\underset{\underset{\sim}{\approx}}{\boldsymbol{a}}$ amount to prescribing the constraints (60) which reduce the second micromorphic medium to a second strain gradient material. Odd-rank tensors of elasticity are excluded in the previous laws by the hypothesis of centro-symmetry [52]. The full classification of anisotropy classes remains to be done for second-order micromorphic media and is not discussed here [53].

It is shown now how the three linearized balance laws (42) to (44) can be combined, after substitution of the constitutive equations (108) to (112), to derive a single partial differential equation involving all degrees of freedom and in which internal constraints can be enforced. The derivation is essentially formal due to the complexity of tensor combinations. Eliminating the coupling stress $\underset{\sim}{\boldsymbol{T}}$ in equation (42) by means of equation (43) yields

$$\text{div}\,(\underset{\sim}{\boldsymbol{\sigma}} + \text{div}\,\underset{\approx}{\boldsymbol{M}} - \rho I \ddot{\underset{\sim}{\boldsymbol{\chi}}}) = \rho \ddot{\underline{\boldsymbol{u}}}$$

The third-rank stress tensor can be eliminated from the previous expression using the constitutive law (110) and the balance law (44):

$$\underset{\approx}{\boldsymbol{M}} = \underset{\underset{\sim}{\approx}}{\boldsymbol{b}} \,\vdots\, \underset{\approx}{\boldsymbol{K}} = \underset{\underset{\sim}{\approx}}{\boldsymbol{b}} \,\vdots\, \left(\underset{\approx}{\boldsymbol{\chi}} + \underset{\underset{\sim}{\approx}}{\boldsymbol{a}}^{-1} \vdots\, \underset{\approx}{\boldsymbol{T}} \right) = \underset{\underset{\sim}{\approx}}{\boldsymbol{b}} \,\vdots\, \left(\underset{\approx}{\boldsymbol{\chi}} + \underset{\underset{\sim}{\approx}}{\boldsymbol{a}}^{-1} \vdots \left(\text{div}\,\underset{\approx}{\boldsymbol{M}} - \rho \ddot{\underset{\approx}{\boldsymbol{\chi}}} : \underset{\approx}{\boldsymbol{I}} \right) \right)$$

and finally

$$\operatorname{div}\left(\underset{\sim}{\boldsymbol{\sigma}}+\operatorname{div}\left(\underset{\approx\!\approx}{\boldsymbol{b}}\vdots\left(\underset{\approx}{\boldsymbol{\chi}}+\underset{\approx\!\approx}{\boldsymbol{a}}^{-1}\vdots\left(\operatorname{div}\underset{\approx}{\boldsymbol{M}}-\rho\,\underset{\approx}{\ddot{\boldsymbol{\chi}}}:\underset{\approx}{\boldsymbol{I}}\right)\right)\right)-\rho I\,\underset{\sim}{\ddot{\boldsymbol{\chi}}}\right)=\rho\,\underline{\ddot{\boldsymbol{u}}} \tag{113}$$

At this stage, the linearized internal constraints (60)

$$\underset{\sim}{\boldsymbol{\chi}}\equiv\underline{\boldsymbol{u}}\otimes\nabla,\quad \underset{\approx}{\boldsymbol{\chi}}\equiv\underline{\boldsymbol{u}}\otimes\nabla\otimes\nabla \tag{114}$$

are substituted into (113) to obtain the dynamical equations of second strain gradient media. This equation is written in symbolic form in order to draw attention to the order of the space and time derivatives of the displacement variable in each term:

$$Au^{(2)}+Bu^{(4)}+Cu^{(6)}-\rho I\ddot{u}^{(4)}-D\ddot{u}^{(2)}-\rho\ddot{u}=0 \tag{115}$$

where the integers in parentheses denote the order of the spatial derivative and the coefficients symbolically represent appropriate combinations of elastic moduli. It is remarkable that the dynamical equation contains spatial derivatives of even order up to six and mixed space and time derivatives of orders two to six. The fourth-order terms are characteristic of Mindlin's first strain gradient dynamics. The direct derivation of the mixed term $\ddot{u}^{(2)}$ in the strain gradient model requires the introduction of a hypermomentum tensor that is very often forgotten, especially in the dynamics of Euler–Bernoulli beams; see [54, 55]. It arises here as a natural consequence of the micromorphic dynamics. The introduction of the second-order microdeformation leads to the terms of sixth order. The previous equation represents one of the most general ones for the analysis of the dispersion of waves in gradient media [56, 57]. It contains terms that have been introduced heuristically in several dynamical equations in the literature; see the discussions in [58, 59, 60, 61].

5.3. Dynamics of linear stress gradient media

Some specific aspects of the dynamical behaviour of linear stress gradient media are now unravelled by combining the linear balance and constitutive equations including inertia contributions. The linearized dynamical equations of stress gradient media are deduced from equations (75) and (76) as follows:

$$\operatorname{div}\underset{\sim}{\boldsymbol{\sigma}}=\rho\,\underline{\ddot{\boldsymbol{u}}},\quad \underset{\approx}{\boldsymbol{R}}-\underset{\sim}{\boldsymbol{\sigma}}\otimes\nabla=\rho I\,\underset{\approx}{\ddot{\boldsymbol{\Phi}}} \tag{116}$$

in the absence of volume forces. Note that the inertia coefficient I in equation $(116)_2$ has no physical dimension since $\underline{\boldsymbol{u}}$ and $\underset{\approx}{\boldsymbol{\Phi}}$ share the same physical dimension of length. The linearized elasticity laws are given by equation (96). They relate the stress and stress gradients to the strain and microdisplacements. For the sake of brevity, the derivation of the dynamic equation for displacement is presented in the one-dimensional case. Equations (116) then reduce to

$$\sigma'=\rho\ddot{u},\quad R-\sigma'=\rho I\ddot{\Phi} \tag{117}$$

where $(\bullet)'$ denotes the one-dimensional derivative. The constitutive laws (96) become

$$\sigma=E(u'+\Phi'),\quad R=D\Phi \tag{118}$$

where E and D are constitutive elasticity moduli, in Pa. Substitution of the constitutive laws into the balance equation leads to the generalized Navier equations

$$E(u^{(2)}+\Phi^{(2)})=\rho\ddot{u},\quad D\Phi-E(u^{(2)}+\Phi^{(2)})=\rho I\ddot{\Phi} \tag{119}$$

From the two previous equations, the following consequences are respectively deduced:

$$\Phi^{(2)}=\frac{\rho}{E}\ddot{u}-u^{(2)},\quad D\Phi^{(2)}-\rho\ddot{u}^{(2)}=\rho I\ddot{\Phi}^{(2)} \tag{120}$$

Substitution of the former into the latter equation provides the linear differential equation for the displacement variable:

$$Du^{(2)}+\rho(1-I)\ddot{u}^{(2)}+\frac{\rho^2 I}{E}\ddddot{u}-\frac{D\rho}{E}\ddot{u}=0 \tag{121}$$

The dispersion of one-dimensional elastic waves according to equations (115) and (121) should be analysed in a way similar to the work done by Metrikine [54] and Berezovski [62] for other microstructured continua. Metrikine [54] examined the question of the causality of such equations. He postulates that *a partial differential equation that governs the dynamical behaviour of a one-dimensional model must be of the same order with respect to the spatial coordinate and with respect to time*. This is not the case for equation (115) due to the higher-order spatial derivation. This feature is well known in strain gradient media and is encountered in the Euler–Bernoulli beam model. It corresponds to the possibility of waves travelling with infinite speed. This paradox is solved by the use of a Timoshenko beam model. Equivalently, the unconstrained micromorphic media can be used to regularize the wave behaviour of gradient continua. The postulated causality condition is not satisfied either for stress gradient continua, as can be seen from equation (121), but for a different reason. The highest time derivative is four whereas the spatial order is two. This is a remarkable feature distinguishing the stress gradient from the strain gradient models.

6. Conclusions

The main contributions of the present work are as follows.

- *A second-order micromorphic model at finite deformation* introducing generalized Boussinesq stress tensors for balance and boundary conditions, and generalized Lagrangian strain measures for constitutive laws. It is more general than the previous Green and Naghdi, and Eringen and Germain theories. The higher-order microdeformations are introduced as relaxed deformation gradients of suitable order.
- *A reduction of the general micromorphic model to the grade n continuum model* by suitable constraints on the microdeformations tensors. In strain gradient theories, the higher-order stress tensor is *not* equal to the gradient of the usual stress tensor. That is why strain and stress gradient theories are distinct models of the continuum.
- *A stress gradient theory at finite deformation including inertia terms*. It involves a third-order tensor of additional degrees of freedom like the second-order micromorphic model. However, its mechanical meaning is different: it represents *microdisplacements* and *not* a relaxed strain gradient.
- *A one-dimensional linearized dispersion equation for the stress gradient model* that essentially differs from the strain gradient prediction. The highest order in time (respectively, space) derivation is two orders larger than the spatial (respectively, time) derivation order in linear stress (respectively, strain) gradient media.

These new theories are excellent candidates for the study of wave dispersion in generalized continua, as started in [63,2] for first-order micromorphic continua. Micromorphic models are necessary to overcome the paradox of the infinite wave speed of some elastic waves in gradient media and to obtain both acoustic and optical branches in the dispersion diagrams [61].

A central remaining question is the determination of the higher-order elastic moduli arising in such theories. Enhanced homogenization methods have been proposed recently to derive them from the microstructure of periodic heterogeneous material; see [64,65,66] for the construction of gradient and micromorphic models from the underlying heterogeneous Cauchy materials. Most approaches remain heuristic and the question is still largely open. Nassar et al. [67] and Reda et al. [68] recently proposed alternative asymptotic methods for the determination of dynamical properties of gradient and micromorphic media with application to composite materials. The found properties are inevitably anisotropic, which has implications for the dispersion of waves; see [55].

The strain gradient and stress gradient models emerge as distinct and rather complementary approaches to material behaviour. Stiffening effects are expected in strain gradient media at small scales, whereas softening effects were derived in the recent homogenization results by Tran [51]. It seems that both models should be combined in order to represent the competition of stiffening and softening effects present in heterogeneous materials, as sketched in Section 5.1.

The proposed theories were presented within the finite deformation framework for application in the size-dependent plasticity and fracture of metals that generally occur at large deformations. They could be applied to the ductile fracture of porous materials as initiated in [69,70] or to the deformation of metal polycrystals [71].

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References

[1] Hütter, G, Mühlich, U, and Kuna, M. Micromorphic homogenization of a porous medium: Elastic behavior and quasi-brittle damage. *Contin Mech Thermodyn* 2015; 27: 1059–1072.
[2] Madeo, A, Barbagallo, G, d'Agostino, MV, et al. First evidence of non-locality in real band-gap metamaterials: Determining parameters in the relaxed micromorphic model. *Proc R Soc Lond A* 2016; 472: 20160169.
[3] Mindlin, R. Micro-structure in linear elasticity. *Arch Rat Mech Anal* 1964; 16: 51–78.
[4] Eringen, A, and Suhubi, E. Nonlinear theory of simple microelastic solids. *Int J Eng Sci* 1964; 2: 189–203, 389–404.
[5] Germain, P. The method of virtual power in continuum mechanics. Part 2: Microstructure. *SIAM J Appl Math* 1973; 25: 556–575.
[6] Eringen, A. Mechanics of micromorphic continua. In: Kröner, E (ed.) *Mechanics of generalized continua*. Berlin: Springer-Verlag, 18–35.
[7] Eringen, A. Laws of micromorphic mechanics. *Int J Eng Sci* 1970; 8: 819–828.
[8] Green, A, and Rivlin, R. Multipolar continuum mechanics. *Arch Rat Mech Anal* 1964; 17: 113–147.
[9] Mindlin, R. Second gradient of strain and surface-tension in linear elasticity. *Int J Solid Struct* 1965; 1: 417–438.
[10] Bleustein, J. A note on the boundary conditions of Toupin's strain-gradient theory. *Int J Solid Struct* 1967; 3: 1053–1057.
[11] Madeo, A, Ghiba, I, Neff, P, et al. A new view on boundary conditions in the Grioli-Koiter-Mindlin-Toupin indeterminate couple stress model. *Eur J Mech A/Solid* 2016; 59: 294–322.
[12] Broese, C, Tsakmakis, C, and Beskos, D. Mindlin's micro-structural and gradient elasticity theories and their thermodynamics. *J Elast* 2016; 125: 87–132.
[13] Javili A, dell'Isola, F, and Steinmann, P. Geometrically nonlinear higher-gradient elasticity with energetic boundaries. *J Mech Phys Solid* 2013; 61: 2381–2401.
[14] Cordero, N, Forest, S, and Busso, E. Second strain gradient elasticity of nano-objects. *J Mech Phys Solid* 2016; 97: 92–124.
[15] Dillard, T, Forest, S, and Ienny, P. Micromorphic continuum modelling of the deformation and fracture behaviour of nickel foams. *Eur J Mech A/Solid* 2006; 25: 526–549.
[16] Enakoutsa, K and Leblond, J. Numerical implementation and assessment of the GLPD micromorphic model of ductile rupture. *Eur J Mech A/Solid* 2009; 28: 445–460.
[17] Ferretti, M, Madeo, A, dell'Isola, F, et al. Modeling the onset of shear boundary layers in fibrous composite reinforcements by second-gradient theory. *Z Angew Math Phys* 2014; 65: 587–612.
[18] Madeo, A, Ferretti, M, dell'Isola, F, et al. Thick fibrous composite reinforcements behave as special second-gradient materials: Three-point bending of 3D interlocks. *Z Angew Math Phys* 2015; 66: 2041–2060.
[19] Madeo, A, Barbagallo, G, D'Agostino, MV, et al. Continuum and discrete models for unbalanced woven fabrics. *Int J Solid Struct* 2016; 94–95: 263–284.
[20] Bergheau, J, Leblond, J and Perrin, G. A new numerical implementation of a second-gradient model for plastic porous solids, with an application to the simulation of ductile rupture tests. *Comput Meth Appl Mech Eng* 2014; 268: 105–125.
[21] Reiher, JC, Giorgio, I, and Bertram, A. Finite-element analysis of polyhedra under point and line forces in second-strain gradient elasticity. *Int J Eng Mech* 2017; 143: 04016112–1–13.
[22] Forest, S, Cordero, N, and Busso, E. First vs. second gradient of strain theory for capillarity effects in an elastic fluid at small length scales. *Comput Mater Sci* 2011; 50: 1299–1304.
[23] Forest, S and Sab, K. Continuum stress gradient theory. *Mech Res Commun* 2012; 40: 16–25.
[24] Sab, K, Legoll, F, and Forest, S. Stress gradient elasticity theory: Existence and uniqueness of solution. *J Elast* 2016; 123: 179–201.
[25] Lebée, A and Sab, K. A bending-gradient model for thick plates, Part I: Theory. *Int J Solid Struct* 2011; 48: 2878–2888.
[26] Lebée, A, and Sab, K. A bending-gradient model for thick plates, Part II: Closed-form solutions for cylindrical bending of laminates. *Int J Solid Struct* 2011; 48: 2889–2901.
[27] Polizzotto, C. Stress gradient versus strain gradient constitutive models within elasticity. *Int J Solid Struct* 2014; 51: 1809–1818.
[28] Polizzotto, C. A unifying variational framework for stress gradient and strain gradient elasticity theories. *Eur J Mech A/Solid* 2015; 49: 430–440.
[29] Forest, S, and Sievert, R. Elastoviscoplastic constitutive frameworks for generalized continua. *Acta Mech* 2003; 160: 71–111.
[30] Forest, S, and Sievert, R. Nonlinear 30 theories. *Int J Solid Struct* 2006; 43: 7224–7245.
[31] Regueiro, R. On finite strain micromorphic elastoplasticity. *Int J Solid Struct* 2010; 47: 786–800.
[32] Sansour, C, Skatulla, S, and Zbib, H. A formulation for the micromorphic continuum at finite inelastic strains. *Int J Solid Struct* 2010; 47: 1546–1554.

[33] Bertram, A. Finite gradient elasticity and plasticity: A constitutive mechanical framework. *Contin Mech Thermodyn* 2015; 27: 1039–1058.
[34] Bertram, A. Finite gradient elasticity and plasticity: A constitutive thermodynamical framework. *Contin Mech Thermodyn* 2016; 28: 819–883.
[35] Bertram, A, and Glüge, R. Gradient materials with internal constraints. *Math Mech Compl Syst* 2016; 4: 1–15.
[36] Maugin, GA. The method of virtual power in continuum mechanics: Application to coupled fields. *Acta Mech* 1980; 35: 1–70.
[37] Green, AE and Rivlin, RS. The relation between director and multipolar theories in continuum mechanics. *Z Angew Math Phys ZAMP* 1967; 18: 208–218.
[38] Green, A, Naghdi, P, and Rivlin, R. Directors and multipolar displacements in continuum mechanics. *Int J Eng Sci* 1965; 2: 611–620.
[39] Eringen, A. *Microcontinuum field theories*. New York, NY: Springer, 1999.
[40] Germain, P. La méthode des puissances virtuelles en mécanique des milieux continus, première partie: Théorie du second gradient. *J Méc* 1973; 12: 235–274.
[41] Liu, IS. *Continuum mechanics*. New York, NY: Springer, 2002.
[42] Seppecher, P. Second gradient theory: Application to Cahn–Hilliard fluids. In: Maugin, G, Drouot, R and Sidoroff, F (eds.) *Continuum thermomechanics: The art and science of modelling material behaviour, Paul Germain's anniversary volume*. Norwell, MA: Kluwer Academic Publishers, 379–388.
[43] Dell'Isola, F, and Seppecher, P. The relationship between edge contact forces, double forces and interstitial working allowed by the principle of virtual power. *CR Acad Sci Par IIb* 1995; 321: 303–308.
[44] Dell'Isola F, Seppecher P and Madeo A. How contact interactions may depend on the shape of Cauchy cuts in *N*th gradient continua: Approach "à la D'Alembert". *Zeitschrift für Angewandte Mathematik und Physik* 2012; 63: 1119–1141.
[45] Dell'Isola, F, Madeo, A, and Seppecher, P. Cauchy tetrahedron argument applied to higher contact interactions. *Arch Rat Mech Anal* 2016; 219: 1305–1341.
[46] Eringen, AC. On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves. *J Appl Phys* 1983; 54: 4703–4710.
[47] Forest, S, and Aifantis, EC. Some links between recent gradient thermo-elasto-plasticity theories and the thermomechanics of generalized continua. *Int J Solid Struct* 2010; 47: 3367–3376.
[48] Ru, C and Aifantis, E. A simple approach to solve boundary value problems in gradient elasticity. *Acta Mech* 1993; 101: 59–68.
[49] Gutkin, MY, and Aifantis, EC. Dislocations in the theory of gradient elasticity. *Script Mater* 1999; 40: 559–566.
[50] Aifantis, E. Update on a class of gradient theories. *Mech Mater* 2003; 35: 259–280.
[51] Tran, VP. *Stochastic modeling of random heterogeneous materials*. PhD Thesis, Université Paris Est, France, 2016.
[52] Auffray, N, Le Quang, H, and He, Q. Matrix representations for 3D strain-gradient elasticity. *J Mech Phys Solid* 2013; 61: 1202–1223.
[53] Olive, M, and Auffray, N. Symmetry classes for even-order tensors. *Math Mech Compl Syst* 2013; 1: 177–210.
[54] Metrikine, A. On causality of the gradient elasticity models. *J Sound Vibr* 2006; 297: 727–742.
[55] Rosi, G, and Auffray, N. Anisotropic and dispersive wave propagation within strain-gradient framework. *Wav Mot* 2016; 63: 120–134.
[56] Mindlin, R. Elasticity, piezoelectricity and crystal lattice dynamics. *J Elast* 1972; 2: 217–282.
[57] Askes, H, and Aifantis, EC. Gradient elasticity in statics and dynamics: An overview of formulations, length scale identification procedures, finite element implementations and new results. *Int J Solid Struct* 2011; 48: 1962–1990.
[58] Engelbrecht, J, Berezovski, A, Pastrone, F, et al. Waves in microstructured materials and dispersion. *Phil Mag* 2005; 85: 4127–4141.
[59] Askes, H, Wang, B, and Bennett, T. Element size and time step selection procedures for the numerical analysis of elasticity with higher-order inertia. *J Sound Vib* 2008; 314: 650–656.
[60] Askes, H, Metrikine, AV, Pichugin, AV, et al. Four simplified gradient elasticity models for the simulation of dispersive wave propagation. *Phil Mag* 2008; 88: 3415–3443.
[61] Papargyri-Beskou, S, Polyzos, D, and Beskos, D. Wave dispersion in gradient elastic solids and structures: A unified treatment. *Int J Solid Struct* 2009; 46: 3751–3759.
[62] Berezovski, A. Nonlinear dispersive wave equations for microstructured solids. *Proc Eston Acad Sci* 2015; 64: 203–211.
[63] Madeo, A, Neff, P, Ghiba, ID, et al. Wave propagation in relaxed micromorphic continua: Modeling metamaterials with frequency band-gaps. *Contin Mech Thermodyn* 2014; 27: 551–570.
[64] Forest, S, and Trinh, DK. Generalized continua and non-homogeneous boundary conditions in homogenization methods. *ZAMM* 2011; 91: 90–109.
[65] Trinh, DK, Jänicke, R, Auffray, N, et al. Evaluation of generalized continuum substitution models for heterogeneous materials. *Int J Multisc Comput Eng* 2012; 10: 527–549.
[66] Jänicke, R, and Steeb, H. Minimal loading conditions for higher order numerical homogenisation schemes. *Arch Appl Mech* 2012; 82: 1075–1088.
[67] Nassar, H, He, QC, and Auffray, N. A generalized theory of elastodynamic homogenization for periodic media. *Int J Solid Struct* 2016; 84: 139–146.

[68] Reda, H, Goda, I, Ganghoffer, J, et al. Dynamical analysis of homogenized second gradient anisotropic media for textile composite structures and analysis of size effects. *Compos Struct* 2017; 161: 540–551.
[69] Hütter, G. Homogenization of a Cauchy continuum towards a micromorphic continuum. *J Mech Phys Solid* 2017; 99: 394–408.
[70] Hütter, G. A micromechanical gradient extension of Gurson's model of ductile damage within the theory of microdilatational media. *Int J Solid Struct* 2017; 110–111: 15–23.
[71] Poh, L. Scale transition of a higher order plasticity model – A consistent homogenization theory from meso to macro. *J Mech Phys Solid* 2013; 61: 2692–2710.