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Nonlinear microstrain theories

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Abstract

A hierarchy of higher order continua is presented that introduces additional degrees of freedom accounting for volume changes, rotation and straining of an underlying microstructure. An increase in the number of degrees of freedom represents a refinement of the material description. In addition to available nonlinear Cosserat and micromorphic theories, general formulations of elastoviscoplastic behaviour are proposed for microdilatation and microstretch continua. A microstrain theory is introduced that is based on six additional degrees of freedom describing the pure straining of the microstructural element. In each case, balance equations and boundary conditions are derived, decompositions of the finite strain measures into elastic and plastic parts are provided. The formulation of finite deformation elastoviscoplastic constitutive equations relies on the introduction of the free energy and dissipation potentials, thus complying with requirements of continuum thermodynamics. Some guidelines for the selection of a suitable higher order model for a given material close the discussion.

Keywords: Continuum mechanics; Higher order continua; Micromorphic; Cosserat; Elastoviscoplasticity; Finite deformation

1. Introduction

Incorporating some effects of characteristic lengths of the materials microstructure into constitutive modeling is possible via the mechanics of generalized continua. Generalized continua based on the assumption of local action, meaning that stress at a point depends only on quantities defined at that point, can be classified into two main groups (Trostel, 1988; Forest, 2005). Higher grade continua are characterized by higher order spatial derivatives of the displacement field as in the second gradient theory of Mindlin (Mindlin and Eshel, 1968; Germain, 1973a; Maugin, 1979), or by the gradient of internal variables as in strain gradient plasticity (Aifantis, 1987). Higher order continua are endowed with additional degrees of freedom that are a priori independent from the usual translational degrees of freedom, namely the three components of the displacement vector of the continuum point. The most recent and comprehensive account of the mechanics of higher order

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continua can be found in the reference (Eringen, 1999). The *classical Cauchy* continuum can be enhanced by introducing 3 rotational degrees of freedom, which leads to the *Cosserat mechanics*. In the *micromorphic theory* of Eringen (Eringen and Suhubi, 1964; Mindlin, 1964), the underlying microstructure at a material point can rotate and deform and has therefore 9 additional degrees of freedom. The microstructure evolution is represented by the deformation of a triad of vectors which generally are not continuum line elements. An intermediate theory, called *microstretch* continuum, combining Cosserat effects and microvolume changes, i.e. rotation and stretch of the volume delimited by the triad of microvectors, was proposed by Eringen (1990) in order to limit the number of additional degrees of freedom. The description of higher order continua is usually based on the first gradient theory, meaning that the first gradient of all degrees of freedom is supposed to play a role in the material response.

The choice of the proper higher order model for a given material depends on the relevant microscopic deformation mechanisms. The Cosserat model is well-suited for granular media made of constituents that can rotate independently from the macromotion (Mühlhaus and Vardoulakis, 1987). It is still a pertinent approach when the grains can deform plastically like in polycrystals (Lachner et al., 1994; Forest et al., 2000). This continuum model is also appropriate for the description of materials with a bending-stiff substructure, like reinforcements of composites [as fibres in Besdo and Dorau (1988), Shu and Fleck (1995) or as layers or laminates in Mühlhaus (1995), Forest and Sab (1998)]. However, in porous media like soils, polymers and metal foams, the rotation of cells is not the unique deformation mode so that the full micromorphic medium is required (Chambon et al., 2001; Forest and Lorentz, 2004; Forest et al., 2005). The micromorphic approach was also used recently in the case of the deformation of crystals (Grekova and Maugin, 2005). A good compromise in terms of numerical efficiency and precision of microstructure description is to use the microstretch approach, as done in soft bio-materials and fluids in Ariman (1971) and Rosenberg and Cimrman (2003).

The mechanics of higher order continua is especially relevant to address nonlinear phenomena like large deformations and nonlinear behavior (viscoplasticity and fracture). Geometrically nonlinear formulations of the Cosserat, micromorphic and microstretch theories are available since the early 1970s. More recently, general elastoviscoplastic constitutive frameworks have been proposed for Cosserat and micromorphic media (Chambon et al., 2001; Forest and Sievert, 2003,). These theories are extensions of the classical anisotropic finite strain elastoviscoplasticity model settled also during the 1970s and based on the multiplicative decomposition of the deformation gradient (Rice, 1971; Mandel, 1971, 1973; Maugin and Epstein, 1998).

The objective of the present work is to propose a hierarchical view of the nonlinear mechanics of higher order continua. The previous picture is completed by the introduction of higher order media with intermediated numbers of degrees of freedom. The three new points put forward in this work are:

- A *microdilatation* theory is introduced that accounts for microvolume changes only. It is the cheapest model in terms of number of degrees of freedom, with a total of 4. Closely related one-parameter theories have been handled in the literature, for instance in Steeb and Diebels (2003). The case of finite deformation elastoviscoplastic microdilatation media is explored in this work.
- The microstretch theory is extended to elastoviscoplasticity at finite deformation.
- A *microstrain* continuum theory is proposed that takes the change of shape of the parallelogram formed by the triad of microvectors into account, but not its rotation. The full micromorphic theory can then be seen as the combination of the Cosserat and microstrain theories.

The methodology for deriving balance equations and formulating constitutive equations is the same as that used in the reference (Forest and Sievert, 2003). The *method of virtual power* enables us to obtain the equations of balance of momentum and of moment of momentum, but also the proper form of the boundary conditions for a finite body (Germain, 1973a,b; Maugin, 1980). Constitutive equations are formulated via the choice of two potentials, namely the Helmholtz free energy and the dissipation potential. One advantage of the theories of generalized continua based on the assumption of local action is that they fall into the well-established framework of *continuum thermodynamics* (Germain et al., 1983).

In this work, zeroth, first, second and third order tensors are denoted by $a, \underline{\mathbf{a}}, \underset{\sim}{\mathbf{a}}, \underset{\approx}{\mathbf{a}}$, respectively. The simple, double and triple contractions are written as ., : and $\vdots$, respectively. In index form with respect to an orthonormal Cartesian basis, these notations correspond to

$$\underline{\mathbf{a}}.\underline{\mathbf{b}} = a_i b_i,\quad \underset{\sim}{\mathbf{a}} : \underset{\sim}{\mathbf{b}} = a_{ij}b_{ij},\quad \underset{\approx}{\mathbf{a}} \,\vdots\, \underset{\approx}{\mathbf{b}} = a_{ijk}b_{ijk} \tag{1}$$

where repeated indices are summed up. The tensor product is denoted by $\otimes$. For example, the component $ijkl$ of $\underset{\sim}{\mathbf{a}} \otimes \underset{\sim}{\mathbf{b}}$ is $a_{ij}b_{kl}$. A modified tensor product $\boxtimes$ is used at some places: the component $ijkl$ of $\underset{\sim}{\mathbf{a}} \boxtimes \underset{\sim}{\mathbf{b}}$ is $a_{ik}b_{jl}$. The nabla operator with respect to the reference configuration is denoted by $\mathbf{\nabla}$. The nabla operator with respect to the current configuration is denoted by $\mathbf{\nabla}^{\mathrm{c}}$. For example $\underset{\sim}{\boldsymbol{\sigma}}.\mathbf{\nabla}^{\mathrm{c}}$ is the divergence of the second order tensor $\underset{\sim}{\boldsymbol{\sigma}}$. The index form of $\underset{\sim}{\boldsymbol{\sigma}}.\mathbf{\nabla}^{\mathrm{c}}$ is $\sigma_{ij,j}$. Similarly, $\underline{\mathbf{u}} \otimes \mathbf{\nabla}$ means $u_{i,j}$. The sign $:=$ defines the quantity on the left-hand side. Complements on this notation system can be found in Forest and Sievert (2003) and Trostel (1993).

2. A hierarchy of higher order continua

An improved description of material behaviour by introducing gradually more and more degrees of freedom is proposed accounting for microstructure evolution. The available Cosserat and micromorphic theories are first recalled to settle the methodology and notations.

2.1. Cosserat media

The material points of a Cosserat medium possess translational degrees of freedom represented by the displacement vector $\underline{\mathbf{u}}$, and independent rotational degrees of freedom, represented by the orthogonal tensor $\underset{\sim}{\boldsymbol{R}}$, called Cosserat rotation or microrotation. The relative deformation tensor ${}^{\sharp}\underset{\sim}{\boldsymbol{F}}$ and the invariant curvature tensor ${}^{\sharp}\underset{\sim}{\boldsymbol{\Gamma}}$ were proposed in Kafadar and Eringen (1971). Their expressions are recalled in Table 1 as functions of the first gradient of displacement and Cosserat rotation. Generalized stress tensors are introduced as the dual force quantities in the expression of power density $p^{(i)}$ of internal forces written in Table 1. The balance equations for momentum and moment of momentum follow. Volume forces and couples are not included for the sake of brevity. Surface forces $\underline{\boldsymbol{t}}$ and couples $\underline{\boldsymbol{m}}$ can be prescribed at the boundary. They are linked to the force and couple stress tensors via the relations given in Table 1. Note that the Cosserat rotation $\underset{\sim}{\boldsymbol{R}}$ is not constrained to follow the mean rotation of the displacement field. Displacement and microrotation are related only at the balance and constitutive levels.

Hyperelastic Cosserat media were studied in Kafadar and Eringen (1971) and Maugin (1998). In the elastoviscoplastic case, the multiplicative decomposition of the relative deformation and the quasi-additive decomposition of the curvature tensor given in Table 1 are adopted, according to Sievert (1992) and Sievert et al. (1998). The Helmholtz free energy is a function of the elastic parts of deformation and curvature, and also

Table 1
Summary of balance and constitutive equations for nonlinear Cosserat continua (after Forest and Sievert, 2003)

Cosserat media

$DOF = \{\underline{\mathbf{u}},\ \underset{\sim}{\boldsymbol{R}}\}$, $STRAIN = \{{}^{\sharp}\underset{\sim}{\boldsymbol{F}} := \underset{\sim}{\boldsymbol{R}}^{T}.\underset{\sim}{\boldsymbol{F}},\ {}^{\sharp}\underset{\sim}{\boldsymbol{\Gamma}} := -\frac{1}{2}\underset{\approx}{\boldsymbol{\varepsilon}} : (\underset{\sim}{\boldsymbol{R}}^{T}.(\underset{\sim}{\boldsymbol{R}}\otimes\mathbf{\nabla}))\}$

Balance equations

Relative force and couple stress tensors: ${}^{\sharp}\underset{\sim}{\boldsymbol{\sigma}} := \underset{\sim}{\boldsymbol{R}}^{T}.\underset{\sim}{\boldsymbol{\sigma}}.\underset{\sim}{\boldsymbol{R}},\quad {}^{\sharp}\underset{\sim}{\boldsymbol{\mu}} := \underset{\sim}{\boldsymbol{R}}^{T}.\underset{\sim}{\boldsymbol{\mu}}.\underset{\sim}{\boldsymbol{R}}$

$p^{(i)} = {}^{\sharp}\underset{\sim}{\boldsymbol{\sigma}} : ({}^{\sharp}\dot{\underset{\sim}{\boldsymbol{F}}}.{}^{\sharp}\underset{\sim}{\boldsymbol{F}}) + {}^{\sharp}\underset{\sim}{\boldsymbol{\mu}} : ({}^{\sharp}\dot{\underset{\sim}{\boldsymbol{\Gamma}}}.{}^{\sharp}\underset{\sim}{\boldsymbol{F}}^{-1})$

$\underset{\sim}{\boldsymbol{\sigma}}.\mathbf{\nabla} = 0,\quad \underset{\sim}{\boldsymbol{\mu}}.\mathbf{\nabla} - \underset{\approx}{\boldsymbol{\varepsilon}} : \underset{\sim}{\boldsymbol{\sigma}} = 0,\quad \forall \underline{\boldsymbol{x}} \in \mathscr{D}$

$\underline{\boldsymbol{t}} = \underset{\sim}{\boldsymbol{\sigma}}.\underline{\boldsymbol{n}},\quad \underline{\boldsymbol{m}} = \underset{\sim}{\boldsymbol{\mu}}.\underline{\boldsymbol{n}},\quad \forall \underline{\boldsymbol{x}} \in \partial\mathscr{D}$

State laws

${}^{\sharp}\underset{\sim}{\boldsymbol{F}} = {}^{\sharp}\underset{\sim}{\boldsymbol{F}}^{e}.{}^{\sharp}\underset{\sim}{\boldsymbol{F}}^{p},\ {}^{\sharp}\underset{\sim}{\boldsymbol{\Gamma}} = {}^{\sharp}\underset{\sim}{\boldsymbol{\Gamma}}^{e}.{}^{\sharp}\underset{\sim}{\boldsymbol{F}}^{p} + {}^{\sharp}\underset{\sim}{\boldsymbol{\Gamma}}^{p}$

$\Psi({}^{\sharp}\underset{\sim}{\boldsymbol{F}}^{e}, {}^{\sharp}\underset{\sim}{\boldsymbol{\Gamma}}^{e}, q)$

${}^{\sharp}\underset{\sim}{\boldsymbol{\sigma}} = \rho\dfrac{\partial\Psi}{\partial{}^{\sharp}\underset{\sim}{\boldsymbol{F}}^{e}}.{}^{\sharp}\underset{\sim}{\boldsymbol{F}}^{eT},\quad {}^{\sharp}\underset{\sim}{\boldsymbol{\mu}} = \rho\dfrac{\partial\Psi}{\partial{}^{\sharp}\underset{\sim}{\boldsymbol{\Gamma}}^{e}}.{}^{\sharp}\underset{\sim}{\boldsymbol{F}}^{eT},\quad R = -\rho\dfrac{\partial\Psi}{\partial q}$

Dissipation potential

$\underset{\sim}{\boldsymbol{\Sigma}} = {}^{\sharp}\underset{\sim}{\boldsymbol{F}}^{eT}.{}^{\sharp}\underset{\sim}{\boldsymbol{\sigma}}.{}^{\sharp}\underset{\sim}{\boldsymbol{F}}^{e-T} + {}^{\sharp}\underset{\sim}{\boldsymbol{\Gamma}}^{eT}.{}^{\sharp}\underset{\sim}{\boldsymbol{\mu}}.{}^{\sharp}\underset{\sim}{\boldsymbol{F}}^{e-T},\quad \underset{\sim}{\boldsymbol{M}} = {}^{\sharp}\underset{\sim}{\boldsymbol{\mu}}.{}^{\sharp}\underset{\sim}{\boldsymbol{F}}^{e-T}$

$\Omega(\underset{\sim}{\boldsymbol{\Sigma}}, \underset{\sim}{\boldsymbol{M}}, R)$

${}^{\sharp}\dot{\underset{\sim}{\boldsymbol{F}}}^{p}.{}^{\sharp}\underset{\sim}{\boldsymbol{F}}^{p-1} = \dfrac{\partial\Omega}{\partial\underset{\sim}{\boldsymbol{\Sigma}}},\quad {}^{\sharp}\dot{\underset{\sim}{\boldsymbol{\Gamma}}}^{p}.{}^{\sharp}\underset{\sim}{\boldsymbol{F}}^{p-1} = \dfrac{\partial\Omega}{\partial\underset{\sim}{\boldsymbol{M}}},\quad \dot{q} = \dfrac{\partial\Omega}{\partial R}$

of some hardening variables q that can be of any tensorial nature (isotropic or kinematic hardening variables for instance). The state laws give the relations between the generalized stress tensors and the arguments of the free energy function. The thermodynamic force associated with q is denoted by R.

The residual dissipation rate is then

$$D = \boldsymbol{\Sigma} : ({}^\sharp\dot{\boldsymbol{F}}^p . {}^\sharp\boldsymbol{F}^{p-1}) + \boldsymbol{M} : ({}^\sharp\dot{\boldsymbol{\Gamma}}^p . {}^\sharp\boldsymbol{F}^{p-1}) + R\dot{q} \geqslant 0 \tag{2}$$

where ${}^\sharp\boldsymbol{F}^p$ and ${}^\sharp\boldsymbol{\Gamma}^p$ are the plastic deformation and plastic curvature tensors. Generalized stress tensors $\boldsymbol{\Sigma}$ and $\boldsymbol{M}$ are introduced following the work of Mandel (1973). They are the driving forces for plastic flow. Their expressions are given in Table 1. In the classical framework, they are called *Mandel stress tensors* by Haupt (2000). This denomination is kept in this work.

To identically fulfill the positivity of the residual dissipation, a dissipation potential $\Omega(\boldsymbol{\Sigma}, \boldsymbol{M}, R)$, convex with respect to its arguments, can be introduced from which the flow and hardening rules are derived (Germain et al., 1983), as shown in the last line of Table 1.

2.2. Micromorphic media

The micromorphic continuum is endowed with 3 translational degrees of freedom $\underline{\mathbf{u}}$, and by 9 microdeformation degrees of freedom represented by the generally non-symmetric second order tensor $\boldsymbol{\chi}$. The tensor field $\boldsymbol{\chi}(\underline{\boldsymbol{X}})$ is generally not compatible. The invariant generalized strain measures selected by Eringen are the classical right Cauchy–Green tensor $\boldsymbol{C}$, the relative deformation tensor $\boldsymbol{\Upsilon} := \boldsymbol{\chi}^{-1}.\boldsymbol{F}$ and the gradient of microdeformation $\boldsymbol{K} := \boldsymbol{\chi}^{-1}.(\boldsymbol{\chi} \otimes \nabla)$ (Eringen, 1999). The three generalized stress tensors are, respectively, associated to the deformation rate, to the relative deformation rate and to the gradient of microdeformation rate according to

$$p^{(i)} = \boldsymbol{\sigma} : (\dot{\boldsymbol{F}}.\boldsymbol{F}^{-1}) + \boldsymbol{s} : (\dot{\boldsymbol{F}}.\boldsymbol{F}^{-1} - \dot{\boldsymbol{\chi}}.\boldsymbol{\chi}^{-1}) + \boldsymbol{S} \vdots ((\dot{\boldsymbol{\chi}}.\boldsymbol{\chi}^{-1}) \otimes \nabla^c). \tag{3}$$

The balance equations for momentum and moment of momentum in a domain $\mathscr{D}$ of the body can be deduced by application of the principle of virtual power, as done in Germain (1973b). They are given in Table 2. Volume forces, couples, double or triple forces are not included for the sake of brevity. The equations involve

Table 2
Summary of balance and constitutive equations for nonlinear micromorphic continua (after Forest and Sievert, 2003)

Micromorphic media

$$DOF = \{\underline{\mathbf{u}},\ \boldsymbol{\chi}\}, \quad STRAIN = \{\boldsymbol{C} := \boldsymbol{F}^T.\boldsymbol{F}, \boldsymbol{\Upsilon} := \boldsymbol{\chi}^{-1}.\boldsymbol{F}, \boldsymbol{K} := \boldsymbol{\chi}^{-1}.(\boldsymbol{\chi} \otimes \nabla)\}$$

Balance equations

$$p^{(i)} = \boldsymbol{\sigma} : (\dot{\boldsymbol{F}}.\boldsymbol{F}^{-1}) + \boldsymbol{s} : (\dot{\boldsymbol{F}}.\boldsymbol{F}^{-1} - \dot{\boldsymbol{\chi}}.\boldsymbol{\chi}^{-1}) + \boldsymbol{S} \vdots ((\dot{\boldsymbol{\chi}}.\boldsymbol{\chi}^{-1}) \otimes \nabla^c)$$

$$= \boldsymbol{\sigma} : (\dot{\boldsymbol{F}}.\boldsymbol{F}^{-1}) + \boldsymbol{s} : (\boldsymbol{\chi}.\dot{\boldsymbol{\Upsilon}}.\boldsymbol{F}^{-1}) + \boldsymbol{S} \vdots (\boldsymbol{\chi}.\dot{\boldsymbol{K}} : (\boldsymbol{\chi}^{-1} \boxtimes \boldsymbol{F}^{-1}))$$

$$(\boldsymbol{\sigma} + \boldsymbol{s}).\nabla = 0, \quad \boldsymbol{S}.\nabla + \boldsymbol{s} = 0 \quad \forall \underline{\boldsymbol{x}} \in \mathscr{D}$$

$$\underline{\boldsymbol{t}} = (\boldsymbol{\sigma} + \boldsymbol{s}).\underline{\boldsymbol{n}}, \quad \boldsymbol{m} = \boldsymbol{S}.\underline{\boldsymbol{n}} \quad \forall \underline{\boldsymbol{x}} \in \partial\mathscr{D}$$

State laws

$$\boldsymbol{F} = \boldsymbol{F}^e.\boldsymbol{F}^p, \quad \boldsymbol{\chi} = {}^\chi\boldsymbol{R}^e.{}^\chi\boldsymbol{U}^e.\boldsymbol{\chi}^p, \quad \boldsymbol{K} = \boldsymbol{K}^e + \boldsymbol{K}^p$$

$$\Psi(\boldsymbol{C}^e := \boldsymbol{F}^{eT}.\boldsymbol{F}^e, \boldsymbol{\Upsilon}^e := {}^\chi\boldsymbol{U}^{e-1}.{}^\chi\boldsymbol{R}^{eT}.\boldsymbol{F}^e, \boldsymbol{K}^e, q)$$

$$\boldsymbol{\sigma} = 2\boldsymbol{F}^e.\rho\frac{\partial\Psi}{\partial\boldsymbol{C}^e}.\boldsymbol{F}^{eT}, \quad \boldsymbol{s} = {}^\chi\boldsymbol{R}^e.{}^\chi\boldsymbol{U}^{e-1}.\rho\frac{\partial\Psi}{\partial\boldsymbol{\Upsilon}^e}.\boldsymbol{F}^{eT}$$

$$\boldsymbol{S} = \boldsymbol{\chi}^{-T}.\rho\frac{\partial\Psi}{\partial{}^\chi\boldsymbol{K}^e} : (\boldsymbol{\chi}^T \boxtimes \boldsymbol{F}^T), \quad R = -\rho\frac{\partial\Psi}{\partial q}$$

Dissipation potential

$$\boldsymbol{\Sigma} = \boldsymbol{F}^{eT}.(\boldsymbol{\sigma} + \boldsymbol{s}).\boldsymbol{F}^{e-T}, \quad \mathscr{S} = -{}^\chi\boldsymbol{U}^e.{}^\chi\boldsymbol{R}^{eT}.\boldsymbol{s}.{}^\chi\boldsymbol{R}^e.{}^\chi\boldsymbol{U}^{e-1}, \quad \boldsymbol{S}_0 = \boldsymbol{\chi}^T.\boldsymbol{S} : (\boldsymbol{\chi}^{-T} \boxtimes \boldsymbol{F}^{-T})$$

$$\Omega(\boldsymbol{\Sigma}, \mathscr{S}, \boldsymbol{S}_0)$$

$$\dot{\boldsymbol{F}}^p.\boldsymbol{F}^{p-1} = \frac{\partial\Omega}{\partial\boldsymbol{\Sigma}}, \quad \dot{\boldsymbol{\chi}}^p.\boldsymbol{\chi}^{p-1} = \frac{\partial\Omega}{\partial\mathscr{S}}, \quad \dot{K}^p = \frac{\partial\Omega}{\partial\boldsymbol{S}_0}, \quad \dot{q} = \frac{\partial\Omega}{\partial R}$$

the divergence of the force stress tensor $\boldsymbol{\sigma}$ and of the hyperstress tensor $\underset{\approx}{\boldsymbol{S}}$. They are coupled by the presence of the relative stress tensor $\underset{\sim}{\boldsymbol{s}}$, which represents a penalty to strong deviations of the micromotion from the macromotion. Surface simple forces and double forces can be prescribed at the boundary $\partial\mathscr{D}$.

Constitutive frameworks for finite strain elastoviscoplastic micromorphic materials were developed in Sansour (1998) and Forest and Sievert (2003). According to the latter, multiplicative decompositions of the macro and microdeformations into elastic and plastic parts are adopted:

$$\underset{\sim}{\boldsymbol{F}} = \underset{\sim}{\boldsymbol{F}}^e . \underset{\sim}{\boldsymbol{F}}^p, \quad \underset{\sim}{\boldsymbol{\chi}} = \underset{\sim}{\boldsymbol{\chi}}^e . \underset{\sim}{\boldsymbol{\chi}}^p = {}^{\chi}\underset{\sim}{\boldsymbol{R}}^e . {}^{\chi}\underset{\sim}{\boldsymbol{U}}^e . \underset{\sim}{\boldsymbol{\chi}}^p \tag{4}$$

where the polar decomposition of $\underset{\sim}{\boldsymbol{\chi}}^e$ into a rotation ${}^{\chi}\underset{\sim}{\boldsymbol{R}}^e$ and a symmetric stretch ${}^{\chi}\underset{\sim}{\boldsymbol{U}}^e$ was introduced. The primary variable $\underset{\sim}{\boldsymbol{\chi}}$ represents in average the motion of particles relative to the center of mass of a material element, that moves with the displacement of a continuum point (Eringen, 1999; Germain, 1973b; Trostel, 1988). This microdeformation is split into elastic, plastic and rotation parts, because it includes strain parts, which contribute also without a gradient to strain energy and dissipation, and this is in contrast to the displacement. It shows that the triad of microvectors characterizing the underlying microstructure can rotate like in a Cosserat theory, but also change in shape. Two alternative decompositions of the microdeformation gradient $\underset{\approx}{\boldsymbol{K}}$ into elastic and plastic contributions were proposed in Forest and Sievert (2003):

$$\underset{\approx}{\boldsymbol{K}} = \underset{\approx}{\boldsymbol{K}}^e + \underset{\approx}{\boldsymbol{K}}^p, \quad \text{or} \quad \underset{\approx}{\boldsymbol{K}} = \underset{\sim}{\boldsymbol{\chi}}^{p-1} . \underset{\simeq}{\boldsymbol{K}}^e : (\underset{\sim}{\boldsymbol{\chi}}^p \boxtimes \underset{\sim}{\boldsymbol{F}}^p) + \underset{\approx}{\boldsymbol{K}}^p. \tag{5}$$

The second decomposition in (5) corresponds to a hyperelastic constitutive equation for the conjugate stress $\underset{\approx}{\boldsymbol{S}}$ in the current configuration, that has also at large plastic deformations the same form as for pure hyperelastic behaviour. The hyperelastic relationships corresponding to the purely additive decomposition (5)a of the microdeformation gradient are given in Table 2. Again, Mandel stress tensors appear in the expression of the residual dissipation rate:

$$D = \underset{\sim}{\boldsymbol{\Sigma}} : (\dot{\underset{\sim}{\boldsymbol{F}}}^p . \underset{\sim}{\boldsymbol{F}}^{p-1}) + \underset{\sim}{\mathscr{S}} : (\dot{\underset{\sim}{\boldsymbol{\chi}}}^p . \underset{\sim}{\boldsymbol{\chi}}^{p-1}) + \underset{\approx}{\boldsymbol{S}}_0 \vdots \dot{\underset{\approx}{\boldsymbol{K}}}^p + R\dot{q}. \tag{6}$$

The expressions of the Mandel stress tensors as functions of the generalized stress tensors $\underset{\sim}{\boldsymbol{\sigma}}$, $\underset{\sim}{\boldsymbol{s}}$ and $\underset{\approx}{\boldsymbol{S}}$ are given in Table 2. The Mandel stress tensors are the arguments of a dissipation potential that can be introduced to derive the evolution rules for plastic flow and hardening.

2.3. Proposed hierarchy

A complete hierarchy of higher order continua including a total number of degrees of freedom ranging from 3 to 12 is presented in Table 3. The main available theories, namely the Cauchy, Cosserat, microstretch

Table 3
A hierarchy of higher order continua from Cauchy to micromorphic media: number and type of degrees of freedom

Name	Number of DOF	DOF (finite case)	DOF (infinitesimal case)	References
Cauchy	3	$\underline{\mathbf{u}}$	$\underline{\mathbf{u}}$	Truesdell and Toupin (1960)
Microdilatation	4	$\underline{\mathbf{u}}, \chi$	$\underline{\mathbf{u}}, \chi$	–
–	5	–	–	–
Cosserat	6	$\underline{\mathbf{u}}, \underset{\sim}{\boldsymbol{R}}$	$\underline{\mathbf{u}}, \underline{\boldsymbol{\Phi}}$	Kafadar and Eringen (1971)
Microstretch	7	$\underline{\mathbf{u}}, \chi, \underset{\sim}{\boldsymbol{R}}$	$\underline{\mathbf{u}}, \chi, \underline{\boldsymbol{\Phi}}$	Eringen (1990)
Incompressible microstrain	8	$\underline{\mathbf{u}}, {}^{\chi}\underset{\sim}{\boldsymbol{C}}$	$\underline{\mathbf{u}}, {}^{\chi}\underset{\sim}{\boldsymbol{\varepsilon}}$	–
		$\det {}^{\chi}\underset{\sim}{\boldsymbol{C}} = 1$	$\operatorname{trace} {}^{\chi}\underset{\sim}{\boldsymbol{\varepsilon}} = 0$	
Microstrain	9	$\underline{\mathbf{u}}, {}^{\chi}\underset{\sim}{\boldsymbol{C}}$	$\underline{\mathbf{u}}, {}^{\chi}\underset{\sim}{\boldsymbol{\varepsilon}}$	This work
–	10	–	–	–
Incompressible micromorphic	11	$\underline{\mathbf{u}}, \underset{\sim}{\boldsymbol{\chi}}$	$\underline{\mathbf{u}}, \underset{\sim}{\boldsymbol{\chi}}$	–
		$\det \underset{\sim}{\boldsymbol{\chi}} = 1$	$\operatorname{trace} \underset{\sim}{\boldsymbol{\chi}} = 0$	–
Micromorphic	12	$\underline{\mathbf{u}}, \underset{\sim}{\boldsymbol{\chi}}$	$\underline{\mathbf{u}}, \underset{\sim}{\boldsymbol{\chi}}^s + \underset{\sim}{\boldsymbol{\chi}}^a$	Eringen and Suhubi (1964) Mindlin (1964)

In some cases, specific notations are introduced in the finite and infinitesimal situations. DOF stands for degree of freedom.

and micromorphic theories, respectively, introduce 3, 6, 7 and 12 degrees of freedom. We introduce in Sections 3 and 5 the microdilatation and microstrain continuum theories which involve 4 and 9 degrees of freedom, respectively. Intermediate numbers of degrees of freedom can be obtained if additional internal constraints are considered. The incompressibility of macro or microdeformation is one such possible constraint. Such incompressible higher order media are presented in Table 3 as examples of theories with 8 and 11 degrees of freedom. Other internal constraints could be enforced.

In the presentation of each theory, the degrees of freedom are gathered in the set *DOF*. Only the first gradient of these degrees of freedom is assumed to play a role. In each case, a set of generalized strain measures *STRAIN* is selected that are invariant with respect to Euclidean transformations. The associated stress tensors are introduced via the power density $p^{(i)}$ of internal forces which is supposed to be a linear form in the introduced strain rates, following the method of virtual power. Hyperelastic and elastoviscoplastic constitutive frameworks are then necessary. In the latter case, decompositions of the generalized strain measures into elastic and plastic parts are proposed. The free energy density is then a function of the elastic contributions and, possibly, of some internal variables. Hyperelastic relationships are derived from the exploitation of the second principle à la Coleman–Noll (Liu, 2002). Viscoplastic flow rules and hardening rules are then necessary for the closure of the problem. An efficient way of expressing them is to assume the existence of a dissipation potential depending on generalized stress tensors that are the driving forces for plastic flow and hardening. If the chosen potential is convex with respect to its arguments, the positivity of the residual dissipation is then ensured (Germain et al., 1983). The existence of such a potential is not a prerequisite of the theory but represents a useful tool for establishing the structure of the constitutive equations.

Sections 3–5, respectively, deal with nonlinear microdilatation, microstretch and microstrain continua. Guidelines for the choice of a suitable higher order theory in the case of a given material are presented in the conclusions.

3. Continua with microdilatation

3.1. Kinematics and balance equations

There are 4 degrees of freedom in this theory, namely the displacement vector $\underline{\mathbf{u}}$ and a scalar microdilatation χ, i.e. the microvolume stretch:

$$DOF = \{\underline{\boldsymbol{u}}, \chi\}. \tag{7}$$

The medium is thus endowed with a microdeformation field having the simple form:

$$\underset{\sim}{\boldsymbol{\chi}} = \chi \underset{\sim}{\mathbf{1}}. \tag{8}$$

The related microvolume change rate $\dot{\chi}/\chi$ is to be compared to the material one $(\det \underset{\sim}{\boldsymbol{F}})^{\bullet}/\det \underset{\sim}{\boldsymbol{F}}$. This leads to the following expression of the power of internal forces:

$$p^{(i)} = \underset{\sim}{\boldsymbol{\sigma}} : (\underset{\sim}{\dot{\boldsymbol{F}}}.\underset{\sim}{\boldsymbol{F}}^{-1}) + s\left(\frac{(\det \underset{\sim}{\boldsymbol{F}})^{\bullet}}{\det \underset{\sim}{\boldsymbol{F}}} - \frac{\dot{\chi}}{\chi}\right) + \underline{\boldsymbol{S}}.\left(\boldsymbol{\nabla}^{c}\left(\frac{\dot{\chi}}{\chi}\right)\right) \tag{9}$$

$$= (\underset{\sim}{\boldsymbol{\sigma}} + s\,\underset{\sim}{\mathbf{1}}) : (\underline{\dot{\boldsymbol{u}}} \otimes \boldsymbol{\nabla}^{c}) - s\frac{\dot{\chi}}{\chi} + \underline{\boldsymbol{S}}.\left(\boldsymbol{\nabla}^{c}\left(\frac{\dot{\chi}}{\chi}\right)\right) \tag{10}$$

where generalized stress tensors $\underset{\sim}{\boldsymbol{\sigma}}$, s and $\underline{\boldsymbol{S}}$ are introduced as dual quantities to the gradient rates. The total power of internal forces on any subdomain $\mathscr{D}$ of the body reads

$$\begin{aligned}\mathscr{P}^{(i)} &= \int_{\mathscr{D}} p^{(i)}\,\mathrm{d}V = \int_{\mathscr{D}} ((\underline{\dot{\boldsymbol{u}}}.(\underset{\sim}{\boldsymbol{\sigma}} + s\underset{\sim}{\mathbf{1}})).\boldsymbol{\nabla}^{c} - \underline{\dot{\boldsymbol{u}}}.((\underset{\sim}{\boldsymbol{\sigma}} + s\underset{\sim}{\mathbf{1}}).\boldsymbol{\nabla}^{c}))\,\mathrm{d}V + \int_{\mathscr{D}} \left(\left(\frac{\dot{\chi}}{\chi}\underline{\boldsymbol{S}}\right).\boldsymbol{\nabla}^{c} - \frac{\dot{\chi}}{\chi}(\underline{\boldsymbol{S}}.\boldsymbol{\nabla}^{c} + s)\right)\mathrm{d}V \\ &= -\int_{\mathscr{D}} ((\underset{\sim}{\boldsymbol{\sigma}} + s\underset{\sim}{\mathbf{1}}).\boldsymbol{\nabla}^{c}).\underline{\dot{\boldsymbol{u}}}\,\mathrm{d}V - \int_{\mathscr{D}} (\underline{\boldsymbol{S}}.\boldsymbol{\nabla}^{c} + s)\frac{\dot{\chi}}{\chi}\,\mathrm{d}V + \int_{\partial\mathscr{D}} \left(\underline{\dot{\boldsymbol{u}}}.(\underset{\sim}{\boldsymbol{\sigma}} + s\underset{\sim}{\mathbf{1}}).\underline{\boldsymbol{n}} + \frac{\dot{\chi}}{\chi}\underline{\boldsymbol{S}}.\underline{\boldsymbol{n}}\right)\mathrm{d}S.\end{aligned} \tag{11}$$

The previous expression indicates that the power of contact forces must have the following form:

$$p^{(c)} = \underline{\boldsymbol{t}}.\underline{\dot{\boldsymbol{u}}} + m\dot{\chi} \tag{12}$$

where $\underline{t}$ and m, respectively, are prescribed surface traction and surface double force densities. For the sake of conciseness, volume densities of simple, double or triples forces are not introduced in this work. This can be done in a straightforward manner following Germain, 1973b. The application of the method of virtual power then leads to the following field equations and boundary conditions:

$$(\underset{\sim}{\boldsymbol{\sigma}} + s\ \underset{\sim}{\mathbf{1}}).\nabla^c = 0 \quad \forall \underline{\boldsymbol{x}} \in \mathscr{D} \tag{13}$$

$$\underline{\boldsymbol{S}}.\nabla^c + s = 0 \quad \forall \underline{\boldsymbol{x}} \in \mathscr{D} \tag{14}$$

$$\underline{\boldsymbol{t}} = (\underset{\sim}{\boldsymbol{\sigma}} + s\,\underset{\sim}{\mathbf{1}}).\underline{\boldsymbol{n}}, \quad m = \underline{\boldsymbol{S}}.\underline{\boldsymbol{n}} \quad \forall \underline{\boldsymbol{x}} \in \partial\mathscr{D} \tag{15}$$

A set of strain measures which are invariant with respect to Euclidean transformations are selected:

$$STRAIN = \left\{ \underset{\sim}{\boldsymbol{C}} := \underset{\sim}{\boldsymbol{F}}^T.\underset{\sim}{\boldsymbol{F}}, \quad e := \frac{\det \underset{\sim}{\boldsymbol{F}}}{\chi}, \quad \underline{\boldsymbol{K}} := \frac{\nabla\chi}{\chi} \right\} \tag{16}$$

The right Cauchy–Green tensor is $\underset{\sim}{\boldsymbol{C}}$, the relative volume change is e and $\underline{\boldsymbol{K}}$ is the relative gradient of microdilatation. The power of internal forces can be written in terms of the rates of these strain measures:

$$p^{(i)} = \underset{\sim}{\boldsymbol{\sigma}} : \left(\frac{1}{2} \underset{\sim}{\boldsymbol{F}}^{-T}.\dot{\underset{\sim}{\boldsymbol{C}}}.\underset{\sim}{\boldsymbol{F}}^{-1} \right) + s\, \frac{\chi}{\det \underset{\sim}{\boldsymbol{F}}}\, \dot{e} + \underline{\boldsymbol{S}}.(\dot{\underline{\boldsymbol{K}}}.\underset{\sim}{\boldsymbol{F}}^{-1}). \tag{17}$$

3.2. Hyperelastic materials with microdilatation

Microdilatation materials are said to be hyperelastic if they admit a Helmholtz free energy density function Ψ that depends solely on the strain measures (16):

$$\Psi(\underset{\sim}{\boldsymbol{C}}, e, \underline{\boldsymbol{K}}). \tag{18}$$

For hyperelastic materials, the dissipation density D must identically vanish:

$$D = p^{(i)} - \rho\dot{\Psi} = \left(\frac{1}{2} \underset{\sim}{\boldsymbol{F}}^{-1}.\underset{\sim}{\boldsymbol{\sigma}}.\underset{\sim}{\boldsymbol{F}}^{-T} - \rho \frac{\partial \Psi}{\partial \underset{\sim}{\boldsymbol{C}}} \right) : \dot{\underset{\sim}{\boldsymbol{C}}} + \left(s\, \frac{\chi}{\det \underset{\sim}{\boldsymbol{F}}} - \rho \frac{\partial \Psi}{\partial e} \right) \dot{e} + \left(\underset{\sim}{\boldsymbol{F}}^{-1}.\underline{\boldsymbol{S}} - \rho \frac{\partial \Psi}{\partial \underline{\boldsymbol{K}}} \right).\dot{\underline{\boldsymbol{K}}} = 0 \tag{19}$$

This leads to the following hyperelastic state laws:

$$\underset{\sim}{\boldsymbol{\sigma}} = 2\,\underset{\sim}{\boldsymbol{F}}.\rho \frac{\partial \Psi}{\partial \underset{\sim}{\boldsymbol{C}}}.\underset{\sim}{\boldsymbol{F}}^T \tag{20}$$

$$s = e\rho \frac{\partial \Psi}{\partial e} \tag{21}$$

$$\underline{\boldsymbol{S}} = \underset{\sim}{\boldsymbol{F}}.\rho \frac{\partial \Psi}{\partial \underline{\boldsymbol{K}}} \tag{22}$$

3.3. Elastoviscoplastic materials with microdilatation

According to classical multiplicative elastoplasticity (Mandel, 1973), the deformation gradient can be split into elastic and plastic parts:

$$\underset{\sim}{\boldsymbol{F}} = \underset{\sim}{\boldsymbol{F}}^e.\underset{\sim}{\boldsymbol{F}}^p. \tag{23}$$

The uniqueness of this decomposition is bound to the choice of an isoclinic intermediate configuration. A triad of rigid directors must be selected in such a medium. The isoclinic configuration is such that the orientation of the directors is the same in the reference configuration and in the intermediate one. A similar multiplicative decomposition is plausible for the microdeformation (Forest and Sievert, 2003). When microdilatations only are considered, it amounts to

$$\chi = \chi^e\,\chi^p. \tag{24}$$

The relative microvolume change is then

$$e = \frac{\det \underset{\sim}{\boldsymbol{F}}}{\chi} = \frac{\det \underset{\sim}{\boldsymbol{F}}^e}{\chi^e}\,\frac{\det \underset{\sim}{\boldsymbol{F}}^p}{\chi^p} = \frac{\det \underset{\sim}{\boldsymbol{F}}^p}{\chi^p}\,\Upsilon^e, \quad \text{with } \Upsilon^e := \frac{\det \underset{\sim}{\boldsymbol{F}}^e}{\chi^e}. \tag{25}$$

An additive decomposition of the gradient of microdilatation is adopted:

$$\underline{\boldsymbol{K}} = \underline{\boldsymbol{K}}^e + \underline{\boldsymbol{K}}^p \tag{26}$$

The power of internal forces can be expressed in terms of the elastic and plastic contributions:

$$p^{(i)} = \underset{\sim}{\boldsymbol{\sigma}} : (\dot{\underset{\sim}{\boldsymbol{F}}}^e.\underset{\sim}{\boldsymbol{F}}^{e-1}) + \left(\underset{\sim}{\boldsymbol{F}}^{eT}.\underset{\sim}{\boldsymbol{\sigma}}.\underset{\sim}{\boldsymbol{F}}^{e-T} + s\,\underset{\sim}{\boldsymbol{1}}\right) : (\dot{\underset{\sim}{\boldsymbol{F}}}^p.\underset{\sim}{\boldsymbol{F}}^{p-1}) + s\,\frac{\dot{\Upsilon}^e}{\Upsilon^e} - s\,\frac{\dot{\chi}^p}{\chi^p} + (\underset{\sim}{\boldsymbol{F}}^{-1}.\underline{\boldsymbol{S}}).(\dot{\underline{\boldsymbol{K}}}^e + \dot{\underline{\boldsymbol{K}}}^p) \tag{27}$$

The free energy density is then a function of all elastic contributions and, possibly, of additional internal variable q (Germain et al., 1983):

$$\Psi(\underset{\sim}{\boldsymbol{C}}^e := \underset{\sim}{\boldsymbol{F}}^{eT}.\underset{\sim}{\boldsymbol{F}}^e, \Upsilon^e, \underline{\boldsymbol{K}}^e, q). \tag{28}$$

The Clausius–Duhem inequality takes then the form

$$\begin{aligned} D = {} & \left(\frac{1}{2}\underset{\sim}{\boldsymbol{F}}^{e-1}.\underset{\sim}{\boldsymbol{\sigma}}.\underset{\sim}{\boldsymbol{F}}^{e-T} - \rho\frac{\partial\Psi}{\partial\underset{\sim}{\boldsymbol{C}}^e}\right) : \dot{\underset{\sim}{\boldsymbol{C}}}^e \\ & + \left(\frac{s}{\Upsilon^e} - \rho\frac{\partial\Psi}{\partial\Upsilon^e}\right)\dot{\Upsilon}^e + \left(\underset{\sim}{\boldsymbol{F}}^{-1}.\underline{\boldsymbol{S}} - \rho\frac{\partial\Psi}{\partial\underline{\boldsymbol{K}}^e}\right).\dot{\underline{\boldsymbol{K}}}^e + (\underset{\sim}{\boldsymbol{F}}^{eT}.\underset{\sim}{\boldsymbol{\sigma}}.\underset{\sim}{\boldsymbol{F}}^{-eT} + s\,\underset{\sim}{\boldsymbol{1}}) : (\dot{\underset{\sim}{\boldsymbol{F}}}^p.\underset{\sim}{\boldsymbol{F}}^{p-1}) \\ & - s\,\frac{\dot{\chi}^p}{\chi^p} + (\underset{\sim}{\boldsymbol{F}}^{-1}.\underline{\boldsymbol{S}}).\dot{\underline{\boldsymbol{K}}}^p - \rho\frac{\partial\Psi}{\partial q}\dot{q} \geqslant 0. \end{aligned} \tag{29}$$

The rates of elastic deformation do not lead to dissipation. This implies the following state laws:

$$\underset{\sim}{\boldsymbol{\sigma}} = 2\underset{\sim}{\boldsymbol{F}}^e.\rho\frac{\partial\Psi}{\partial\underset{\sim}{\boldsymbol{C}}^e}.\underset{\sim}{\boldsymbol{F}}^{eT} \tag{30}$$

$$s = \Upsilon^e\,\rho\frac{\partial\Psi}{\partial\Upsilon^e} \tag{31}$$

$$\underline{\boldsymbol{S}} = \underset{\sim}{\boldsymbol{F}}.\rho\frac{\partial\Psi}{\partial\underline{\boldsymbol{K}}^e} \tag{32}$$

$$R = -\rho\frac{\partial\Psi}{\partial q} \tag{33}$$

Taking the state laws into account, the intrinsic dissipation reduces to

$$D = \underset{\sim}{\boldsymbol{\Sigma}} : (\dot{\underset{\sim}{\boldsymbol{F}}}^p.\underset{\sim}{\boldsymbol{F}}^{p-1}) + \mathscr{S}\,\frac{\dot{\chi}^p}{\chi^p} + \underline{\boldsymbol{S}}_0.\dot{\underline{\boldsymbol{K}}}^p + R\dot{q} \geqslant 0 \tag{34}$$

where the following stress measures have been introduced:

$$\underset{\sim}{\boldsymbol{\Sigma}} = \underset{\sim}{\boldsymbol{F}}^{eT}.(\underset{\sim}{\boldsymbol{\sigma}} + s\,\underset{\sim}{\boldsymbol{1}}).\underset{\sim}{\boldsymbol{F}}^{-eT}, \quad \mathscr{S} = -s, \quad \underline{\boldsymbol{S}}_0 = \underset{\sim}{\boldsymbol{F}}^{-1}.\underline{\boldsymbol{S}}. \tag{35}$$

A systematic way of ensuring the positivity of intrinsic dissipation (34) for any process is to introduce a convex dissipation potential

$$\Omega(\underset{\sim}{\mathbf{\Sigma}}, \mathscr{S}, \underline{\boldsymbol{S}}_0, R) \tag{36}$$

function of all generalized stresses and thermodynamic forces (Halphen and Nguyen, 1975; Germain et al., 1983). The flow rules and evolution equations for hardening variables are derived from the dissipation potential:

$$\dot{\underset{\sim}{\boldsymbol{F}}}^p . \underset{\sim}{\boldsymbol{F}}^{p-1} = \frac{\partial \Omega}{\partial \underset{\sim}{\mathbf{\Sigma}}} \tag{37}$$

$$\dot{\chi}^p \chi^{p-1} = \frac{\partial \Omega}{\partial \mathscr{S}} \tag{38}$$

$$\dot{\underline{\boldsymbol{K}}}^p = \frac{\partial \Omega}{\partial \underline{\boldsymbol{S}}_0} \tag{39}$$

$$\dot{q} = \frac{\partial \Omega}{\partial R} \tag{40}$$

The existence of a dissipation potential is not a requirement of the theory. It is only a simple and systematic way of formulating constitutive equations. More general flow rules and evolution equations identically fulfilling the dissipation inequality are allowed, that may not possess a dissipation potential.

Alternative elastic–plastic decompositions are possible for the microdilatation gradient $\underline{\boldsymbol{K}}$. In particular, similarly to classical finite strain plasticity, one may require that the hyperelastic relation (32) should be replaced by

$$\underline{\boldsymbol{S}} = \underset{\sim}{\boldsymbol{F}}^e . \rho \frac{\partial \Psi}{\partial \underline{\boldsymbol{K}}^e} \tag{41}$$

which has the same form as (22). This amounts to choosing an intermediate configuration for which both force stresses $\underset{\sim}{\boldsymbol{\sigma}} + s \underset{\sim}{\mathbf{1}}$ and hyperstresses $\underline{\boldsymbol{S}}$ are simultaneously released. Such an hyperelastic relation is compatible with the following quasi-additive decomposition of $\underline{\boldsymbol{K}}$ (compare Forest and Sievert, 2003 and Eq. (5)b for full micromorphic media):

$$\underline{\boldsymbol{K}} = \underline{\boldsymbol{K}}^e . \underset{\sim}{\boldsymbol{F}}^p + \underline{\boldsymbol{K}}^p. \tag{42}$$

3.4. Infinitesimal case

The previous thermomechanical framework for media with microdilatation can be linearized based on the assumption of small perturbations (small strains and rotations, small microdilatation gradient). The deformation gradient and the microdeformation of a micromorphic medium can be split into symmetric and skew-symmetric parts:

$$\underset{\sim}{\boldsymbol{F}} = \underset{\sim}{\mathbf{1}} + \underset{\sim}{\boldsymbol{\varepsilon}} + \underset{\sim}{\boldsymbol{\omega}}, \quad \underset{\sim}{\boldsymbol{\chi}} = \underset{\sim}{\boldsymbol{\chi}}^s + \underset{\sim}{\boldsymbol{\chi}}^a = \underset{\sim}{\mathbf{1}} + {}^{\chi}\underset{\sim}{\boldsymbol{\varepsilon}} + \underset{\sim}{\boldsymbol{\chi}}^a, \quad {}^{\chi}\underset{\sim}{\boldsymbol{\varepsilon}} := \underset{\sim}{\boldsymbol{\chi}}^s - \underset{\sim}{\mathbf{1}}. \tag{43}$$

The Jacobians of the transformations are linearized as follows:

$$\det \underset{\sim}{\boldsymbol{F}} \simeq 1 + \text{trace } \underset{\sim}{\boldsymbol{\varepsilon}}, \quad \chi = 1 + \chi_\varepsilon, \quad \chi_\varepsilon := \text{trace } {}^{\chi}\underset{\sim}{\boldsymbol{\varepsilon}} \tag{44}$$

where χ_ε represents the linear microvolume-strain. The degrees of freedom and strain measures of the infinitesimal theory of materials with microdilatation are

$$DOF = \{\underline{\mathbf{u}}, \chi_\varepsilon\}, \quad STRAIN = \{\underset{\sim}{\boldsymbol{\varepsilon}}, e := \text{trace } \underset{\sim}{\boldsymbol{\varepsilon}} - \chi_\varepsilon, \underline{\boldsymbol{K}} := \nabla \chi_\varepsilon\}. \tag{45}$$

The power of internal forces takes then a particularly simple form:

$$p^{(i)} = \underset{\sim}{\boldsymbol{\sigma}} : \underset{\sim}{\dot{\boldsymbol{\varepsilon}}} + s\,(\text{trace}\ \underset{\sim}{\dot{\boldsymbol{\varepsilon}}} - \dot{\chi}_\varepsilon) + \underline{\boldsymbol{S}}.\boldsymbol{\nabla}\dot{\chi}_\varepsilon = \underset{\sim}{\boldsymbol{\sigma}} : \underset{\sim}{\dot{\boldsymbol{\varepsilon}}} + s\,\dot{e} + \underline{\boldsymbol{S}}.\underline{\dot{\boldsymbol{K}}}. \tag{46}$$

The balance equations (14) are unchanged, providing that the Eulerian nabla operator $\boldsymbol{\nabla}^c$ is replaced by the gradient operator $\boldsymbol{\nabla}$ with respect to the reference configuration. The decompositions (23)–(26) of strain measures into elastic and plastic parts become:

$$\underset{\sim}{\boldsymbol{\varepsilon}} = \underset{\sim}{\boldsymbol{\varepsilon}}^e + \underset{\sim}{\boldsymbol{\varepsilon}}^p, \quad \chi_\varepsilon = \chi_\varepsilon^e + \chi_\varepsilon^p \tag{47}$$

$$e = (\text{trace}\ \underset{\sim}{\boldsymbol{\varepsilon}}^e - \chi_\varepsilon^e) + (\text{trace}\ \underset{\sim}{\boldsymbol{\varepsilon}}^p - \chi_\varepsilon^p) = e^e + e^p \tag{48}$$

$$\underline{\boldsymbol{K}} = \underline{\boldsymbol{K}}^e + \underline{\boldsymbol{K}}^p \tag{49}$$

The free energy density is a function $\Psi(\underset{\sim}{\boldsymbol{\varepsilon}}^e, e^e, \underline{\boldsymbol{K}}^e, q)$ from which the state laws are derived:

$$\underset{\sim}{\boldsymbol{\sigma}} = \rho \frac{\partial \Psi}{\partial \underset{\sim}{\boldsymbol{\varepsilon}}^e} \tag{50}$$

$$s = \rho \frac{\partial \Psi}{\partial e^e} \tag{51}$$

$$\underline{\boldsymbol{S}} = \rho \frac{\partial \Psi}{\partial \underline{\boldsymbol{K}}^e} \tag{52}$$

$$R = -\rho \frac{\partial \Psi}{\partial q} \tag{53}$$

The residual dissipation within the small perturbation framework is

$$D = \underset{\sim}{\boldsymbol{\sigma}} : \underset{\sim}{\dot{\boldsymbol{\varepsilon}}}^p + s\,\dot{e}^p + \underline{\boldsymbol{S}}.\underline{\dot{\boldsymbol{K}}}^p + R\dot{q} \tag{54}$$

$$= (\underset{\sim}{\boldsymbol{\sigma}} + s\,\underset{\sim}{\mathbf{1}}) : \underset{\sim}{\dot{\boldsymbol{\varepsilon}}}^p - s\,\dot{\chi}_\varepsilon^p + \underline{\boldsymbol{S}}.\underline{\dot{\boldsymbol{K}}}^p + R\dot{q} \geqslant 0 \tag{55}$$

The existence of a convex dissipation potential $\Omega(\underset{\sim}{\boldsymbol{\Sigma}} := \underset{\sim}{\boldsymbol{\sigma}} + s\,\underset{\sim}{\mathbf{1}}, -s, \underline{\boldsymbol{S}}, R)$ can be postulated from which flow rules and evolution equations can be derived:

$$\underset{\sim}{\dot{\boldsymbol{\varepsilon}}}^p = \frac{\partial \Omega}{\partial \underset{\sim}{\boldsymbol{\Sigma}}}, \quad \dot{\chi}_\varepsilon^p = -\frac{\partial \Omega}{\partial s}, \quad \underline{\dot{\boldsymbol{K}}}^p = \frac{\partial \Omega}{\partial \underline{\boldsymbol{S}}}, \quad \dot{q} = \frac{\partial \Omega}{\partial R} \tag{56}$$

An alternative representation of the flow rule for plastic microdilatation amounts to considering the functional dependence $\Omega(\underset{\sim}{\boldsymbol{\sigma}}, s, \underline{\boldsymbol{S}}, R)$ with

$$\underset{\sim}{\dot{\boldsymbol{\varepsilon}}}^p = \frac{\partial \Omega}{\partial \underset{\sim}{\boldsymbol{\sigma}}}, \quad \dot{e}^p = \frac{\partial \Omega}{\partial s} \tag{57}$$

4. Microstretch continua

Continua possessing a microdilatation as well as a microrotation are called by Eringen microstretch continua (Eringen, 1990). The theory is extended here to finite strain elastoviscoplasticity.

4.1. Kinematics and balance equations

In this theory the microdeformation capabilities of the medium are reduced to

$$\underset{\sim}{\boldsymbol{\chi}} = \chi\ \underset{\sim}{\boldsymbol{R}}, \tag{58}$$

i.e., a Cosserat rotation $\underset{\sim}{\boldsymbol{R}}$ and a microdilatation χ. The whole set of degrees of freedom is then:

$$DOF = \{\underline{\mathbf{u}}, \chi, \underset{\sim}{\boldsymbol{R}}\} \tag{59}$$

The microdeformation rate takes the form

$$\dot{\underset{\sim}{\boldsymbol{\chi}}}\cdot\underset{\sim}{\boldsymbol{\chi}}^{-1} = \frac{\dot{\chi}}{\chi}\,\underset{\sim}{\mathbf{1}} + \dot{\underset{\sim}{\boldsymbol{R}}}.\underset{\sim}{\boldsymbol{R}}^{T} \tag{60}$$

where microdilatation and Cosserat effects are clearly separated. The power of internal forces is therefore a straightforward combination of the microdilatation and Cosserat theories:

$$p^{(i)} = {}^{\sharp}\underset{\sim}{\boldsymbol{\sigma}} : ({}^{\sharp}\dot{\underset{\sim}{\boldsymbol{F}}}.{}^{\sharp}\underset{\sim}{\boldsymbol{F}}^{-1}) + s\left(\frac{(\det \underset{\sim}{\boldsymbol{F}})^{\bullet}}{\det \underset{\sim}{\boldsymbol{F}}} - \frac{\dot{\chi}}{\chi}\right) + {}^{\sharp}\underset{\sim}{\boldsymbol{\mu}} : ({}^{\sharp}\dot{\underset{\sim}{\boldsymbol{\Gamma}}}.{}^{\sharp}\underset{\sim}{\boldsymbol{F}}^{-1}) + \underline{\boldsymbol{S}}.\left(\boldsymbol{\nabla}^{c}\left(\frac{\dot{\chi}}{\chi}\right)\right) \tag{61}$$

The relative deformation ${}^{\sharp}\underset{\sim}{\boldsymbol{F}}$ and curvature tensors ${}^{\sharp}\underset{\sim}{\boldsymbol{\Gamma}}$ are defined in Table 1. The application of the method of virtual power then leads to the following field equations and boundary conditions:

$$(\underset{\sim}{\boldsymbol{\sigma}} + s\,\underset{\sim}{\mathbf{1}}).\boldsymbol{\nabla}^{c} = 0 \quad \forall \underline{\boldsymbol{x}} \in \mathscr{D} \tag{62}$$

$$\underline{\boldsymbol{S}}.\boldsymbol{\nabla}^{c} + s = 0 \quad \forall \underline{\boldsymbol{x}} \in \mathscr{D} \tag{63}$$

$$\underset{\sim}{\boldsymbol{\mu}}.\boldsymbol{\nabla}^{c} - \underset{\simeq}{\boldsymbol{\epsilon}} : \underset{\sim}{\boldsymbol{\sigma}} = 0 \quad \forall \underline{\boldsymbol{x}} \in \mathscr{D} \tag{64}$$

$$\underline{\boldsymbol{t}} = (\underset{\sim}{\boldsymbol{\sigma}} + s\,\underset{\sim}{\mathbf{1}}).\underline{\boldsymbol{n}}, \quad m = \underline{\boldsymbol{S}}.\underline{\boldsymbol{n}}, \quad \underline{\boldsymbol{m}} = \underset{\sim}{\boldsymbol{\mu}}.\underline{\boldsymbol{n}} \quad \forall \underline{\boldsymbol{x}} \in \partial\mathscr{D} \tag{65}$$

The spatial stress tensors are $\underset{\sim}{\boldsymbol{\sigma}} = \underset{\sim}{\boldsymbol{R}}.{}^{\sharp}\underset{\sim}{\boldsymbol{\sigma}}.\underset{\sim}{\boldsymbol{R}}^{T}$, $\underset{\sim}{\boldsymbol{\mu}} = \underset{\sim}{\boldsymbol{R}}.{}^{\sharp}\underset{\sim}{\boldsymbol{\mu}}.\underset{\sim}{\boldsymbol{R}}^{T}$. A set of strain measures which are invariant with respect to Euclidean transformations are selected:

$$STRAIN = \left\{ {}^{\sharp}\underset{\sim}{\boldsymbol{F}} := \underset{\sim}{\boldsymbol{R}}^{T}.\underset{\sim}{\boldsymbol{F}}, \quad e := \frac{\det \underset{\sim}{\boldsymbol{F}}}{\chi}, \quad \underline{\boldsymbol{K}} := \frac{\boldsymbol{\nabla}\chi}{\chi}, \quad {}^{\sharp}\underset{\sim}{\boldsymbol{\Gamma}} := -\frac{1}{2}\underset{\simeq}{\boldsymbol{\epsilon}} : (\underset{\sim}{\boldsymbol{R}}^{T}.(\underset{\sim}{\boldsymbol{R}} \otimes \boldsymbol{\nabla})) \right\} \tag{66}$$

4.2. Elastoviscoplastic microstretch materials

The following decompositions into elastic and plastic parts of the strain measures of microdilatation and Cosserat media are adopted:

$${}^{\sharp}\underset{\sim}{\boldsymbol{F}} = {}^{\sharp}\underset{\sim}{\boldsymbol{F}}^{e}.{}^{\sharp}\underset{\sim}{\boldsymbol{F}}^{p}, \quad \chi = \chi^{e}\,\chi^{p}, \quad \underline{\boldsymbol{K}} = \underline{\boldsymbol{K}}^{e}.{}^{\sharp}\underset{\sim}{\boldsymbol{F}}^{p} + \underline{\boldsymbol{K}}^{p}, \quad {}^{\sharp}\underset{\sim}{\boldsymbol{\Gamma}} = {}^{\sharp}\underset{\sim}{\boldsymbol{\Gamma}}^{e}.{}^{\sharp}\underset{\sim}{\boldsymbol{F}}^{p} + {}^{\sharp}\underset{\sim}{\boldsymbol{\Gamma}}^{p} \tag{67}$$

The decomposition of the relative volume change e is still given by (24). Note that we have chosen here the decomposition (42) of the microdilatation gradient to be consistent with the decomposition of the torsion-curvature tensor ${}^{\sharp}\underset{\sim}{\boldsymbol{\Gamma}}$ proposed in Table 1. Different elastic and plastic contributions can now be distinguished in the power of internal forces:

$$\begin{aligned} p^{(i)} = {} & {}^{\sharp}\underset{\sim}{\boldsymbol{\sigma}} : ({}^{\sharp}\dot{\underset{\sim}{\boldsymbol{F}}}^{e}.{}^{\sharp}\underset{\sim}{\boldsymbol{F}}^{e-1}) + ({}^{\sharp}\underset{\sim}{\boldsymbol{F}}^{eT}.{}^{\sharp}\underset{\sim}{\boldsymbol{\sigma}}.{}^{\sharp}\underset{\sim}{\boldsymbol{F}}^{e-T}) : ({}^{\sharp}\dot{\underset{\sim}{\boldsymbol{F}}}^{p}.{}^{\sharp}\underset{\sim}{\boldsymbol{F}}^{p-1}) \\ & + s\,\dot{\Upsilon}^{e}\Upsilon^{e-1} - s\,\dot{\chi}^{p}\,\chi^{p-1} + s \operatorname{trace}\,({}^{\sharp}\dot{\underset{\sim}{\boldsymbol{F}}}^{p}.{}^{\sharp}\underset{\sim}{\boldsymbol{F}}^{p-1}) + \underline{\boldsymbol{S}}.(\dot{\underline{\boldsymbol{K}}}^{e}.\underset{\sim}{\boldsymbol{F}}^{e-1}) \\ & + \underline{\boldsymbol{S}}.\left(\underline{\boldsymbol{K}}^{e}.({}^{\sharp}\dot{\underset{\sim}{\boldsymbol{F}}}^{p}.{}^{\sharp}\underset{\sim}{\boldsymbol{F}}^{p-1}.\underset{\sim}{\boldsymbol{F}}^{e-1})\right) + \underline{\boldsymbol{S}}.(\dot{\underline{\boldsymbol{K}}}^{p}.\underset{\sim}{\boldsymbol{F}}^{-1}) + {}^{\sharp}\underset{\sim}{\boldsymbol{\mu}} : \left({}^{\sharp}\dot{\underset{\sim}{\boldsymbol{\Gamma}}}^{e}.{}^{\sharp}\underset{\sim}{\boldsymbol{F}}^{e-1}\right) \\ & + {}^{\sharp}\underset{\sim}{\boldsymbol{\mu}} : \left({}^{\sharp}\dot{\underset{\sim}{\boldsymbol{\Gamma}}}^{p}.{}^{\sharp}\underset{\sim}{\boldsymbol{F}}^{-1}\right) + {}^{\sharp}\underset{\sim}{\boldsymbol{\mu}} : ({}^{\sharp}\underset{\sim}{\boldsymbol{\Gamma}}^{e}.{}^{\sharp}\dot{\underset{\sim}{\boldsymbol{F}}}^{p}.{}^{\sharp}\underset{\sim}{\boldsymbol{F}}^{p-1}.{}^{\sharp}\underset{\sim}{\boldsymbol{F}}^{e-1}) \end{aligned} \tag{68}$$

where $\underset{\sim}{\boldsymbol{F}}^{e} := \underset{\sim}{\boldsymbol{R}}.{}^{\sharp}\underset{\sim}{\boldsymbol{F}}^{e}$ has been introduced. Accordingly, the free energy is taken as a function of the following set of variables:

$$\Psi\left({}^{\sharp}\underset{\sim}{\boldsymbol{F}}^{e}, \Upsilon^{e} := \frac{\det \underset{\sim}{\boldsymbol{F}}^{e}}{\chi^{e}}, \underline{\boldsymbol{K}}^{e}, {}^{\sharp}\underset{\sim}{\boldsymbol{\Gamma}}^{e}, \quad q\right) \tag{69}$$

The expression of the power $p^{(i)}$ and of the dependence of the free energy density are substituted into the entropy inequality, leading to the Clausius–Duhem inequality:

$$D = \left({}^\sharp\underset{\sim}{\boldsymbol{\sigma}} . {}^\sharp\underset{\sim}{\boldsymbol{F}}^{e-T} - \rho \frac{\partial \Psi}{\partial\, {}^\sharp\underset{\sim}{\boldsymbol{F}}^e} \right) : {}^\sharp\dot{\underset{\sim}{\boldsymbol{F}}}^e + \left(s\Upsilon^{e-1} - \rho \frac{\partial \Psi}{\partial \Upsilon^e} \right) \dot{\Upsilon}^e + \left(\underline{\boldsymbol{S}} . \underset{\sim}{\boldsymbol{F}}^{e-T} - \rho \frac{\partial \Psi}{\partial \underline{\boldsymbol{K}}^e} \right) . \dot{\underline{\boldsymbol{K}}}^e + \left({}^\sharp\underset{\sim}{\boldsymbol{\mu}} . {}^\sharp\underset{\sim}{\boldsymbol{F}}^{e-T} - \rho \frac{\partial \Psi}{\partial\, {}^\sharp\underset{\sim}{\boldsymbol{\Gamma}}^e} \right) : {}^\sharp\dot{\underset{\sim}{\boldsymbol{\Gamma}}}^e + D_r \geqslant 0 \tag{70}$$

where D_r is the residual dissipation rate to be explicited in the next paragraph. The exploitation of the second principle according to Coleman–Noll's procedure (Liu, 2002) leads to the state laws:

$$ {}^\sharp\underset{\sim}{\boldsymbol{\sigma}} = \rho \frac{\partial \Psi}{\partial\, {}^\sharp\underset{\sim}{\boldsymbol{F}}^e} . {}^\sharp\underset{\sim}{\boldsymbol{F}}^{eT} \tag{71}$$

$$ s = \Upsilon^e \rho \frac{\partial \Psi}{\partial \Upsilon^e} \tag{72}$$

$$ \underline{\boldsymbol{S}} = \rho \frac{\partial \Psi}{\partial \underline{\boldsymbol{K}}^e} . \underset{\sim}{\boldsymbol{F}}^{eT} \tag{73}$$

$$ {}^\sharp\underset{\sim}{\boldsymbol{\mu}} = \rho \frac{\partial \Psi}{\partial {}^\sharp\underset{\sim}{\boldsymbol{\Gamma}}^e} . {}^\sharp\underset{\sim}{\boldsymbol{F}}^{eT} \tag{74}$$

$$ R = -\rho \frac{\partial \Psi}{\partial q} \tag{75}$$

The dissipation due to inelastic processes is

$$D_r = \underset{\sim}{\boldsymbol{\Sigma}} : ({}^\sharp\dot{\underset{\sim}{\boldsymbol{F}}}^p . {}^\sharp\underset{\sim}{\boldsymbol{F}}^{p-1}) + \mathscr{S}\, \dot{\chi}^p \chi^{p-1} + \underline{\boldsymbol{S}}_0 . (\dot{\underline{\boldsymbol{K}}}^p . \underset{\sim}{\boldsymbol{F}}^{p-1}) + \underset{\sim}{\boldsymbol{M}} : \left({}^\sharp\dot{\underset{\sim}{\boldsymbol{\Gamma}}}^p . {}^\sharp\underset{\sim}{\boldsymbol{F}}^{p-1} \right) + R\dot{q} \tag{76}$$

where generalized stress tensors were introduced. These generalized Mandel stresses read:

$$\underset{\sim}{\boldsymbol{\Sigma}} := {}^\sharp\underset{\sim}{\boldsymbol{F}}^{eT} . ({}^\sharp\underset{\sim}{\boldsymbol{\sigma}} + s\, \underset{\sim}{\boldsymbol{1}}) . {}^\sharp\underset{\sim}{\boldsymbol{F}}^{e-T} + \underline{\boldsymbol{K}}^e \otimes (\underline{\boldsymbol{S}} . \underset{\sim}{\boldsymbol{F}}^{e-T}) + {}^\sharp\underset{\sim}{\boldsymbol{\Gamma}}^{eT} . {}^\sharp\underset{\sim}{\boldsymbol{\mu}} . {}^\sharp\underset{\sim}{\boldsymbol{F}}^{e-T} \tag{77}$$

$$\mathscr{S} := -s, \quad \underline{\boldsymbol{S}}_0 := \underline{\boldsymbol{S}} . \underset{\sim}{\boldsymbol{F}}^{e-T}, \quad \underset{\sim}{\boldsymbol{M}} := {}^\sharp\underset{\sim}{\boldsymbol{\mu}} . {}^\sharp\underset{\sim}{\boldsymbol{F}}^{e-T} \tag{78}$$

To ensure the positivity of the intrinsic dissipation D_r, the existence of a convex dissipation potential may be postulated:

$$\Omega(\underset{\sim}{\boldsymbol{\Sigma}}, \mathscr{S}, \underline{\boldsymbol{S}}_0, \underset{\sim}{\boldsymbol{M}}, R) \tag{79}$$

from which the flow rule and evolution equations are derived

$$ {}^\sharp\dot{\underset{\sim}{\boldsymbol{F}}}^p . {}^\sharp\underset{\sim}{\boldsymbol{F}}^{p-1} = \frac{\partial \Omega}{\partial \underset{\sim}{\boldsymbol{\Sigma}}} \tag{80}$$

$$ \dot{\chi}^p \chi^{p-1} = \frac{\partial \Omega}{\partial \mathscr{S}} \tag{81}$$

$$ \dot{\underline{\boldsymbol{K}}}^p . \underset{\sim}{\boldsymbol{F}}^{p-1} = \frac{\partial \Omega}{\partial \underline{\boldsymbol{S}}_0} \tag{82}$$

$$ {}^\sharp\dot{\underset{\sim}{\boldsymbol{\Gamma}}}^p . {}^\sharp\underset{\sim}{\boldsymbol{F}}^{p-1} = \frac{\partial \Omega}{\partial \underset{\sim}{\boldsymbol{M}}} \tag{83}$$

$$ \dot{q} = \frac{\partial \Omega}{\partial R} \tag{84}$$

4.3. Infinitesimal case

In the infinitesimal case, the Cosserat rotation is represented by the pseudo-vector:

$$\underline{\phi} = -\frac{1}{2}\underset{\approx}{\varepsilon} : \underset{\sim}{R} \tag{85}$$

As a result, the degrees of freedom and strain measures of the infinitesimal microstretch theory are

$$DOF = \{\underline{\mathbf{u}}, \chi_\varepsilon, \underline{\phi}\} \tag{86}$$

$$STRAIN = \{\underset{\sim}{e} := \underline{\mathbf{u}} \otimes \nabla + \underset{\approx}{\varepsilon}.\underline{\phi}, e := \text{trace }\underset{\sim}{\varepsilon} - \chi_\varepsilon, \underline{K} := \nabla\chi_\varepsilon, \underset{\sim}{\kappa} := \underline{\phi} \otimes \nabla\} \tag{87}$$

where χ_ε is the linear microvolume strain (compare Eq. (44)), $\underset{\sim}{e}$ and e are the linear relative deformations and $\underset{\sim}{\kappa}$ the linear curvature tensor. The linear version of the power of internal forces is then

$$p^{(i)} = \underset{\sim}{\sigma} : \underset{\sim}{\dot{e}} + s(\text{trace }\underset{\sim}{\dot{\varepsilon}} - \dot{\chi}_\varepsilon) + \underset{\sim}{\mu} : \underset{\sim}{\dot{\kappa}} + \underline{S}.\nabla\dot{\chi}_\varepsilon = \underset{\sim}{\sigma} : \underset{\sim}{\dot{e}} + s\,\dot{e} + \underset{\sim}{\mu} : \underset{\sim}{\dot{\kappa}} + \underline{S}.\nabla\dot{\chi}_\varepsilon \tag{88}$$

The balance and boundary conditions are still given formally by the Eqs. (62)–(65) by replacing ∇^c by ∇. The decompositions (67) are linearized into

$$\underset{\sim}{e} = \underset{\sim}{e}^e + \underset{\sim}{e}^p, \quad e = e^e + e^p, \quad \underline{K} = \underline{K}^e + \underline{K}^p, \quad \underset{\sim}{\kappa} = \underset{\sim}{\kappa}^e + \underset{\sim}{\kappa}^p \tag{89}$$

where e^e and e^p are according to Eq. (48). The free energy density is the function

$$\Psi(\underset{\sim}{e}^e, e^e, \underline{K}^e, \underset{\sim}{\kappa}^e, q)$$

from which the elastic relations are derived:

$$\underset{\sim}{\sigma} = \rho\frac{\partial\Psi}{\partial\underset{\sim}{e}^e}, \quad s = \rho\frac{\partial\Psi}{\partial e^e}, \quad \underline{S} = \rho\frac{\partial\Psi}{\partial\underline{K}^e}, \quad \underset{\sim}{\mu} = \rho\frac{\partial\Psi}{\partial\underset{\sim}{\kappa}^e} \tag{90}$$

The residual dissipation is then

$$D = \underset{\sim}{\sigma} : \underset{\sim}{\dot{e}}^p + s\,\dot{e}^p + \underline{S}.\underline{\dot{K}}^p + \underset{\sim}{\mu} : \underset{\sim}{\dot{\kappa}}^p + R\dot{q} \tag{91}$$

5. Microstrain continua

The microstrain continuum is endowed with 6 additional degrees of freedom accounting for the pure strain part of the microdeformation of the underlying microstructure. The rotational part of microdeformation is assumed to play no significant role in the constitutive behaviour of the material. As a result, 3 degrees of freedom can be spared compared to a full micromorphic medium.

5.1. Kinematics and balance equations

The additional degrees of freedom are the components of the symmetric right Cauchy–Green strain tensor ${}^\chi\underset{\sim}{C}$ associated with the microdeformation $\underset{\sim}{\chi}$:

$${}^\chi\underset{\sim}{C} := \underset{\sim}{\chi}^T \cdot \underset{\sim}{\chi} = {}^\chi\underset{\sim}{U}^2 \tag{92}$$

where ${}^\chi\underset{\sim}{U}$ is the right stretch tensor in the polar decomposition of the microdeformation. The set of degrees of freedom of the theory is

$$DOF = \{\underline{\mathbf{u}}, {}^\chi\underset{\sim}{C}\} \tag{93}$$

The power of internal forces is then a linear form in the rates of the degrees of freedom and their spatial gradients:

$$p^{(i)} = \frac{1}{2}\,\underset{\sim}{\boldsymbol{\tau}} : (\boldsymbol{F}^{-T} . \dot{\underset{\sim}{\boldsymbol{C}}} . \boldsymbol{F}^{-1}) - \frac{1}{2}\,\underset{\sim}{\boldsymbol{s}} : {}^{\chi}\dot{\underset{\sim}{\boldsymbol{C}}} + \underset{\approx}{\boldsymbol{S}} \;\vdots\; ({}^{\chi}\dot{\underset{\sim}{\boldsymbol{C}}} \otimes \boldsymbol{\nabla}^{c}) \tag{94}$$

where the second order stress tensor $\underset{\sim}{\boldsymbol{s}}$ is symmetric. The third order hyperstress tensor $\underset{\approx}{\boldsymbol{S}}$ is here symmetric with respect to its first two indices.

After noting that the term $1/2\underset{\sim}{\boldsymbol{\chi}}^{-T} . {}^{\chi}\dot{\underset{\sim}{\boldsymbol{C}}} . \underset{\sim}{\boldsymbol{\chi}}^{-1}$ is nothing but the symmetric part of the microdeformation rate $\dot{\underset{\sim}{\boldsymbol{\chi}}} . \underset{\sim}{\boldsymbol{\chi}}^{-1}$, the previous expression of $p^{(i)}$ can be written in the form

$$p^{(i)} = (\underset{\sim}{\boldsymbol{\tau}} - \underset{\sim}{\boldsymbol{\chi}} . \underset{\sim}{\boldsymbol{s}} . \underset{\sim}{\boldsymbol{\chi}}^{T}) : (\dot{\underset{\sim}{\boldsymbol{F}}} . \underset{\sim}{\boldsymbol{F}}^{-1}) + (\underset{\sim}{\boldsymbol{\chi}} . \underset{\sim}{\boldsymbol{s}} . \underset{\sim}{\boldsymbol{\chi}}^{T}) : (\dot{\underset{\sim}{\boldsymbol{F}}} . \underset{\sim}{\boldsymbol{F}}^{-1} - \dot{\underset{\sim}{\boldsymbol{\chi}}} . \underset{\sim}{\boldsymbol{\chi}}^{-1}) + \underset{\approx}{\boldsymbol{S}} \;\vdots\; ({}^{\chi}\dot{\underset{\sim}{\boldsymbol{C}}} \otimes \boldsymbol{\nabla}^{c}) \tag{95}$$

It can then be compared to the corresponding expression of $p^{(i)}$ for a full micromorphic medium given in Table 2. Comparing (95) with (3) shows that only the effect of the symmetric part of the relative deformation rate $\dot{\underset{\sim}{\boldsymbol{F}}} . \underset{\sim}{\boldsymbol{F}}^{-1} - \dot{\underset{\sim}{\boldsymbol{\chi}}} . \underset{\sim}{\boldsymbol{\chi}}^{-1}$ and the gradient of the microstrain rate have been retained. A formal equivalence between the stress tensors of the microstrain and micromorphic theories can be drawn. The tensor $\underset{\sim}{\boldsymbol{\tau}} - \underset{\sim}{\boldsymbol{\chi}} . \underset{\sim}{\boldsymbol{s}} . \underset{\sim}{\boldsymbol{\chi}}^{T}$ plays the role of $\underset{\sim}{\boldsymbol{\sigma}}$ in (3). The additional second order stress tensor $\underset{\sim}{\boldsymbol{s}}$ is related to the symmetric part of the relative stress tensor, i.e. the one conjugate to the relative deformation rate in (3). This comparison motivates the minus sign introduced in front of $\underset{\sim}{\boldsymbol{s}}$ in the linear form (94). The third-order hyperstress tensor $\underset{\approx}{\boldsymbol{S}}$ also has additional symmetry properties, with respect to its first two indices, compared to its counterpart in the micromorphic balance.

The method of virtual power is used again to derive the spatial balance equations of moment and generalized moment of momentum:

$$\underset{\sim}{\boldsymbol{\tau}} . \boldsymbol{\nabla}^{c} = 0, \quad \forall \underline{\boldsymbol{x}} \in \mathscr{D} \tag{96}$$

$$\underset{\approx}{\boldsymbol{S}} . \boldsymbol{\nabla}^{c} + \underset{\sim}{\boldsymbol{s}} = 0, \quad \forall \underline{\boldsymbol{x}} \in \mathscr{D} \tag{97}$$

$$\underline{\boldsymbol{t}} = \underset{\sim}{\boldsymbol{\tau}} . \underline{\boldsymbol{n}} \quad \underset{\sim}{\boldsymbol{m}} = \underset{\approx}{\boldsymbol{S}} . \underline{\boldsymbol{n}}, \quad \forall \underline{\boldsymbol{x}} \in \partial\mathscr{D} \tag{98}$$

where surface traction $\underline{\boldsymbol{t}}$ and double force tensor $\underset{\sim}{\boldsymbol{m}}$ in the current configuration were introduced. It turns out that no apparent coupling between the stresses associated with macro and microdeformation occurs in the equations of equilibrium for the microstrain medium. In contrast the relative stress tensor couples the balance of momentum and of moment of momentum for a micromorphic continuum (see Table 2). The apparent coupling is restored by introducing the stress tensor $\underset{\sim}{\boldsymbol{\sigma}} := \underset{\sim}{\boldsymbol{\tau}} - \underset{\sim}{\boldsymbol{\chi}} . \underset{\sim}{\boldsymbol{s}} . \underset{\sim}{\boldsymbol{\chi}}^{T}$ of the power (95) in the balance (96). In fact, the effective coupling of macro and microdeformation by the field equations depends on the adopted constitutive relationships between the stress tensors and the strain measures. This is discussed in the next subsection.

The strain measures of the theory are:

$$STRAIN = \{\underset{\sim}{\boldsymbol{C}}, {}^{\chi}\underset{\sim}{\boldsymbol{C}}, \underset{\approx}{\boldsymbol{K}} := {}^{\chi}\underset{\sim}{\boldsymbol{C}} \otimes \boldsymbol{\nabla}\} \tag{99}$$

5.2. Hyperelastic microstrain materials

Microstrain materials are said to be hyperelastic when the free energy density is a function of the strain measures (99):

$$\Psi(\underset{\sim}{\boldsymbol{C}}, {}^{\chi}\underset{\sim}{\boldsymbol{C}}, \underset{\approx}{\boldsymbol{K}}) \tag{100}$$

No dissipation is associated with the deformation of hyperelastic materials, so that

$$\rho\dot{\Psi} = p^{(i)} = \rho\frac{\partial\Psi}{\partial\underset{\sim}{\boldsymbol{C}}} : \dot{\underset{\sim}{\boldsymbol{C}}} + \rho\frac{\partial\Psi}{\partial{}^{\chi}\underset{\sim}{\boldsymbol{C}}} : {}^{\chi}\dot{\underset{\sim}{\boldsymbol{C}}} + \rho\frac{\partial\Psi}{\partial\underset{\approx}{\boldsymbol{K}}} \;\vdots\; \dot{\underset{\approx}{K}} \tag{101}$$

The hyperelastic relations follow:

$$\underset{\sim}{\tau} = 2\,\underset{\sim}{F}.\rho \frac{\partial \Psi}{\partial \underset{\sim}{C}}.\underset{\sim}{F}^T \tag{102}$$

$$\underset{\sim}{s} = -2\rho \frac{\partial \Psi}{\partial^{\chi} \underset{\sim}{C}} \tag{103}$$

$$\underset{\approx}{S} = \rho \frac{\partial \Psi}{\partial \underset{\approx}{K}}.\underset{\sim}{F}^T \tag{104}$$

If the expression of Ψ contains no coupling terms between macro and microstrains, like mixed products of $\underset{\sim}{C}$ and ${}^{\chi}\underset{\sim}{C}$, the previous hyperelastic relations, when inserted into the field Eqs. (96) and (97), lead to totally uncoupled partial differential equations for macro and micromotions. Such a situation has been noticed in a different context, namely turbulence theory of micromorphic fluids (Görn, 1996; Eringen, 1999). This would mean that the solutions of the classical boundary value problem with the associated Cauchy continuum are unaffected by the micromotion and essentially unchanged. This is surely not what one aims at when introducing additional degrees of freedom. The possible existence of coupling terms is indeed contained in the general constitutive framework (100) but it can be made more visible by choosing the following form of the functional dependence of the free energy:

$$\widehat{\Psi}(\underset{\sim}{C}, \underset{\sim}{e} := \underset{\sim}{C} - {}^{\chi}\underset{\sim}{C}, \underset{\approx}{K}) := \Psi(\underset{\sim}{C}, \underset{\sim}{C} - \underset{\sim}{e}, \underset{\approx}{K}) \equiv \Psi(\underset{\sim}{C}, {}^{\chi}\underset{\sim}{C}, \underset{\approx}{K}) \tag{105}$$

The relative strain measure $\underset{\sim}{e} := \underset{\sim}{C} - {}^{\chi}\underset{\sim}{C}$ can be regarded as the counterpart, within the context of the microstrain theory, of the relative deformation $\underset{\sim}{\chi}^{-1}.\underset{\sim}{F}$ used for micromorphic materials. The hyperelastic relations (102)–(104) can then be equivalently written in the form:

$$\underset{\sim}{\tau} = 2\,\underset{\sim}{F}.\rho \left(\frac{\partial \widehat{\Psi}}{\partial \underset{\sim}{C}} + \frac{\partial \widehat{\Psi}}{\partial \underset{\sim}{e}} \right).\underset{\sim}{F}^T \tag{106}$$

$$\underset{\sim}{s} = 2\rho \frac{\partial \widehat{\Psi}}{\partial \underset{\sim}{e}} \tag{107}$$

$$\underset{\approx}{S} = \rho \frac{\partial \widehat{\Psi}}{\partial \underset{\approx}{K}}.\underset{\sim}{F}^T \tag{108}$$

where the coupling term is apparent.

5.3. Elastoviscoplastic microstrain continua

The deformation gradient is decomposed into elastic and plastic parts:

$$\underset{\sim}{F} = \underset{\sim}{F}^e.\underset{\sim}{F}^p = \underset{\sim}{R}^e.\underset{\sim}{U}^e.\underset{\sim}{F}^p \tag{109}$$

assuming the existence of a privileged triad of directors controlling the anisotropy of the elastic–plastic behaviour and uniquely defining an intermediate isoclinic configuration. In the micromorphic theory of Table 2, the microdeformation admits a similar multiplicative decomposition into elastic and plastic parts. It can be used also in the microstrain theory provided that an assumption is made to connect the microrotation with the macromotion. A possible choice is

$$\underset{\sim}{\chi} = \underset{\sim}{\chi}^e.\underset{\sim}{\chi}^p = \underset{\sim}{R}^e.{}^{\chi}\underset{\sim}{U}^e.\underset{\sim}{\chi}^p \tag{110}$$

The rotation part in the polar decomposition of the elastic microdeformation $\underset{\sim}{\chi}^e$ is assumed to be equal to the macrorotation part $\underset{\sim}{R}^e$ which is the rotation part in the polar decomposition of the elastic deformation $\underset{\sim}{F}^e$. Using the notations of Table 2 and of Eq. (4), it corresponds to

$$ {}^{\chi}\underset{\sim}{R}^e \equiv \underset{\sim}{R}^e \tag{111}$$

This means that materials are considered here which do not possess an independent microrotation. Besides straining by $\underset{\sim}{\chi}^p$ and ${}^{\chi}\underset{\sim}{U}^e$, the micromaterial element rotates as the macromaterial element. In other words, the Cosserat rotation is fixed to be equal to $\underset{\sim}{R}^e$. The decomposition of the macro and microstrain tensors are then

$$ \underset{\sim}{C} = \underset{\sim}{U}^2 = \underset{\sim}{F}^{pT}.\underset{\sim}{C}^e.\underset{\sim}{F}^p, \quad \text{with } \underset{\sim}{C}^e := \underset{\sim}{U}^{e2} \tag{112}$$

$$ {}^{\chi}\underset{\sim}{C} = {}^{\chi}\underset{\sim}{U}^2 = \underset{\sim}{\chi}^{pT}.{}^{\chi}\underset{\sim}{C}^e.\underset{\sim}{\chi}^p, \quad \text{with } {}^{\chi}\underset{\sim}{C}^e := {}^{\chi}\underset{\sim}{U}^{e2} \tag{113}$$

The (non-necessarily symmetric) plastic microdeformation tensor $\underset{\sim}{\chi}^p$ will result from an evolution equation to be defined. The elastic microstrain ${}^{\chi}\underset{\sim}{C}^e$, or equivalently ${}^{\chi}\underset{\sim}{U}^e$, then follows from the decomposition (113) of ${}^{\chi}\underset{\sim}{C}$.

An additive decomposition of the microstrain gradient is first assumed:

$$ \underset{\approx}{K} = \underset{\approx}{K}^e + \underset{\approx}{K}^p \tag{114}$$

The power of internal forces can now be rewritten in terms of elastic and plastic contributions:

$$ \begin{aligned} p^{(i)} = {} & \underset{\sim}{\tau} : \left(\frac{1}{2}\underset{\sim}{F}^{e-T}.\dot{\underset{\sim}{C}}^e.\underset{\sim}{F}^{e-1}\right) - \underset{\sim}{s} : \left(\frac{1}{2}\underset{\sim}{\chi}^{pT}.{}^{\chi}\dot{\underset{\sim}{C}}^e.\underset{\sim}{\chi}^p\right) + \underset{\approx}{S} \vdots (\dot{\underset{\approx}{K}}^e.\underset{\sim}{F}^{-1}) \\ & + (\underset{\sim}{F}^{eT}.\underset{\sim}{\tau}.\underset{\sim}{F}^{e-T}) : (\dot{\underset{\sim}{F}}^p.\underset{\sim}{F}^{p-1}) - ({}^{\chi}\underset{\sim}{C}^e.\underset{\sim}{\chi}^p.\underset{\sim}{s}) : \dot{\underset{\sim}{\chi}}^p + \underset{\approx}{S} \vdots (\dot{\underset{\approx}{K}}^p.\underset{\sim}{F}^{-1}) \end{aligned} \tag{115}$$

It is assumed that energy can be stored via the elastic strain $\underset{\sim}{C}^e$, the elastic microstrain ${}^{\chi}\underset{\sim}{C}^e$ and the elastic part of microstrain gradient $\underset{\approx}{K}^e$. The free energy density is then a function of all the elastic strain measures and, possibly, of additional internal variables:

$$ \Psi(\underset{\sim}{C}^e, {}^{\chi}\underset{\sim}{C}^e, \underset{\approx}{K}^e, q) \tag{116}$$

The rate of dissipation is evaluated as:

$$ \begin{aligned} D &= p^{(i)} - \rho\dot{\Psi} \\ &= \left(\frac{1}{2}\underset{\sim}{F}^{e-1}.\underset{\sim}{\tau}.\underset{\sim}{F}^{e-T} - \rho\frac{\partial\Psi}{\partial\underset{\sim}{C}^e}\right) : \dot{\underset{\sim}{C}}^e - \left(\frac{1}{2}\underset{\sim}{\chi}^p.\underset{\sim}{s}.\underset{\sim}{\chi}^{pT} + \rho\frac{\partial\Psi}{\partial{}^{\chi}\underset{\sim}{C}^e}\right) : {}^{\chi}\dot{\underset{\sim}{C}}^e \\ &\quad + \left(\underset{\approx}{S}.\underset{\sim}{F}^{-T} - \rho\frac{\partial\Psi}{\partial\underset{\approx}{K}^e}\right) \vdots \dot{\underset{\approx}{K}}^e + \left(\underset{\sim}{F}^{eT}.\underset{\sim}{\tau}.\underset{\sim}{F}^{e-T}\right) : (\dot{\underset{\sim}{F}}^p.\underset{\sim}{F}^{p-1}) \\ &\quad - ({}^{\chi}\underset{\sim}{C}^e.\underset{\sim}{\chi}^p.\underset{\sim}{s}.\underset{\sim}{\chi}^{pT}) : (\dot{\underset{\sim}{\chi}}^p.\underset{\sim}{\chi}^{p-1}) + (\underset{\approx}{S}.\underset{\sim}{F}^{-T}) \vdots \dot{\underset{\approx}{K}}^p - \frac{\partial\Psi}{\partial q}\dot{q} \end{aligned} \tag{117}$$

The hyperelasticity state laws are then deduced from the exploitation of the second principle:

$$ \underset{\sim}{\tau} = 2\underset{\sim}{F}^e.\rho\frac{\partial\Psi}{\partial\underset{\sim}{C}^e}.\underset{\sim}{F}^{eT} \tag{118}$$

$$ \underset{\sim}{s} = -2\underset{\sim}{\chi}^{p-1}.\rho\frac{\partial\Psi}{\partial{}^{\chi}\underset{\sim}{C}^e}.\underset{\sim}{\chi}^{p-T} \tag{119}$$

$$ \underset{\approx}{S} = \rho\frac{\partial\Psi}{\partial\underset{\approx}{K}^e}.\underset{\sim}{F}^T \tag{120}$$

$$ R = -\rho\frac{\partial\Psi}{\partial q} \tag{121}$$

Mandel stress tensors can be introduced as driving forces for the dissipative mechanisms:

$$\underset{\sim}{\boldsymbol{\Sigma}} = \underset{\sim}{\boldsymbol{F}}^{eT}.\underset{\sim}{\boldsymbol{\tau}}.\underset{\sim}{\boldsymbol{F}}^{e-T} \tag{122}$$

$$\underset{\sim}{\mathscr{S}} = -{}^{\chi}\underset{\sim}{\boldsymbol{C}}^{e}.\underset{\sim}{\boldsymbol{\chi}}^{p}.\underset{\sim}{\boldsymbol{s}}.\underset{\sim}{\boldsymbol{\chi}}^{pT} \tag{123}$$

$$\underset{\approx}{\boldsymbol{S}}_0 = \underset{\approx}{\boldsymbol{S}}.\underset{\sim}{\boldsymbol{F}}^{-T} \tag{124}$$

so that the residual dissipation simply reads

$$D = \underset{\sim}{\boldsymbol{\Sigma}} : (\dot{\underset{\sim}{\boldsymbol{F}}}^{p}.\underset{\sim}{\boldsymbol{F}}^{p-1}) + \underset{\sim}{\mathscr{S}} : (\dot{\underset{\sim}{\boldsymbol{\chi}}}^{p}.\underset{\sim}{\boldsymbol{\chi}}^{p-1}) + \underset{\approx}{\boldsymbol{S}}_0 \vdots \dot{\underset{\approx}{\boldsymbol{K}}}^{p} + R\dot{q} \tag{125}$$

A simple way of identically fulfilling the dissipation inequality is to introduce a convex dissipation potential $\Omega(\underset{\sim}{\boldsymbol{\Sigma}}, \underset{\sim}{\mathscr{S}}, \underset{\approx}{\boldsymbol{S}}_0, R)$ that depends on the Mandel stress tensors and other thermodynamical forces. The flow rules and evolution equations follow:

$$\dot{\underset{\sim}{\boldsymbol{F}}}^{p}.\underset{\sim}{\boldsymbol{F}}^{p-1} = \frac{\partial \Omega}{\partial \underset{\sim}{\boldsymbol{\Sigma}}} \tag{126}$$

$$\dot{\underset{\sim}{\boldsymbol{\chi}}}^{p}.\underset{\sim}{\boldsymbol{\chi}}^{p-1} = \frac{\partial \Omega}{\partial \underset{\sim}{\mathscr{S}}} \tag{127}$$

$$\dot{\underset{\approx}{\boldsymbol{K}}}^{p} = \frac{\partial \Omega}{\partial \underset{\approx}{\boldsymbol{S}}} \tag{128}$$

$$\dot{q} = \frac{\partial \Omega}{\partial R} \tag{129}$$

When plastic flow is not allowed, the previous framework identically reduces to the pure hyperelastic case depicted in Section 5.2. The Cosserat rotation, identified with $\underset{\sim}{\boldsymbol{R}}^{e}$ in this section, then coincides with the rotation part in the polar decomposition of the deformation gradient (see Eq. (109)). This specific choice was not mentioned in Section 5.2, because, in the pure hyperelastic case, the proposed theory does not depend *in fine* on the choice of the microrotation.

5.3.1. A more specific functional dependence of the free energy

In applications of such enhanced continuum theories, the predicted size effects are the result of intimate coupling between macro and microdeformation. Such a coupling is not obvious when looking at the balance and constitutive equations of the microstrain theory, although it is actually contained in the general functional dependence of the free energy (116). It becomes more apparent when a relative elastic strain $\underset{\sim}{\boldsymbol{e}}^{e}$ is introduced, defined as:

$$\underset{\sim}{\boldsymbol{e}}^{e} := \underset{\sim}{\boldsymbol{C}}^{e} - {}^{\chi}\underset{\sim}{\boldsymbol{C}}^{e} \tag{130}$$

It can be regarded as the counterpart, within the microstrain theory, of the relative elastic microdeformation $\underset{\sim}{\boldsymbol{\Upsilon}}^{e}$ introduced in the theory of elastoviscoplastic micromorphic materials (see Table 2). The free energy can then be taken as a function of the form:

$$\widehat{\Psi}(\underset{\sim}{\boldsymbol{C}}^{e}, \underset{\sim}{\boldsymbol{e}}^{e}, \underset{\approx}{\boldsymbol{K}}^{e}, q) := \Psi(\underset{\sim}{\boldsymbol{C}}^{e}, \underset{\sim}{\boldsymbol{C}}^{e} - \underset{\sim}{\boldsymbol{e}}^{e}, \underset{\approx}{\boldsymbol{K}}^{e}, q) \equiv \Psi(\underset{\sim}{\boldsymbol{C}}^{e}, {}^{\chi}\underset{\sim}{\boldsymbol{C}}^{e}, \underset{\approx}{\boldsymbol{K}}^{e}, q) \tag{131}$$

The hyperelastic relations (118) and (119) then become

$$\underset{\sim}{\boldsymbol{\tau}} = 2\underset{\sim}{\boldsymbol{F}}^{e}.\rho\left(\frac{\partial \widehat{\Psi}}{\partial \underset{\sim}{\boldsymbol{C}}^{e}} + \frac{\partial \widehat{\Psi}}{\partial \underset{\sim}{\boldsymbol{e}}^{e}}\right).\underset{\sim}{\boldsymbol{F}}^{eT} \tag{132}$$

$$\underset{\sim}{\boldsymbol{s}} = 2\underset{\sim}{\boldsymbol{\chi}}^{p-1}.\rho\frac{\partial \widehat{\Psi}}{\partial \underset{\sim}{\boldsymbol{e}}^{e}}.\underset{\sim}{\boldsymbol{\chi}}^{p-T} \tag{133}$$

where the interplay between elastic macro and microstrains is visible. The introduction of the relative strain measure $\underset{\sim}{\boldsymbol{e}}^e$ can be considered as a standard approach in order to achieve coupling terms in the strain energy function.

5.3.2. Alternative decomposition of the microstrain gradient

The existence of an intermediate configuration for which all stress measures are released simultaneously requires a decomposition of the microstrain gradient different from (114). An alternative decomposition similar to that proposed for microdilatation media in Eq. (42) allowing such a definition of the intermediate configuration is

$$\underset{\approx}{\boldsymbol{K}} = \underset{\approx}{\boldsymbol{K}}^e . \underset{\sim}{\boldsymbol{F}}^p + \underset{\approx}{\boldsymbol{K}}^p \tag{134}$$

It leads to a modified hyperelastic relation (121):

$$\underset{\approx}{\boldsymbol{S}} = \rho \frac{\partial \Psi}{\partial \underset{\approx}{\boldsymbol{K}}^e} . \underset{\sim}{\boldsymbol{F}}^{eT} \tag{135}$$

which has the same form as for pure hyperelastic microstrain materials (see (104)). When the decomposition (134) is adopted, the definition of the Mandel stress tensor (122), which is the driving force for plastic flow, is modified as follows:

$$\underset{\sim}{\boldsymbol{\Sigma}} = \underset{\sim}{\boldsymbol{F}}^{eT} . \underset{\sim}{\boldsymbol{\tau}} . \underset{\sim}{\boldsymbol{F}}^{e-T} + \underset{\approx}{\boldsymbol{K}}^{eT} : \underset{\approx}{\boldsymbol{S}} . \underset{\sim}{\boldsymbol{F}}^{e-T} \tag{136}$$

In this formula, the transpose of the elastic microstrain gradient is defined as: $K^e_{ijm} = K^{eT}_{mij}$. The residual dissipation still takes the form (125). When a dissipation potential $\Omega(\underset{\sim}{\boldsymbol{\Sigma}}, \underset{\sim}{\mathscr{S}}, \underset{\approx}{\boldsymbol{S}}_0, R)$ exists, the flow rules are still given by the relations (126)–(129). When furthermore the potential is convex with respect to all its arguments, the positivity of dissipation as predicted by the model is ensured.

5.4. Infinitesimal case

The geometrically linear microstrain theory can now be derived from the previous finite strain situation. The context of small perturbations is adopted implying small deformations and microstrains. The additional degrees of freedom are the components of the symmetric second order tensor ${}^\chi\underset{\sim}{\boldsymbol{\varepsilon}} = \underset{\sim}{\boldsymbol{\chi}}^s - \underset{\sim}{\boldsymbol{1}}$ (see Eq. (43)):

$$DOF = \{\underline{\mathbf{u}}, {}^\chi\underset{\sim}{\boldsymbol{\varepsilon}}\} \tag{137}$$

$$STRAIN = \{\underset{\sim}{\boldsymbol{\varepsilon}}, \underset{\sim}{\boldsymbol{e}} := \underset{\sim}{\boldsymbol{\varepsilon}} - {}^\chi\underset{\sim}{\boldsymbol{\varepsilon}}, \underset{\approx}{\boldsymbol{K}} := {}^\chi\underset{\sim}{\boldsymbol{\varepsilon}} \otimes \boldsymbol{\nabla}\} \tag{138}$$

The linear relative strain $\underset{\sim}{\boldsymbol{e}}$ has been retained in the set of strain measures to make coupling terms more apparent in the following linearized balance and constitutive equations. Indeed, it is the deviation of the microstrain from the macrostrain which actually requires additional work. This represents no loss of generality. Within the small perturbation framework, the power of internal forces simply reads:

$$p^{(i)} = \underset{\sim}{\boldsymbol{\sigma}} : \dot{\underset{\sim}{\boldsymbol{\varepsilon}}} + \underset{\sim}{\boldsymbol{s}} : (\dot{\underset{\sim}{\boldsymbol{\varepsilon}}} - {}^\chi\dot{\underset{\sim}{\boldsymbol{\varepsilon}}}) + \underset{\approx}{\boldsymbol{S}} \,\vdots\, {}^\chi\dot{\underset{\sim}{\boldsymbol{\varepsilon}}} \otimes \boldsymbol{\nabla} = \underset{\sim}{\boldsymbol{\sigma}} : \dot{\underset{\sim}{\boldsymbol{\varepsilon}}} + \underset{\sim}{\boldsymbol{s}} : \dot{\underset{\sim}{e}} + \underset{\approx}{\boldsymbol{S}} \,\vdots\, \dot{\underset{\approx}{\boldsymbol{K}}} \tag{139}$$

The effective stress tensor $\underset{\sim}{\boldsymbol{\sigma}} + \underset{\sim}{\boldsymbol{s}}$ is nothing but the linear version of $\underset{\sim}{\boldsymbol{\tau}}$ introduced in 5.1. All strain measures are decomposed into elastic and plastic parts as

$$\underset{\sim}{\boldsymbol{\varepsilon}} = \underset{\sim}{\boldsymbol{\varepsilon}}^e + \underset{\sim}{\boldsymbol{\varepsilon}}^p, \quad \underset{\sim}{\boldsymbol{e}} = \underset{\sim}{\boldsymbol{e}}^e + \underset{\sim}{\boldsymbol{e}}^p, \quad \underset{\approx}{\boldsymbol{K}} = \underset{\approx}{\boldsymbol{K}}^e + \underset{\approx}{\boldsymbol{K}}^p \tag{140}$$

The decompositions are mere linearizations of (112)–(114). In particular, the decomposition of the relative strain tensor is motivated by the linearization of the microstrain tensor (113):

$$ {}^\chi\underset{\sim}{\boldsymbol{C}} = \underset{\sim}{\boldsymbol{\chi}}^T . \underset{\sim}{\boldsymbol{\chi}} = (\underset{\sim}{\boldsymbol{1}} + {}^\chi\underset{\sim}{\boldsymbol{\varepsilon}} - \underset{\sim}{\boldsymbol{\chi}}^a).(\underset{\sim}{\boldsymbol{1}} + {}^\chi\underset{\sim}{\boldsymbol{\varepsilon}} + \underset{\sim}{\boldsymbol{\chi}}^a) \simeq \underset{\sim}{\boldsymbol{1}} + 2\,{}^\chi\underset{\sim}{\boldsymbol{\varepsilon}} \tag{141}$$

$$= \underset{\sim}{\boldsymbol{\chi}}^{pT} . {}^\chi\underset{\sim}{\boldsymbol{C}}^e . \underset{\sim}{\boldsymbol{\chi}}^p = (\underset{\sim}{\boldsymbol{1}} + {}^\chi\underset{\sim}{\boldsymbol{\varepsilon}}^p - \underset{\sim}{\boldsymbol{\chi}}^{ap}).(\underset{\sim}{\boldsymbol{1}} + 2\,{}^\chi\underset{\sim}{\boldsymbol{\varepsilon}}^e).(\underset{\sim}{\boldsymbol{1}} + {}^\chi\underset{\sim}{\boldsymbol{\varepsilon}}^p + \underset{\sim}{\boldsymbol{\chi}}^{ap})$$

$$\simeq \underset{\sim}{\boldsymbol{1}} + 2\,{}^\chi\underset{\sim}{\boldsymbol{\varepsilon}}^e + 2\,{}^\chi\underset{\sim}{\boldsymbol{\varepsilon}}^p \tag{142}$$

where the following notations were introduced: ${}^{\chi}\underset{\sim}{\boldsymbol{\varepsilon}}^p$ (resp. $\underset{\sim}{\boldsymbol{\chi}}^{ap}$) is the (resp. skew–) symmetric part of the finite plastic microstrain $\underset{\sim}{\boldsymbol{\chi}}^p - \underset{\sim}{\mathbf{1}}$. The tensor $({}^{\chi}\underset{\sim}{\boldsymbol{C}}^e - 1)/2$ is here ${}^{\chi}\underset{\sim}{\boldsymbol{\varepsilon}}^e$. The elastic and plastic part of the relative deformation can then be identified:

$$ {}^{\chi}\underset{\sim}{\boldsymbol{\varepsilon}} = {}^{\chi}\underset{\sim}{\boldsymbol{\varepsilon}}^e + {}^{\chi}\underset{\sim}{\boldsymbol{\varepsilon}}^p, \quad \underset{\sim}{\boldsymbol{e}}^e := \underset{\sim}{\boldsymbol{\varepsilon}}^e - {}^{\chi}\underset{\sim}{\boldsymbol{\varepsilon}}^e, \quad \underset{\sim}{\boldsymbol{e}}^p := \underset{\sim}{\boldsymbol{\varepsilon}}^p - {}^{\chi}\underset{\sim}{\boldsymbol{\varepsilon}}^p \tag{143} $$

The free energy is a function

$$ \Psi(\underset{\sim}{\boldsymbol{\varepsilon}}^e, \underset{\sim}{\boldsymbol{e}}^e, \underset{\approx}{\boldsymbol{K}}^e, q) \tag{144} $$

The linearized hyperelastic relations read

$$ \underset{\sim}{\boldsymbol{\sigma}} = \rho \frac{\partial \Psi}{\partial \underset{\sim}{\boldsymbol{\varepsilon}}^e} \tag{145} $$

$$ \underset{\sim}{\boldsymbol{s}} = \rho \frac{\partial \Psi}{\partial \underset{\sim}{\boldsymbol{e}}^e} \tag{146} $$

$$ \underset{\approx}{\boldsymbol{S}} = \rho \frac{\partial \Psi}{\partial \underset{\approx}{\boldsymbol{K}}^e} \tag{147} $$

$$ R = -\rho \frac{\partial \Psi}{\partial q} \tag{148} $$

The residual dissipation is evaluated as

$$ D = \underset{\sim}{\boldsymbol{\sigma}} : \underset{\sim}{\dot{\boldsymbol{\varepsilon}}}^p + \underset{\sim}{\boldsymbol{s}} : \underset{\sim}{\dot{\boldsymbol{e}}}^p + \underset{\approx}{\boldsymbol{S}} \;\vdots\; \underset{\approx}{\dot{\boldsymbol{K}}} + R\dot{q} \tag{149} $$

The flow rules and evolution equations are assumed to derive from a convex dissipation potential $\Omega(\underset{\sim}{\boldsymbol{\sigma}},\ \underset{\sim}{\boldsymbol{s}},\ \underset{\approx}{\boldsymbol{S}},\ R)$:

$$ \underset{\sim}{\dot{\boldsymbol{\varepsilon}}}^p = \frac{\partial \Omega}{\partial \underset{\sim}{\boldsymbol{\sigma}}}, \quad \underset{\sim}{\dot{\boldsymbol{e}}}^p = \frac{\partial \Omega}{\partial \underset{\sim}{\boldsymbol{s}}}, \quad \underset{\approx}{\dot{\boldsymbol{K}}}^p = \frac{\partial \Omega}{\partial \underset{\approx}{\boldsymbol{S}}}, \quad \dot{q} = \frac{\partial \Omega}{\partial R} \tag{150} $$

6. Conclusions

Available higher order theories and a new one have been cast into a hierarchy of continua involving a number of additional degrees of freedom ranging from 1 to 9. The additional degrees of freedom account for the volume change, the rotation and the change in shape of the parallelogram delimited by a triad of vectors attached to a microstructure underlying the material point. The following points are put forward:

1. The continuum theory of media with microdilatation is the cheapest one since it involves a single additional degree of freedom compared to the standard Cauchy theory. It requires three generalized stress tensors of orders 0, 1 and 2 conjugate to the strain rates. A multiplicative decomposition of the microdilatation and a (quasi-)additive decomposition of the microdeformation gradient into elastic and plastic parts were proposed. The corresponding hyperelastic relationships and a proposal for the form of the flow and hardening rules were provided.
2. The microstretch continuum theory combines the features of the microdilatation and Cosserat theories. The proposed finite deformation elastoviscoplasticity theory for microstretch materials is based on a multiplicative decomposition of the relative deformation gradient and (quasi-)additive decompositions of the curvature and microdeformation gradient. The driving force for macro and microplastic flow is represented by generalized Mandel stress tensors that were expressed in terms of the stress tensors present in the balance equations.
3. The microstrain continuum is the missing link between Cosserat (or microstretch) and micromorphic theories. The microstrain theory accounts only for the change of shape of the triad of microvectors and

postulates that its rotation does not influence material response. It introduces 6 additional degrees of freedom associated with the components of a symmetric second order tensor, namely the right Cauchy–Green tensor associated with the microdeformation. In contrast to the full micromorphic theories, the additional stress tensors exhibit symmetry properties. A decomposition of the microstrain tensor into elastic and plastic parts has been deduced from the multiplicative decomposition of the microdeformation. Generalized Mandel stress tensors have also been defined. Micro and macrodeformation are coupled in balance partial differential equations for instance via the introduction of a relative microstrain measure in the free energy function and, possibly, by coupled dissipative hardening mechanisms.

The choice of a well-suited higher order continuum to describe the mechanical response of a given material represents a compromise between the selection of the proper microdeformation mechanism to be accounted for in the modeling, on the one hand, and the computational cost induced by the introduction of additional degrees of freedom, on the other hand. There is currently no systematic nor unique way of selecting such a higher order model for a given microstructure. However, guidelines can be given to motivate such a choice. The first step is to define a triad of microvectors associated with the heterogeneities inside an idealized representative volume element and then to assess its main deformation mode. If microvolume changes are dominant, the microdilatation theory can be a cheap and efficient model. In this spirit, a theory introducing one additional scalar order parameter was applied to model the deformation of polymer films in the reference (Steeb and Diebels, 2004). Problems of growth and remodeling can be addressed by microdilatation related theories as done in Epstein and Maugin (2000) and Steeb and Diebels (2003). When rotation effects are dominant, the Cosserat continuum has proved to be a reliable tool especially in the simulation of strain localization in granular media (Nübel and Huang, 2004). The microstretch theory has found interesting applications for the deformation of biomaterials (Rosenberg and Cimrman, 2003).

The microstrain theory is a good candidate for the modeling of strain localization phenomena in metallic foams (Forest et al., 2005; Dillard et al., 2006). The plastic deformation of such materials is associated with the formation and propagation of intense strain bands. The critical hardening modulus for band formation and the orientation of the bands can be predicted based on bifurcation analyzes in classical compressible elastoplasticity. The post-bifurcation regime, especially the simulation of the development of finite size localization bands, requires the use of a regularization procedure (Forest and Lorentz, 2004). In the specific case of aluminium and nickel foams, it can be shown that the rotation of the cells is not the main deformation mechanisms but rather the crushing or tearing of the cells. As a result, the Cosserat continuum is not appropriate. The microdilatation model is insufficient to affect all kinds of combined shear and opening/compression modes in localization bands. The microstrain continuum seems to be the well suited model, thus sparing 3 degrees of freedom compared to the full micromorphic approach.

An alternative framework for the continuum modeling of size effects in materials is the use of gradient of internal variable theories. Models involving gradients of internal variables are also based on the assumption of local action. In such models the additional variable attached to each material point and called internal variable in the initial classical theory, must be treated as an actual degree of freedom. Accordingly, the variable is not "internal" any more but can be controlled for instance at the boundary of the body. Only the first gradient of this variable is considered in the further development of the theory. This is actually not the presentation adopted in Aifantis (1987) but it has been actualized in Forest et al. (2002) and Forest and Sievert (2003). As a result, the formulation of gradient of internal variable models is similar to that of higher order theories with the difference that the additional degrees of freedom are not necessarily of kinematic nature. Another alternative is to resort to higher grade theories like the second gradient medium presented in Mindlin and Eshel (1968). There are strong links between higher order and higher grade theories. A second grade medium can be regarded as a micromorphic medium with the internal constraint that the microdeformation coincides with the deformation gradient.

The identification of the numerous material parameters involved in the constitutive equations for higher order media is one of the main obstacles to the use of such theories. Two types of recent achievements in this field should lead us to revise this argument. Firstly, the identification of intrinsic length scale parameters from strain field measurements, as exemplified in Geers et al. (1999), is more efficient than looking only at overall material responses. Secondly, computational homogenization methods can be used to predict in a systematic way the parameters of the macromodel starting from a precise description of a representative volume element of the considered heterogeneous material to be replaced by a homogeneous generalized continuum. Examples of extensions of classical homogenization schemes to higher order continua can be found in Forest and Sab (1998), Xun et al. (2004), and Hu et al. (2005).

Two other recent scientific developments plead for an increased use of higher order continua. On the one hand, the initial and deformed microstructures of some materials can be known with unprecedented precision, for instance using 3D microtomography (Dillard et al., 2005). The deformation modes of individual heterogeneities can be studied with great precision to select the best-suited macroscopic continuum model. On the other hand, the increasing capacity of computing facilities and especially parallel computing makes it possible to address finite element problems with a considerable number of degrees of freedom (of the order of more than 1 million even for nonlinear problems, cf. Feyel et al., 1997). The finite element formulation of problems involving higher order continua is rather straightforward, especially in the infinitesimal case. The formal structure of the variational formulation of the problem is identical to the classical one, provided that the number of lines and rows in the vectors of unknowns and the tangent matrices are properly extended (Forest and Lorentz, 2004). As a result the number of additional degrees of freedom at each node in FE computations based on higher order continua should not be a prohibitive obstacle any longer. It is clear that this number should always be kept as small as possible. That is why it is recommended to make use of the full hierarchy listed in Table 3 and to increase progressively the number of additional degrees of freedom for improving gradually material description.

Appendix A. Supplementary data

Supplementary data associated with this article can be found, in the online version, at doi:10.1016/j.ijsolstr.2006.05.012.

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