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COSSERAT OVERALL MODELING OF HETEROGENEOUS MATERIALS

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Introduction

Classical homogenization methods enable one to replace a heterogeneous material with periodic microstructure by a homogeneous equivalent medium. The mathematical theory of homogenization [1] establisches the validity of this procedure providing that the size of the heterogenities is much smaller than the characteristic size of the considered structure or, more precisely, than the wavelength of the static mechanical loading. When these prerequisites are not fulfilled, the use of asymptotic methods is generally recommended [2].

The aim of this work is to propose an alternative methodology consisting in replacing the heterogeneous medium by a generalized continuum. Such continua involve additional degrees of freedom (Cosserat, micromorphic media [3]) or higher order gradients of the displacement field (second grade materials [4]...). The advantages of replacing a composite material by a homogeneous generalized continuum have been shown in [5] for instance, but the explicit links to homogenization theory appeared only recently [6,7,8].

The proposed scheme is general : it consists in developing the macroscopic displacement field into a polynomial main field and a periodic perturbation. The minimum order and number of terms of the retained polynomial are dictated by the number of macroscopic degrees of freedom and generalized strain measures to be prescribed on a unit cell. An additional scale invariance property of this polynomial is also required. For the sake of simplicity, the method is presented in the two-dimensional case and for a square unit cell. This heterogeneous Cauchy medium is replaced by a homogeneous Cosserat continuum. A simple application of the method to multilayer components is then presented to illustrate the predictive capability of a Cosserat effective medium. The results are compared with the response of a conventional Cauchy continuum.

In the sequel, $\underline{x}$, $\underset{\sim}{x}$, $\underset{\approx}{x}$ respectively denote a vector, second-rank and fourth-rank tensors. Vector product and transposition are denoted by $\times$ and ^{t}. The space of symmetric linear applications on $\mathbf{R}^2$ is $\mathcal{L}_s(\mathbf{R}^2)$.

Kinematics at the micro and macro scales

The kinematics of the heterogeneous material is given by the displacement field $\underline{\mathbf{u}} = u_1\underline{\mathbf{e}}_1 + u_2\underline{\mathbf{e}}_2$, whereas the effective medium is described by both a displacement field $\underline{\mathbf{U}} = U_1\underline{\mathbf{e}}_1 + U_2\underline{\mathbf{e}}_2$

and an independent rotation field $\underline{\Phi} = \Phi \underline{e}_3$. The relations between $\underline{u}$ and $(\underline{U}, \underline{\Phi})$ are first investigated.

Let $V_l(\underline{X})$ be a square, l its edge length and $\underline{X} = X_1\underline{e}_1 + X_2\underline{e}_2$ its center. We postulate that $(\underline{U}, \underline{\Phi})$ characterize the rigid body motion that best fits the actual displacement field $\underline{u}$ on $V_l(\underline{X})$. They are more precisely defined as the arguments for which

$$\int_{x_1=X_1-l/2}^{x_1=X_1+l/2} \int_{x_2=X_2-l/2}^{x_2=X_2+l/2} \mid \underline{u}(\underline{x}) - \underline{U}(\underline{X}) - \underline{\Phi}(\underline{X}) \times (\underline{x} - \underline{X}) \mid^2 dx_1 dx_2, \tag{1}$$

reaches a minimum. A straightforward calculation gives

$$\underline{U}(\underline{X}) =< \underline{u} >_{V_l(\underline{X})} \tag{2}$$

$$\underline{\Phi}(\underline{X}) = \frac{6}{l^2} < (\underline{x} - \underline{X}) \times \underline{u} >_{V_l(\underline{X})} \tag{3}$$

where $< . >_{V_l(\underline{X})}$ denotes volume averaging. It can then be proved that

$$\underset{\sim}{\nabla}_X \underline{U}(\underline{X}) =< \underset{\sim}{\nabla}_x \underline{u} >_{V_l(\underline{X})}, \tag{4}$$

$$\underline{K} = \frac{\partial \Phi}{\partial X_1}\underline{e}_1 + \frac{\partial \Phi}{\partial X_2}\underline{e}_2 = \underline{\nabla}_X \Phi = \frac{6}{l^2} < \underline{\nabla}_x((\underline{x} \times \underline{u}).\underline{e}_3) >_{V_l(\underline{X})} - \frac{6}{l^2}\underline{\nabla}_X((\underline{X} \times \underline{U}).\underline{e}_3) \tag{5}$$

where $\underline{K}$ denotes the torsion-curvature tensor in the two-dimensional case. The following notations are introduced :

$$\underset{\sim}{\varepsilon}(\underline{x}) = \frac{1}{2}(\underset{\sim}{\nabla}_x \underline{u} + \underset{\sim}{\nabla}_x^t \underline{u})(\underline{x}) \tag{6}$$

$$\underset{\sim}{E}(\underline{X}) = \frac{1}{2}(\underset{\sim}{\nabla}_X \underline{U} + \underset{\sim}{\nabla}_X^t \underline{U})(\underline{X}) \tag{7}$$

$$\underline{\Omega}(\underline{X}) = \frac{1}{2}(\frac{\partial U_2}{\partial X_1} - \frac{\partial U_1}{\partial X_2})\underline{e}_3 = \Omega(\underline{X})\underline{e}_3. \tag{8}$$

The elastic strain energy of a linear elastic Cosserat medium is a quadratic form in $(\underset{\sim}{E}, \underline{\Omega} - \underline{\Phi}, \underline{K})$ [3].

Let us now compute $(\underset{\sim}{E}, \underline{\Omega} - \underline{\Phi}, \underline{K})$ according to (2)-(5) when $\underline{u}$ is polynomial of grade 3 in $(\tilde{x}_1 = x_1/l, \tilde{x}_2 = x_2/l)$:

$$\begin{aligned} u_i = A_i + B_{i1}\tilde{x}_1 + B_{i2}\tilde{x}_2 &+ C_{i1}\tilde{x}_1^2 + C_{i2}\tilde{x}_2^2 + 2C_{i3}\tilde{x}_1\tilde{x}_2 \\ &+ D_{i1}\tilde{x}_1^3 + D_{i2}\tilde{x}_2^3 + 3D_{i3}\tilde{x}_1^2\tilde{x}_2 + 3D_{i4}\tilde{x}_1\tilde{x}_2^2 \quad (i = 1, 2). \end{aligned} \tag{9}$$

We find

$$\begin{aligned} U_i &= (A_i + \frac{C_{i1} + C_{i2}}{12}) + (B_{i1} + \frac{D_{i1} + D_{i4}}{4})\tilde{X}_1 + (B_{i2} + \frac{D_{i2} + D_{i3}}{4})\tilde{X}_2 \\ &+ C_{i1}\tilde{X}_1^2 + C_{12}\tilde{X}_2^2 + 2C_{i3}\tilde{X}_1\tilde{X}_2 + D_{i1}\tilde{X}_1^3 + D_{i2}\tilde{X}_2^3 + 3D_{i4}\tilde{X}_1\tilde{X}_2^2, \end{aligned} \tag{10}$$

$$\begin{aligned} l\Phi(\underline{X}) &= \left(\frac{B_{21} - B12}{2} + \frac{3}{40}(D_{21} - D_{12}) + \frac{D_{24} - D_{13}}{8}\right) + (C_{21} - C_{13})\tilde{X}_1 + (C_{23} - C_{12})\tilde{X}_2 \\ &+ 3(D_{23} - D_{14})\tilde{X}_1\tilde{X}_2 + \frac{3}{2}(D_{21} - D_{13})\tilde{X}_1^2 + \frac{3}{2}(D_{24} - D_{12})\tilde{X}_2^2, \end{aligned} \tag{11}$$

$$l(\Phi(\underline{X}) - \Omega(\underline{X})) = \frac{D_{12} - D_{21}}{20}. \tag{12}$$

The following particular cases arise :

- If $\underline{\mathbf{u}}$ is affine, we get $\underset{\sim}{\mathbf{E}}(\underline{\mathbf{X}}) = \underset{\sim}{\varepsilon}(\underline{\mathbf{X}})$ constant, $\Phi - \Omega = 0$ and $\underline{\mathbf{K}} = 0$.
- If $\underline{\mathbf{u}}$ is polynomial of grade 2, we get

$$\underset{\sim}{\mathbf{E}}(\underline{\mathbf{X}}) = \underset{\sim}{\varepsilon}(\underline{\mathbf{X}}) \text{ affine, } \Phi - \Omega = 0, \ \underline{\mathbf{K}} = \frac{C_{21} - C_{13}}{l^2}\underline{\mathbf{e}}_1 + \frac{C_{23} - C_{12}}{l^2}\underline{\mathbf{e}}_2.$$

- If $\underline{\mathbf{u}}$ is polynomial of grade 3, and if we impose that $\underset{\sim}{\mathbf{E}}(\underline{\mathbf{X}}) = \underset{\sim}{\varepsilon}(\underline{\mathbf{X}})$ be of grade 2 and Φ be affine, the following relations must hold

$$D_{21} = -D_{12}, D_{24} = D_{12}, D_{13} = -D_{12}$$

$$D_{14} = -D_{11}, D_{23} = -D_{11}, D_{22} = D_{11}$$

which leads to an expression of the terms of grade 3 involving only two independent constants D_{11} and D_{12}. A strain field such that $\underset{\sim}{\mathbf{E}}(\underline{\mathbf{X}}) = \underset{\sim}{\varepsilon}(\underline{\mathbf{X}})$ is said to be scale invariant in the homogenization process.

In the latter case, the expressions reduce to

$$\underset{\sim}{\mathbf{E}}(\underline{\mathbf{X}}) = \underset{\sim}{\varepsilon}(\underline{\mathbf{X}}) \text{ quadratic,} \tag{13}$$

$$(\Phi - \Omega)(\underline{\mathbf{X}}) = \frac{D_{12}}{10l}, \tag{14}$$

$$\underline{\mathbf{K}}(\underline{\mathbf{X}}) = \frac{C_{21} - C_{13}}{l^2}\underline{\mathbf{e}}_1 + \frac{C_{23} - C_{12}}{l^2}\underline{\mathbf{e}}_2. \tag{15}$$

It must be noted that terms including $D_{11}, C_{11}, C_{22}, (B_{21} - B_{12})/2, (C_{21} + C_{13})/2, (C_{23} - C_{12})/2, A_1$ and A_2 do not appear in the previous expressions (13)-(15) for $\underset{\sim}{\varepsilon}(\underline{\mathbf{x}})$, $\underset{\sim}{\mathbf{E}}(\underline{\mathbf{X}})$, $(\Phi - \Omega)(\underline{\mathbf{X}})$ and $\underline{\mathbf{K}}(\underline{\mathbf{X}})$, or are left indeterminate after prescribing the scale invariance condition, so that they will not be retained in the polynomial development. Finally, we retain the form:

$$\begin{cases} u_1^* &= B_{11}\tilde{x}_1 + B_{12}\tilde{x}_2 - C_{23}\tilde{x}_2^2 + 2C_{13}\tilde{x}_1\tilde{x}_2 + D_{12}(\tilde{x}_2^3 - 3\tilde{x}_1^2\tilde{x}_2), \\ u_2^* &= B_{12}\tilde{x}_1 + B_{22}\tilde{x}_2 - C_{13}\tilde{x}_1^2 + 2C_{23}\tilde{x}_1\tilde{x}_2 - D_{12}(\tilde{x}_1^3 - 3\tilde{x}_1\tilde{x}_2^2). \end{cases} \tag{16}$$

Identification of an overall Cosserat medium

Within the classical framework of periodic homogenization, a single unit cell $V_l(\underline{\mathbf{X}} = 0)$ is considered and, in the case of elasticity, the effective Cauchy medium is obtained by minimizing the elastic strain energy with respect to displacement fields belonging to $\mathcal{C}(\underset{\sim}{\mathbf{E}})$ defined by :

$$\forall \underset{\sim}{\mathbf{E}}^* \in \mathcal{L}_s(\mathbf{R}^2) \quad \underline{\mathbf{u}} \in \mathcal{C}(\underset{\sim}{\mathbf{E}}^*) \Longleftrightarrow \underline{\mathbf{u}}(\underline{\mathbf{x}}) = \underset{\sim}{\mathbf{E}}^*.\underline{\mathbf{x}} + \underline{\mathbf{u}}^{per}(\underline{\mathbf{x}}) \tag{17}$$

where the displacement field $\underline{\mathbf{u}}^{per}$ fulfills periodicity conditions at the boundary of $V_l(0)$ [9]. If $\underset{\approx}{\mathbf{a}}$ denotes the local elasticity tensor field, the effective elastic properties $\underset{\approx}{\mathbf{A}}^{hom}$ are such that

$$\frac{1}{2}\underset{\sim}{\mathbf{E}} : \underset{\approx}{\mathbf{A}}^{hom} : \underset{\sim}{\mathbf{E}} = \min_{\underline{\mathbf{u}} \in \mathcal{C}(\underset{\sim}{\mathbf{E}})} \frac{1}{2} < \underset{\sim}{\varepsilon}(\underline{\mathbf{u}}) : \underset{\approx}{\mathbf{a}}(\underline{\mathbf{x}}) : \underset{\sim}{\varepsilon}(\underline{\mathbf{u}}) >_{V_l(0)} . \tag{18}$$

The previous definition can be extended to the case of an effective Cosserat medium as follows :

$$\forall (\underset{\sim}{\mathbf{E}}^*, \Theta^*, \underline{\mathbf{K}}^*) \in \mathcal{L}_s(\mathbf{R}^2) \times \mathbf{R} \times \mathbf{R}^2, \ (\underline{\mathbf{u}} \in \mathcal{K}(\underset{\sim}{\mathbf{E}}^*, \Theta^*, \underline{\mathbf{K}}^*) \Longleftrightarrow \underline{\mathbf{u}} = \underline{\mathbf{u}}^* + \underline{\mathbf{u}}^{per}) \tag{19}$$

where $\underline{\mathbf{u}}^*$ is polynomial of the form (16) and $\underline{\mathbf{u}}^{per}$ is periodic. Furthermore, $(\underset{\sim}{\mathbf{E}}^*, \Theta^*, \underline{\mathbf{K}}^*)$ describing the kinematics of the Cosserat continuum at $\underline{\mathbf{X}} = 0$, the previous field must be such that

$$\underset{\sim}{\mathbf{E}}(0) = \underset{\sim}{\mathbf{E}}^*, \quad (\Phi - \Omega)(0) = \Theta^*, \quad \underline{\mathbf{K}}(0) = \underline{\mathbf{K}}^*. \tag{20}$$

These conditions imply the following relations for the coefficients of the polynomial :

$$\underset{\sim}{\mathbf{E}}(0) = \underset{\sim}{\mathbf{E}}^* \Longrightarrow B_{ij} = lE^*_{ij}; \tag{21}$$

$$(\Phi - \Omega)(0) = \Theta^* \Longrightarrow \frac{D_{12}}{10l} + \frac{6}{l^2} < \underline{\mathbf{x}} \times \underline{\mathbf{u}}^{per} >_{V_l(0)} = \Theta^*. \tag{22}$$

The last condition $\underline{\mathbf{K}}(0) = \underline{\mathbf{K}}^*$ gives :

$$-\frac{2C_{13}}{l^2} + \frac{6}{l^2} < x_1 u^{per}_{2,1} >_{V_l(0)} = K^*_1, \quad \frac{2C_{23}}{l^2} - \frac{6}{l^2} < x_2 u^{per}_{1,2} >_{V_l(0)} = K^*_2. \tag{23}$$

The strain energy density of the overall Cosserat medium is then defined by

$$\Psi(\underset{\sim}{\mathbf{E}}^*, \Theta^*, \underline{\mathbf{K}}^*) = \min_{\underline{\mathbf{u}} \in \mathcal{K}(\underset{\sim}{\mathbf{E}}^*, \Theta^*, \underline{\mathbf{K}}^*)} \frac{1}{2} < \underset{\sim}{\boldsymbol{\varepsilon}}(\underline{\mathbf{u}}) : \underset{\approx}{\mathbf{a}} : \underset{\sim}{\boldsymbol{\varepsilon}}(\underline{\mathbf{u}}) >_{V_l(0)}. \tag{24}$$

One may notice that $\underline{\mathbf{U}}^{per} = < \underline{\mathbf{u}}^{per} >$ is constant whereas the corresponding $\underline{\boldsymbol{\Phi}}^{per}$ is periodic.

To solve (24), a polynomial $\underline{\mathbf{u}}^*$ is given and the the minimizing procedure proceeds over all periodic fields $\underline{\mathbf{u}}^{per}$. The obtained solution for $\underline{\mathbf{u}}^{per}$ linearly depends on the coefficients of polynomial $\underline{\mathbf{u}}^*$. These coefficients are determined by solving equations (21)-(23). This ensures that $\underline{\mathbf{u}} = \underline{\mathbf{u}}^* + \underline{\mathbf{u}}^{per} \in \mathcal{K}(\underset{\sim}{\mathbf{E}}^*, \Theta^*, \underline{\mathbf{K}}^*)$. The overall force and couple stresses are then obtained by differentiating Ψ with respect to $(\underset{\sim}{\mathbf{E}}^*, \Theta^*, \underline{\mathbf{K}}^*)$. This homogenization scheme is applied in the next section to a specific example.

Example : Deformation of a multilayer material

Let us consider a multilayered material made of a material A and a much softer material B with the following elastic properties :

$$E^A = 210000\text{MPa}, \quad \nu^A = 0.3, \quad E^B = 1000\text{MPa}, \nu^B = 0.49.$$

The volume fraction of each component is 0.5. This heterogenous material is now replaced by a homogeneous Cosserat material according to the previous scheme. The strain measures are $\underset{\sim}{\mathbf{E}}, \Theta, \underline{\mathbf{K}}$. They are related to the associated force and couple stresses $\underset{\sim}{\boldsymbol{\Sigma}}$ and $\underline{\mathbf{M}}$ according to

$$\begin{bmatrix} \Sigma_{11} \\ \Sigma_{22} \\ \Sigma_{12} \\ \Sigma_{21} \\ M_1 \\ M_2 \end{bmatrix} = \begin{bmatrix} Y_{1111} & Y_{1122} & 0 & 0 & 0 & 0 \\ Y_{1122} & Y_{1111} & 0 & 0 & 0 & 0 \\ 0 & 0 & Y_{1212} & Y_{1221} & 0 & 0 \\ 0 & 0 & Y_{1221} & Y_{2121} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{3131} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{3131} \end{bmatrix} \begin{bmatrix} E_{11} \\ E_{22} \\ E_{12} + \Theta \\ E_{12} - \Theta \\ K_1 \\ K_2 \end{bmatrix} \tag{25}$$

where Y denotes the overall elasticity tensor. Its components are determined by prescribing successively the non-vanishing components $B_{11} = 1, B_{22} = 1, B_{12} = 0.5$, and $D_{12} = 1, C_{13} = 1, C_{23} = 1$ and computing the resulting $\underset{\sim}{\mathbf{E}}, \Theta, \underline{\mathbf{K}}$ according to (2)-(5) and strain energy.

The problem is solved using the finite element method in elasticity. The B_{ij}, C_{ij} and D_{ij} are treated as global degrees of freedom attributed to all elements of the cell, whereas the displacements u_i^{per} are the degrees of freedom attributed to each node. This is a straightforward extension of the numerical method used in [10] for conventional periodic homogenization. Periodicity conditions are then simply prescribed at the boundary of the cell. The values of the effective elasticity constants are found to be :

$$Y_{1111} = 131161\text{MPa}, Y_{1122} = 22420\text{Mpa}, Y_{2222} = 32274\text{MPa}.$$

$$Y_{1212} = 637577\text{MPa}, Y_{1221} = -720428\text{MPa}, Y_{2121} = 817085\text{MPa}$$

$$C_{3131} = 2500\text{MPa.mm}^2, C_{3232} = 2987\text{MPa.mm}^2.$$

For this calculation, we have used a unit cell (edge length $l = 1mm$) with a hard core bounded by two soft layers (see also figure 1). The deformed states associated with $C_{13} = 0.5$ (prescribed curvature K_1) and to $D_{12} = 1$ (prescribed Cosserat relative micro-rotation) are shown on figure 1 and 2.

Let us now consider a multilayered structure made of 6x8 cells and submitted to the following loading conditions : the nodes of the right hand side of the structure are fixed in both directions whereas a constant displacement in direction 2 is prescribed on the right hand side. The right hand side is free of forces in direction 1. The obtained deformed state is schown on figure 3. It can be seen that the mechanical loading combines flexion and shear. The same computation is carried out using the conventional Cauchy homogeneous equivalent continuum and with the newly identified Cosserat medium. A comparison between the two responses is given on figure 4 : the displacement U_2 is schown along a line $X_2 = Cst$ belonging to the hard material in the middle of the structure. The Cauchy continuum is seen to give a poor prediction of the real deformation state, it is not able to take the clamping conditions into account. On the contrary, the additional boundary condition $\Phi = 0$ can be prescribed at the left hand side for the Cosserat continuum to more precisely approach the actual situation. In the Cosserat computation, the right hand side is also free of couples.

Another example of the use of a Cosserat effective medium for a structure subjected to strong mechanical loading conditions gradients can be found in [11].

References

1. E. Sanchez-Palencia, Int. J. Engng Sci 12, 331, (1974).
2. J. Sanchez-Hubert and E. Sanchez-Palencia, Introduction aux méthodes asymptotiques et à l'homogénéisation, Masson, Paris, 1992.
3. C. B. Kafadar and A.C. Eringen, Int. J. Engng Sci. 9, 271, (1971).
4. P. Germain, J. de Mécanique 12, 235-274, (1973)
5. D. Besdo and H.-U. Dorau, Z. Angew. Math. u. Mech. 68, T153, (1988).
6. M. Gologanu, J.B. Leblond, G. Perrin, Condinuum Micromechanics, ed. P. Suquet, CISM course, Udine, (1996).
7. S. Forest, Int. Seminar on Geometry, Continuum and Microstructure, G. Maugin ed., *to appear*, Hermann, Paris, (1997).
8. F. Pradel, K. Sab, Euromech-Mecamat 2d Conference on Mechanics of Materials with Intrinsic Length Scales, Magdeburg, J. de Physique IV, *to appear*, (1998).
9. P. Suquet, in Homogenization techniques for composite media, E. Sanchez-Palencia and A. Zaoui ed., Lecture notes in physics 272, Springer-Verlag, (1985).
10. J.M. Besson, M. Jaeger and O. Desbordes, Composite Structures, D. Gay, C. Bord ed., i.i.t.t. International, (1988).
11. S. Forest, 40e Colloque de Métallurgie, Comportement mécanique et effets d'échelle, J. de Physique IV, *to appear*, (1997).

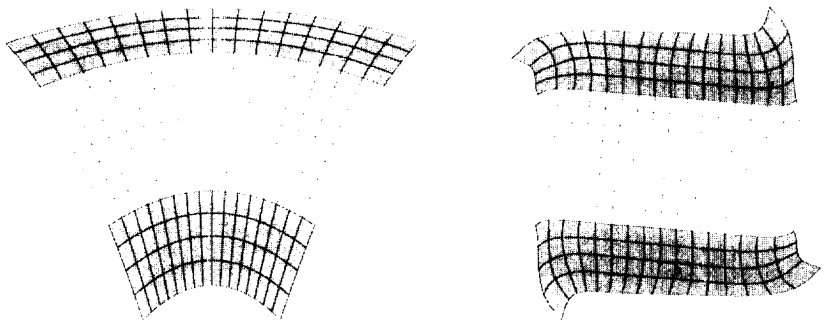

Figure 1 and 2 : Prescribed curvature (left) and relative rotation (right) on the unit cell.

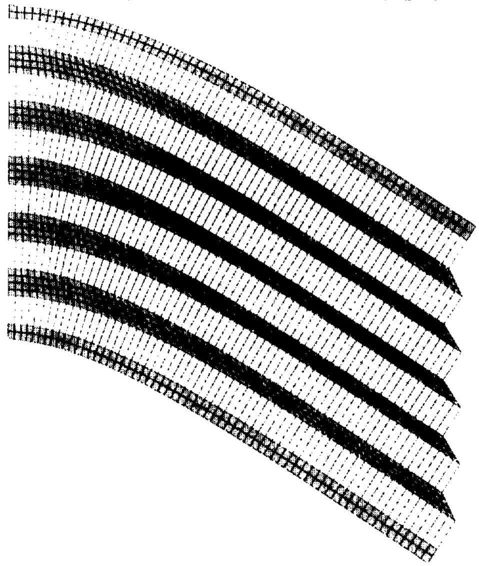

Figure 3 : Deformation of a multilayered structure (material B in gray).

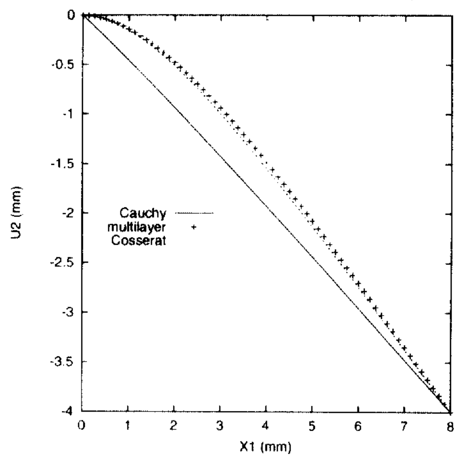

Figure 4 : Comparison between the response of the effective Cosserat continuum, the Cauchy continuum and the reference structure, along a horizontal line in the middle of the specimen within the hard material.