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Stress gradient continuum theory

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ABSTRACT

A stress gradient continuum theory is presented that fundamentally differs from the well-established strain gradient model. It is based on the assumption that the deviatoric part of the gradient of the Cauchy stress tensor can contribute to the free energy density of solid materials. It requires the introduction of so-called micro-displacement degrees of freedom in addition to the usual displacement components. An isotropic linear elasticity theory is worked out for two-dimensional stress gradient media. The analytical solution of a simple boundary value problem illustrates the essential differences between stress and strain gradient models.

1. Introduction

Much attention has been dedicated to strain gradient effects in continuum mechanics and materials sciences in the last fifty years, since the pioneering work of Toupin (1962) and Mindlin (1965). The second gradient theory represents an extension of the classical Cauchy continuum by incorporating the effect of the second gradient of the displacement field into the balance and constitutive equations of the medium, in addition to the usual first gradient of the displacement. It must be noted that the second gradient of the displacement theory and the strain gradient model represent the same continuum, due to compatibility conditions, as shown by Mindlin and Eshel (1968). Higher order stresses, called hyperstresses or double stresses, must be included in the theory as the quantities conjugate to the components of the second gradient of displacement. This results in an extended balance of momentum equation and additional boundary conditions. These equations have been derived first by Toupin and Mindlin using variations of the elastic energy, and then by Germain (1973a) by means of the method of virtual power. A derivation *à la Cauchy*, i.e. based on the representation of generalized contact forces, was established more recently by Noll and Virga (1990) and Dell'Isola and Seppecher (1995, 1997), due to the fact that the Neumann conditions are rather intricate in a second gradient medium.

In contrast, the role of stress gradients has been the subject of little attention, if one excepts its introduction in fatigue crack initiation models at notches and holes of various sizes as studied in the engineering community (Bascoul and Maso, 1981; Lahellec et al., 2005). More recently, a stress-gradient based criterion has been proposed for dislocation nucleation in crystals at a nano-scale (Acharya and Miller, 2004).

Regarding continuum mechanics, there is a long-standing misconception or, at least, ambiguity going through the whole literature on generalized continua, that implicitly considers that the strain gradient theory can also be regarded as a stress gradient model. The stress gradient can be found in Aifantis gradient elasticity model (Aifantis, 1992, 2009; Ru and Aifantis, 1993; Lazar et al., 2006) in the form:

$$\underset{\sim}{\boldsymbol{\tau}} = \underset{\sim}{\boldsymbol{\sigma}} - c\nabla^2 \underset{\sim}{\boldsymbol{\sigma}} \tag{1}$$

where $\underset{\sim}{\boldsymbol{\tau}}$ is an effective stress tensor whose divergence vanishes in the absence of body forces and c is a material parameter associated with a characteristic length. In a Cartesian orthonormal coordinate system the Laplace operator is applied to each component of the matrix. The Laplace term arises as the divergence of the gradient of the stress field. However, it can be shown that the presence of the stress gradient in this model is the result of a specific constitutive assumption made in Mindlin's strain gradient elasticity (Forest and Aifantis, 2010). Accordingly, Aifantis gradient elasticity must be considered as a strain gradient model.

As a result, the question arises whether it is possible to formulate a stress gradient continuum theory describing

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size-dependent material properties and how much it may differ from the well-established strain gradient model. Generalized continuum theories include strain gradient, micromorphic and non-local models (Eringen, 1999, 2002) that introduce higher order strain gradients, additional degrees of freedom or non-local kernels, but not explicitly the stress gradient as a primary variable. To the knowledge of the authors, a stress gradient continuum theory does not exist in the literature. The objective of the present work is to establish the framework of such a stress gradient theory and to illustrate the predicted behavior in the case of linear isotropic elasticity. In particular we will prove that this theory fundamentally differs from Mindlin's strain gradient model: stress gradient and strain gradient models are two distinct representations of the continuum.

The presented stress gradient theory for the 3D continuum is similar to the so-called bending-gradient theory recently proposed by Lebée and Sab (2011a,b) for out-of-plane loaded elastic thick laminated plates. In this plate theory, the stress energy density is a function of the local bending moment and its gradient. Moreover, these authors show that the well-known Reissner plate theory (Reissner, 1945) for out-of-plane loaded elastic thick homogeneous plates actually is a degenerated case of their bending-gradient theory. In the bending-gradient theory the stress energy density is a function of the local bending moment and of the spherical part of its gradient which coincides with the classical shear forces, see also (Cecchi and Sab, 2007; Nguyen et al., 2007, 2008).

A systematic comparison of the new model will be drawn with Mindlin's second gradient theory and Germain's general micromorphic theory (Germain, 1973b). The pros and the cons of each model will be addressed at different stages of the discussion. In particular, both computational and physical, or more precisely micro-mechanical, arguments will be raised to characterize the new approach.

For the sake of brevity, the theory is developed within the small deformation framework and under static conditions. A first construction of the theory is proposed in Section 2 for elastic stress gradient solids. The general theory, independent of the constitutive behavior, is presented based on the method of virtual power in Section 3. A two-dimensional linear isotropic elasticity theory is formulated in Section 4. Finally, the responses of the stress gradient and strain gradient continua are compared in Section 5 in the case of a generic boundary value problem involving periodic body forces.

Tensors of zeroth, first, second, third and fourth ranks are respectively denoted by a, $\underline{\boldsymbol{a}}$, $\underset{\sim}{\boldsymbol{a}}$, $\underset{\sim}{\underline{\boldsymbol{a}}}$ (or $\underset{\simeq}{\boldsymbol{a}}$) and $\underset{\approx}{\boldsymbol{a}}$. The intrinsic notation is usually complemented by the index notation to avoid any confusion. The tensor product is denoted by $\otimes$. We also define the symmetrized tensor product using the following notations:

$$\underline{\boldsymbol{a}} \overset{s}{\otimes} \underline{\boldsymbol{b}} = \frac{1}{2}(\underline{\boldsymbol{a}} \otimes \underline{\boldsymbol{b}} + \underline{\boldsymbol{b}} \otimes \underline{\boldsymbol{a}}), \quad a_{(i}b_{j)} = \frac{1}{2}(a_i b_j + a_j b_i) \tag{2}$$

The nabla operator is denoted by ∇ and operates as follows on a vector field, in a Cartesian orthonormal basis $(\underline{\boldsymbol{e}}_1, \underline{\boldsymbol{e}}_2, \underline{\boldsymbol{e}}_3)$:

$$\underline{\boldsymbol{u}}(\underline{\boldsymbol{x}}) \otimes \nabla = \frac{\partial u_i}{\partial x_j} \underline{\boldsymbol{e}}_i \otimes \underline{\boldsymbol{e}}_j = u_{i,j} \underline{\boldsymbol{e}}_i \otimes \underline{\boldsymbol{e}}_j \tag{3}$$

The Cauchy stress tensor is denoted by $\underset{\sim}{\boldsymbol{\sigma}}$ and has the following components:

$$\underset{\sim}{\boldsymbol{\sigma}} = \sigma_{ij} \underline{\boldsymbol{e}}_i \otimes \underline{\boldsymbol{e}}_j \tag{4}$$

The stress gradient tensor is defined as

$$\underset{\sim}{\boldsymbol{\sigma}} \otimes \nabla = \sigma_{ij,k} \underline{\boldsymbol{e}}_i \otimes \underline{\boldsymbol{e}}_j \otimes \underline{\boldsymbol{e}}_k \tag{5}$$

Its divergence is the vector

$$\underset{\sim}{\boldsymbol{\sigma}} \cdot \nabla = \sigma_{ij,j} \underline{\boldsymbol{e}}_i \tag{6}$$

2. Formulation of a stress gradient elasticity model

2.1. Algebra of deviatoric third rank tensors

The stress gradient tensor is the third rank tensor defined by Eq. (5). Its components are symmetric with respect to the two first indices. In this work, the space of third rank tensors that are symmetric with respect to the first two indices is denoted by $\mathcal{R}$. It is a vector space of dimension 18 which is endowed with the scalar product:

$$\underset{\simeq}{\boldsymbol{R}} \therefore \underset{\simeq}{\boldsymbol{R}} = R_{ijk} R_{ijk}, \quad \forall \underset{\simeq}{\boldsymbol{R}} \in \mathcal{R} \tag{7}$$

Each tensor $\underset{\simeq}{\boldsymbol{R}} \in \mathcal{R}$ can then be decomposed into a spherical part $\underset{\simeq}{\boldsymbol{R}}^s \in \mathcal{S} \subset \mathcal{R}$ and a deviatoric part $\underset{\simeq}{\boldsymbol{R}}^d \in \mathcal{D} \subset \mathcal{R}$:

$$\underset{\simeq}{\boldsymbol{R}} = \underset{\simeq}{\boldsymbol{R}}^s + \underset{\simeq}{\boldsymbol{R}}^d \tag{8}$$

with

$$R^s_{ijk} = \frac{1}{4}(R_{ilm}\delta_{lm}\delta_{jk} + R_{jlm}\delta_{lm}\delta_{ik}) \tag{9}$$

Here, the space $\mathcal{D}$ is the subset of $\mathcal{R}$ containing the deviatoric elements $\underset{\simeq}{\boldsymbol{R}}$ such that

$$\underset{\simeq}{\boldsymbol{R}} : \underset{\sim}{\boldsymbol{1}} = 0, \quad R_{ijk}\delta_{jk} = 0 \tag{10}$$

where $\underset{\sim}{\boldsymbol{1}}$ is the second rank identity tensor and δ_{ij} is the Kronecker symbol. It follows that $\mathcal{S} = \mathcal{D}^{\perp}$ and $\mathcal{R} = \mathcal{D} \oplus \mathcal{S}$.

We finally note that the spherical part of the stress gradient is directly related to the divergence of the stress tensor by

$$(\underset{\sim}{\boldsymbol{\sigma}} \otimes \nabla)^s_{ijk} = \frac{1}{4}(\sigma_{im,m}\delta_{jk} + \sigma_{jm,m}\delta_{ik}) \tag{11}$$

or equivalently,

$$(\underset{\sim}{\boldsymbol{\sigma}} \otimes \nabla) : \underset{\sim}{\boldsymbol{1}} = (\underset{\sim}{\boldsymbol{\sigma}} \otimes \nabla)^s : \underset{\sim}{\boldsymbol{1}} = \underset{\sim}{\boldsymbol{\sigma}} \cdot \nabla \tag{12}$$

The previous definitions are valid in the physical three-dimensional space. However, we will also need expressions in the two-dimensional case. In the purely two-dimensional case, the formula (9) must be replaced by

$$R^s_{ijk} = \frac{1}{3}(R_{ilm}\delta_{lm}\delta_{jk} + R_{jlm}\delta_{lm}\delta_{ik})$$

where the indices i, j, k only take the values 1, 2. In the two-dimensional case, the matrix form of the decomposition (8) becomes

$$\begin{bmatrix} R_{111} \\ R_{122} \\ R_{221} \\ R_{222} \\ R_{211} \\ R_{112} \end{bmatrix} = \begin{bmatrix} R^s_{111} \\ R^s_{122} \\ R^s_{221} \\ R^s_{222} \\ R^s_{211} \\ R^s_{112} \end{bmatrix} + \begin{bmatrix} R^d_{111} \\ R^d_{122} \\ R^d_{221} \\ R^d_{222} \\ R^d_{211} \\ R^d_{112} \end{bmatrix} \tag{13}$$

with

$$\begin{bmatrix} R^s_{111} \\ R^s_{122} \\ R^s_{221} \\ R^s_{222} \\ R^s_{211} \\ R^s_{112} \end{bmatrix} = \begin{bmatrix} \frac{2}{3}(R_{111}+R_{122}) \\ \frac{1}{3}(R_{111}+R_{122}) \\ 0 \\ \frac{2}{3}(R_{211}+R_{222}) \\ \frac{1}{3}(R_{211}+R_{222}) \\ 0 \end{bmatrix} \quad \text{and}$$

$$\begin{bmatrix} R^d_{111} \\ R^d_{122} \\ R^d_{221} \\ R^d_{222} \\ R^d_{211} \\ R^d_{112} \end{bmatrix} = \begin{bmatrix} \frac{1}{3}(R_{111}-2R_{122}) \\ -\frac{1}{3}(R_{111}-2R_{122}) \\ R_{221} \\ \frac{1}{3}(R_{222}-2R_{211}) \\ -\frac{1}{3}(R_{222}-2R_{211}) \\ R_{112} \end{bmatrix} \tag{14}$$

It can be checked that, in the two-dimensional case, a deviatoric tensor $\underset{\approx}{\boldsymbol{R}} \in \mathcal{D}$ is such that

$$R_{111}+R_{122}=0, \quad R_{211}+R_{222}=0 \tag{15}$$

2.2. Construction of an elastic stress gradient material theory

We consider a homogeneous elastic Cauchy material occupying the domain Ω. Clamping conditions are imposed at its boundary $\partial\Omega$ where the displacement vector vanishes: $\underline{\boldsymbol{u}}(\underline{\boldsymbol{x}})=0,\ \forall \underline{\boldsymbol{x}} \in \partial\Omega$. The solid Ω is subjected to body forces $\underline{\boldsymbol{f}}$. The variational formulation of the considered boundary value problem consists in minimizing the complementary energy

$$W^*(\underset{\sim}{\boldsymbol{\sigma}}) = \int_\Omega \frac{1}{2}\underset{\sim}{\boldsymbol{\sigma}} : \underset{\approx}{\boldsymbol{S}} : \underset{\sim}{\boldsymbol{\sigma}}\, dV$$

where $\underset{\approx}{\boldsymbol{S}}$ is the elastic compliance tensor. The minimization takes place with respect to all statically admissible stress fields $\underset{\sim}{\boldsymbol{\sigma}} \in SA$ with

$$SA = \{\underset{\sim}{\boldsymbol{\sigma}} | \underset{\sim}{\boldsymbol{\sigma}}\cdot\nabla + \underline{\boldsymbol{f}} = 0 \text{ in } \Omega\}$$

The proposed stress gradient theory is based on the idea that the deviatoric part of the stress gradient can contribute to the energy, in addition to the stress tensor itself. The space SA^{SG} of statically admissible fields now contains the elements $\underset{\sim}{\boldsymbol{\sigma}}$ and $\underset{\approx}{\boldsymbol{R}}$ that fulfill the following conditions on Ω:

$$\underset{\sim}{\boldsymbol{\sigma}}\cdot\nabla + \underline{\boldsymbol{f}} = 0 \tag{16}$$

$$\underset{\approx}{\boldsymbol{R}} = (\underset{\sim}{\boldsymbol{\sigma}}\otimes\nabla)^d \tag{17}$$

Note that the spherical part of the stress gradient is entirely determined by the first balance equation (16) so that its deviatoric part only, denoted by $\underset{\approx}{\boldsymbol{R}}$, can enter the stress energy potential.

We introduce the stress energy density potential $w^*(\underset{\sim}{\boldsymbol{\sigma}}, \underset{\approx}{\boldsymbol{R}})$. The solution of the boundary value problem considered previously is now obtained by minimizing the complementary energy functional[1]

$$W^{*SG}(\underset{\sim}{\boldsymbol{\sigma}}) = \int_\Omega w^*(\underset{\sim}{\boldsymbol{\sigma}}, \underset{\approx}{\boldsymbol{R}})dV \tag{18}$$

with respect to all $(\underset{\sim}{\boldsymbol{\sigma}}, \underset{\approx}{\boldsymbol{R}}) \in SA^{SG}$.

To obtain the dual variational formulation of the previous stress gradient problem, we multiply Eq. (16) by the displacement vector and integrate by parts over Ω:

$$\int_\Omega \underset{\sim}{\boldsymbol{\sigma}} : \underset{\sim}{\boldsymbol{\varepsilon}}\, dv = \int_\Omega \underline{\boldsymbol{f}}\cdot\underline{\boldsymbol{u}}\, dv + \int_{\partial\Omega} (\underset{\sim}{\boldsymbol{\sigma}}\cdot\underline{\boldsymbol{n}})\cdot\underline{\boldsymbol{u}}\, da,$$

$$\int_\Omega \sigma_{ij}\varepsilon_{ij} dV = \int_\Omega f_i u_i dV + \int_{\partial\Omega} \sigma_{ij} n_j u_i da \tag{19}$$

where $\underset{\sim}{\boldsymbol{\varepsilon}}$ is the strain tensor defined as the symmetric part of the displacement gradient field.

In a similar way, we multiply Eq. (17) by new kinematic variables $\underset{\approx}{\boldsymbol{\Phi}} \in \mathcal{D}$ and integrate over Ω:

$$\begin{aligned} \int_\Omega \underset{\approx}{\boldsymbol{R}} \therefore \underset{\approx}{\boldsymbol{\Phi}} - (\underset{\sim}{\boldsymbol{\sigma}}\otimes\nabla)^d \therefore \underset{\approx}{\boldsymbol{\Phi}}\, dv &= \int_\Omega \underset{\approx}{\boldsymbol{R}} \therefore \underset{\approx}{\boldsymbol{\Phi}} - (\underset{\sim}{\boldsymbol{\sigma}}\otimes\nabla) \therefore \underset{\approx}{\boldsymbol{\Phi}}\, dv \\ &= \int_\Omega R_{ijk}\Phi_{ijk} - \sigma_{ij,k}\Phi_{ijk}\, dv \\ &= 0 \end{aligned} \tag{20}$$

Integration by parts of the previous equation gives:

$$\int_\Omega \underset{\sim}{\boldsymbol{\sigma}} : \underset{\approx}{\boldsymbol{\Phi}}\cdot\nabla + \underset{\approx}{\boldsymbol{R}} \therefore \underset{\approx}{\boldsymbol{\Phi}}\, dv = \int_{\partial\Omega} \underset{\sim}{\boldsymbol{\sigma}} : \underset{\approx}{\boldsymbol{\Phi}}\cdot\underline{\boldsymbol{n}}\, da \tag{21}$$

$$\int_\Omega \sigma_{ij}\Phi_{ijk,k} + R_{ijk}\Phi_{ijk} dV = \int_{\partial\Omega} \sigma_{ij}\Phi_{ijk} n_k da \tag{22}$$

Summing up Eqs. (19) and (21), the following variational formulation is obtained

$$\begin{aligned} &\int_\Omega (\underset{\sim}{\boldsymbol{\sigma}} : (\underset{\sim}{\boldsymbol{\varepsilon}} + \underset{\approx}{\boldsymbol{\Phi}}\cdot\nabla) + \underset{\approx}{\boldsymbol{R}} \therefore \underset{\approx}{\boldsymbol{\Phi}})\, dv \\ &= \int_\Omega \underline{\boldsymbol{f}}\cdot\underline{\boldsymbol{u}}\, dv + \int_{\partial\Omega} \underset{\sim}{\boldsymbol{\sigma}} : (\underline{\boldsymbol{u}} \overset{s}{\otimes} \underline{\boldsymbol{n}} + \underset{\approx}{\boldsymbol{\Phi}}\cdot\underline{\boldsymbol{n}})\, da \end{aligned} \tag{23}$$

$$\begin{aligned} &\int_\Omega \left(\sigma_{ij}(\varepsilon_{ij} + \Phi_{ijk,k}) + R_{ijk}\Phi_{ijk}\right) dV \\ &= \int_\Omega f_i u_i dV + \int_{\partial\Omega} \sigma_{ij}(u_{(i}n_{j)} + \Phi_{ijk}n_k)da \end{aligned} \tag{24}$$

This expression provides the definition of the strain measures that are conjugate to the stress and stress gradient tensors. The stress tensor is conjugate to the following generalized strain measure

$$\underset{\sim}{\boldsymbol{e}} := \underset{\sim}{\boldsymbol{\varepsilon}} + \underset{\approx}{\boldsymbol{\Phi}}\cdot\nabla, \quad e_{ij} = \varepsilon_{ij} + \Phi_{ijk,k} \tag{25}$$

whereas the new kinematic degrees of freedom $\underset{\approx}{\boldsymbol{\Phi}}$ are conjugate to $\underset{\approx}{\boldsymbol{R}}$. It is recalled that both $\underset{\approx}{\boldsymbol{\Phi}}$ and $\underset{\approx}{\boldsymbol{R}}$ are deviatoric third order

[1] *SG* stands for stress gradient.

tensors. The conjugate stress and strain measures are related by the constitutive equations:

$$\underset{\sim}{\boldsymbol{e}} = \frac{\partial w^*(\underset{\sim}{\boldsymbol{\sigma}}, \underset{\approx}{\underline{\boldsymbol{R}}})}{\partial \underset{\sim}{\boldsymbol{\sigma}}} \quad \text{and} \quad \underset{\approx}{\underline{\boldsymbol{\Phi}}} = \frac{\partial w^*(\underset{\sim}{\boldsymbol{\sigma}}, \underset{\approx}{\underline{\boldsymbol{R}}})}{\partial \underset{\approx}{\underline{\boldsymbol{R}}}}$$

The variational formulation (23) also provides the new clamping conditions for the considered boundary value problem:

$$\underline{\boldsymbol{u}} \overset{s}{\otimes} \underline{\boldsymbol{n}} + \underset{\approx}{\underline{\boldsymbol{\Phi}}} \cdot \underline{\boldsymbol{n}} = 0, \quad u_{(i}n_{j)} + \Phi_{ijk}n_k = 0 \tag{26}$$

3. Method of virtual power for the stress gradient medium

The balance equations and general boundary conditions for stress gradient media can also be formulated independently of the type of material behavior. This is done now by means of the method of virtual power as used by Germain in the case of strain gradient media (Germain, 1973a; Maugin, 1980).

3.1. Derivation of balance equations and boundary conditions

The usual set of kinematic degrees of freedom (DOF) available at each material point, namely the components of the displacement vector, is extended to incorporate the components of an element $\underset{\approx}{\underline{\boldsymbol{\Phi}}} \in \mathcal{D}$:

$$\text{DOF} = \{\underline{\boldsymbol{u}},\ \underset{\approx}{\underline{\boldsymbol{\Phi}}}\}, \quad \text{DOF} = \{u_i,\ \Phi_{ijk}\} \tag{27}$$

The two sets of kinematic DOF are assumed to be independent and to both have the dimension of length. The new kinematic DOF Φ_{ijk} are symmetric with respect to the first two indices. They are deviatoric in the sense defined in Section 2.1 for third rank tensors belonging to $\mathcal{D}$. They are called *micro-displacements*, by analogy to the micro-deformation degrees of freedom introduced in the micromorphic theory by Eringen (1999). In the 3D case, the number of degrees of freedom of the theory at a material point is 18, corresponding to 3 components of displacements and 15 micro-displacement components, due to the constraint that $\underset{\approx}{\underline{\boldsymbol{\Phi}}}^s = 0$.

The proposed theory is a first gradient theory with respect to the sets of degrees of freedom. The gradient of the displacement is assumed to work with the usual symmetric Cauchy stress. Furthermore we assume that the divergence of the micro-displacement also contributes to the overall work. The following set is therefore defined:

$$GRAD = \{\underline{\boldsymbol{u}} \otimes \nabla,\ \underset{\approx}{\underline{\boldsymbol{\Phi}}},\ \underset{\approx}{\underline{\boldsymbol{\Phi}}} \cdot \nabla\}, \quad GRAD = \{u_{i,j},\ \Phi_{ijk},\ \Phi_{ijk,k}\} \tag{28}$$

The following form of the virtual work density of internal forces is postulated, as a linear form with respect to the elements of *GRAD*:

$$\begin{aligned} p^{(i)}(\underline{\boldsymbol{u}}^*, \underset{\approx}{\underline{\boldsymbol{\Phi}}}^*) &= \underset{\sim}{\boldsymbol{\sigma}} : \underline{\boldsymbol{u}}^* \otimes \nabla + \underset{\approx}{\underline{\boldsymbol{R}}} \therefore \underset{\approx}{\underline{\boldsymbol{\Phi}}}^* + \underset{\sim}{\boldsymbol{\sigma}} : \underset{\approx}{\underline{\boldsymbol{\Phi}}}^* \cdot \nabla \\ &= \sigma_{ij}u^*_{i,j} + R_{ijk}\Phi^*_{ijk} + \sigma_{ij}\Phi^*_{ijk,k} \end{aligned} \tag{29}$$

where $\underline{\boldsymbol{u}}^*$ and $\underset{\approx}{\underline{\boldsymbol{\Phi}}}^*$ are virtual displacement and micro-displacement fields. The stress conjugate to the gradient of displacement field is the symmetric Cauchy stress. We have assumed that the generalized stress conjugate to the divergence of the micro-displacement tensor is equal to the Cauchy stress itself. The generalized stress $\underset{\approx}{\underline{\boldsymbol{R}}} \in \mathcal{D}$ is conjugate to the micro-displacement tensor and shares the same symmetry properties. Accordingly, both $\underset{\approx}{\underline{\boldsymbol{\Phi}}}$ and $\underset{\approx}{\underline{\boldsymbol{R}}}$ are deviatoric third rank tensors. We can compute the virtual work of internal forces on any sub-domain $V \subset \Omega$ and integrate by parts:

$$\begin{aligned} \mathcal{P}^{(i)}(\underline{\boldsymbol{u}}^*, \underset{\approx}{\underline{\boldsymbol{\Phi}}}^*) &= \int_V p^{(i)}(\underline{\boldsymbol{u}}^*, \underset{\approx}{\underline{\boldsymbol{\Phi}}}^*)\, dv \\ &= \int_{\partial V} \underset{\sim}{\boldsymbol{\sigma}} : (\underline{\boldsymbol{u}}^* \overset{s}{\otimes} \underline{\boldsymbol{n}} + \underset{\approx}{\underline{\boldsymbol{\Phi}}} \cdot \underline{\boldsymbol{n}})\, da - \int_V ((\underset{\sim}{\boldsymbol{\sigma}} \cdot \nabla) \cdot \underline{\boldsymbol{u}}^* \\ &\quad + ((\underset{\sim}{\boldsymbol{\sigma}} \otimes \nabla)^d - \underset{\approx}{\underline{\boldsymbol{R}}}) \therefore \underset{\approx}{\underline{\boldsymbol{\Phi}}})\, dv \end{aligned} \tag{30}$$

This prompts us to introduce the following form for the work density of contact forces at material points on the boundary ∂V, with normal vector $\underline{\boldsymbol{n}}$:

$$p^{(c)}(\underline{\boldsymbol{u}}^*, \underset{\approx}{\underline{\boldsymbol{\Phi}}}^*) = \underset{\sim}{\boldsymbol{T}} : (\underline{\boldsymbol{u}}^* \overset{s}{\otimes} \underline{\boldsymbol{n}} + \underset{\approx}{\underline{\boldsymbol{\Phi}}}^* \cdot \underline{\boldsymbol{n}}) = T_{ij}(u^*_i n_j + \Phi^*_{ijk} n_k) \tag{31}$$

where $\underset{\sim}{\boldsymbol{T}}$ is a symmetric stress tensor prescribed at the surface.

The work density of forces acting at a distance is then

$$p^{(e)}(\underline{\boldsymbol{u}}^*, \underset{\approx}{\underline{\boldsymbol{\Phi}}}^*) = \underline{\boldsymbol{f}} \cdot \underline{\boldsymbol{u}}^* + \underset{\approx}{\underline{\boldsymbol{F}}} \therefore \underset{\approx}{\underline{\boldsymbol{\Phi}}}^* = f_i u^*_i + F_{ijk}\Phi^*_{ijk} \tag{32}$$

where $\underline{\boldsymbol{f}}$ and $\underset{\approx}{\underline{\boldsymbol{F}}}$ are given first rank and third rank volume simple forces, respectively.

In the static case, the principle of virtual work then stipulates that, for all subdomain V and for all virtual fields $\underline{\boldsymbol{u}}^*$ and $\underset{\approx}{\underline{\boldsymbol{\Phi}}}^*$, we have

$$\int_V p^{(i)}(\underline{\boldsymbol{u}}^*, \underset{\approx}{\underline{\boldsymbol{\Phi}}}^*)\, dv = \int_V p^{(e)}(\underline{\boldsymbol{u}}^*, \underset{\approx}{\underline{\boldsymbol{\Phi}}}^*)\, dv + \int_{\partial V} p^{(c)}(\underline{\boldsymbol{u}}^*, \underset{\approx}{\underline{\boldsymbol{\Phi}}}^*)\, da \tag{33}$$

Integration by parts and application of the principle for all subdomains lead to the field equations of the problem

$$\underset{\sim}{\boldsymbol{\sigma}} \cdot \nabla + \underline{\boldsymbol{f}} = 0, \quad (\underset{\sim}{\boldsymbol{\sigma}} \otimes \nabla)^d - \underset{\approx}{\underline{\boldsymbol{R}}} + \underset{\approx}{\underline{\boldsymbol{F}}} = 0, \quad \forall \underline{\boldsymbol{x}} \in \Omega \tag{34}$$

$$\sigma_{ij,j} + f_i = 0, \quad (\sigma_{ij,k})^d - R_{ijk} + F_{ijk} = 0, \quad \forall \underline{\boldsymbol{x}} \in \Omega \tag{35}$$

and the corresponding boundary conditions:

$$\underset{\sim}{\boldsymbol{T}} = \underset{\sim}{\boldsymbol{\sigma}}, \quad T_{ij} = \sigma_{ij} \quad \forall \underline{\boldsymbol{x}} \in \partial\Omega \tag{36}$$

As a result, the stress gradient theory allows to prescribe all the components of the stress tensor at the boundary. There are six dual conditions which amount to fixing the six kinematic components of $\underline{\boldsymbol{u}} \overset{s}{\otimes} \underline{\boldsymbol{n}} + \underset{\approx}{\underline{\boldsymbol{\Phi}}} \cdot \underline{\boldsymbol{n}}$ at the boundary.

At this stage, one may ask why the theory has 18 independent degrees of freedom and only six boundary conditions. This is due to the fact that once the 6 components of the stress tensor σ_{ij} are known at the boundary, the 12 independent components of the tangential derivative of the stress are automatically known.

3.2. Comparison with Mindlin's strain gradient model

A parallel can be drawn between Mindlin's original strain gradient model and the proposed stress gradient theory. Fundamental differences arise. The pros and the cons regarding possible computational advantages of the stress gradient model are pointed out.

Mindlin's strain gradient model or, equivalently, second gradient of displacement theory relies on the usual three displacement degrees of freedom. From the three equivalent formulations reported by Mindlin and Eshel (1968), we adopt the presentation based on the second gradient of displacement. The work density of internal forces takes the following form:

$$p^{(i)}(\underline{\boldsymbol{u}}) = \underset{\sim}{\boldsymbol{\sigma}} : \underline{\boldsymbol{u}} \otimes \nabla + \underset{\approx}{\underline{\boldsymbol{S}}} \therefore \underline{\boldsymbol{u}} \otimes \nabla \otimes \nabla = \sigma_{ij}\, u_{i,j} + S_{ijk}\, u_{i,jk} \tag{37}$$

where the hyper-stress tensor $\underset{\approx}{\boldsymbol{S}}$, symmetric with respect to its two last indices, is introduced. The work density of forces acting at a distance allows the introduction of simple, double and triple body forces (Germain, 1973a):

$$p^{(e)}(\underline{\boldsymbol{u}}) = \underline{\boldsymbol{f}} \cdot \underline{\boldsymbol{u}} + \underset{\sim}{\boldsymbol{F}} : \underline{\boldsymbol{u}} \otimes \nabla + \underset{\approx}{\boldsymbol{F}} \therefore \underline{\boldsymbol{u}} \otimes \nabla \otimes \nabla = f_i\, u_i + F_{ij}\, u_{i,j} + F_{ijk}\, u_{i,jk} \tag{38}$$

where $\underset{\sim}{\boldsymbol{F}}$ generally contains a skew-symmetric part accounting for volume couples. For the representation of contact forces, it is necessary to introduce normal and tangent gradient operators at a boundary. They are called ∇^n and ∇^t, respectively. In the case of a vector field, they read

$$\underline{\boldsymbol{u}} \otimes \nabla = \underline{\boldsymbol{u}} \otimes \nabla^n + \underline{\boldsymbol{u}} \otimes \nabla^t, \quad u_{i,j} = D^n_j u_i + D^t_j u_i \tag{39}$$

with[2]

$$\underline{\boldsymbol{u}} \otimes \nabla^n := ((\underline{\boldsymbol{u}} \otimes \nabla) \cdot \underline{\boldsymbol{n}}) \otimes \underline{\boldsymbol{n}} = D_n \underline{\boldsymbol{u}} \otimes \underline{\boldsymbol{n}}, \quad \text{where } D_n \underline{\boldsymbol{u}} = (\underline{\boldsymbol{u}} \otimes \nabla) \cdot \underline{\boldsymbol{n}}$$

$$D^n_j u_i := u_{i,k} n_k n_j = (D_n u_i) n_j, \quad D^t_j u_i := u_{i,j} - u_{i,k} n_k n_j$$

The displacement and its normal gradient only can be controlled at a boundary. This leads to the following representation of contact forces at a smooth boundary:

$$p^{(c)}(\underline{\boldsymbol{u}}) = \underline{\boldsymbol{t}} \cdot \underline{\boldsymbol{u}} + \underline{\boldsymbol{m}} \cdot D_n \underline{\boldsymbol{u}} = t_i u_i + m_i (D_n \underline{\boldsymbol{u}})_i \tag{40}$$

where $\underline{\boldsymbol{t}}$ and $\underline{\boldsymbol{m}}$ are simple and double traction vectors. The method of virtual work provides then the field equations for the second gradient continuum in the form:

$$\underset{\sim}{\boldsymbol{\tau}} \cdot \nabla + \underline{\boldsymbol{f}} = 0, \quad \text{with } \underset{\sim}{\boldsymbol{\tau}} = \underset{\sim}{\boldsymbol{\sigma}} - \underset{\sim}{\boldsymbol{F}} - (\underset{\approx}{\boldsymbol{S}} - \underset{\approx}{\boldsymbol{F}}) \cdot \nabla \tag{41}$$

$$\tau_{ij,j} + f_i = 0, \quad \text{with } \tau_{ij} = \sigma_{ij} - F_{ij} - S_{ijk,k} + F_{ijk,k} \tag{42}$$

The boundary conditions then involve complex surface terms:

$$\underline{\boldsymbol{t}} = \underset{\sim}{\boldsymbol{\tau}} \cdot \underline{\boldsymbol{n}} + 2R \underset{\approx}{\boldsymbol{S}} : \underline{\boldsymbol{n}} \otimes \underline{\boldsymbol{n}} - \nabla^t \cdot (\underset{\approx}{\boldsymbol{S}} \cdot \underline{\boldsymbol{n}}) \tag{43}$$

$$t_i = \tau_{ij} n_j + 2R S_{ijk} n_j n_k - D^t_j (S_{ijk} n_k)$$
$$\underline{\boldsymbol{m}} = \underset{\approx}{\boldsymbol{S}} : \underline{\boldsymbol{n}} \otimes \underline{\boldsymbol{n}}, \quad m_i = S_{ijk} n_j n_k \tag{44}$$

where R is Gauss mean curvature $2R = \underline{\boldsymbol{n}} \cdot \nabla^t = D^t_k n_k$.

Table 1 shows the similarities and differences between the stress and strain gradient theories. The stress gradient model introduces 15 more degrees of freedom than the strain gradient one. Both theories are based on one second rank and one third rank stress tensors but in the stress gradient model the third rank generalized stress tensor is deviatoric. Both theories require 6 boundary conditions to be prescribed. The 6 essential conditions in strain gradient theory amount to fixing the components of the displacement and normal gradient vector of the displacement. The corresponding essential conditions in the stress gradient case deal with the six components $u_{(i} n_{j)} + \Phi_{ijk} n_k$. The condition (26) involves only the values of the degrees of freedom and the normal vector, and no surface derivative, in contrast to the boundary condition (43). The natural conditions of the stress gradient model consist in prescribing the six stress components. In contrast, the natural conditions of the strain gradient model are quite intricate since the traction vector depends on the surface curvature and on the tangent derivative of the double stress tensor. This fact may represent a significant advantage of the stress gradient model from the point of view of computational mechanics. It turns out that the question of prescribing the boundary conditions (43) has not been discussed in the papers dealing with the finite element implementation of the strain gradient model (Shu et al., 1999; Wei, 2006). Also higher order shape functions or additional Lagrange multipliers must be used in the implementation of the strain gradient element. The main disadvantage of the stress gradient model is the increased number of degrees of freedom. This drawback becomes less and less relevant with the increasing computing power.

3.3. Relation to Germain's general micromorphic theory

It is instructive to compare the previous gradient theories to one of the most general theory of materials with microstructure, namely Germain's general micromorphic continuum model (Germain, 1973b). This generalization of Eringen's micromorphic model (Eringen, 1999) relies on the introduction of additional degrees of freedom in the form of second order, third order and possibly higher order deformation tensors. Let us concentrate on the second order micromorphic theory he proposed, which is based on the following form of the power density of internal forces:

$$p^{(i)} = \underset{\sim}{\boldsymbol{\sigma}} : \underline{\boldsymbol{u}} \otimes \nabla + \overset{1}{\underset{\sim}{\boldsymbol{s}}} : \underset{\sim}{\boldsymbol{\chi}} + \overset{2}{\underset{\approx}{\boldsymbol{s}}} \therefore \underset{\approx}{\boldsymbol{\Phi}} + \overset{1}{\underset{\approx}{\boldsymbol{S}}} \therefore \underset{\sim}{\boldsymbol{\chi}} \otimes \nabla + \overset{2}{\underset{\approx}{\boldsymbol{S}}} :: \underset{\approx}{\boldsymbol{\Phi}} \otimes \nabla = \sigma_{i,j} u_{i,j} + \overset{1}{s}_{ij} \chi_{ij} + \overset{2}{s}_{ijk} \Phi_{ijk} + \overset{1}{S}_{ijk} \chi_{ij,k} + \overset{2}{S}_{ijkl} \Phi_{ijk,l} \tag{45}$$

where the additional degrees of freedom χ_{ij} and Φ_{ijk} are independent second and third order tensor fields. The tensors $\underset{\sim}{\boldsymbol{\chi}}$ and $\underset{\approx}{\boldsymbol{\Phi}}$ are generally non compatible fields, meaning that they do not correspond to gradients of a vector and second order tensor fields, respectively. As a result, they generally do not display any symmetry property with respect to their indices. They are associated with the conjugate stresses $\overset{1}{\underset{\sim}{\boldsymbol{s}}}, \overset{2}{\underset{\approx}{\boldsymbol{s}}}, \overset{1}{\underset{\approx}{\boldsymbol{S}}}, \overset{2}{\underset{\approx}{\boldsymbol{S}}}$ in the power linear form. The principle of virtual power can be used to derive the balance equations fulfilled by the generalized stresses, written here in the static case in the absence of volume forces:

$$\underset{\sim}{\boldsymbol{\sigma}} \cdot \nabla = 0, \quad \overset{1}{\underset{\approx}{\boldsymbol{S}}} \cdot \nabla - \overset{1}{\underset{\sim}{\boldsymbol{s}}} = 0, \quad \overset{2}{\underset{\approx}{\boldsymbol{S}}} \cdot \nabla - \overset{2}{\underset{\approx}{\boldsymbol{s}}} = 0 \tag{46}$$

$$\sigma_{ij,j} = 0, \quad \overset{1}{S}_{ijk,k} - \overset{1}{s}_{ij} = 0, \quad \overset{2}{S}_{ijkl,l} - \overset{2}{s}_{ijk} = 0$$

At this stage, two special cases can be considered:

1 *First order micromorphic model with kinematic internal constraint.* In the micromorphic model obtained when $\underset{\approx}{\boldsymbol{\Phi}} \equiv 0$, the following kinematic internal constraint is enforced:

$$\underset{\sim}{\boldsymbol{\chi}} \equiv \underline{\boldsymbol{u}} \otimes \nabla, \quad \chi_{ij} \equiv u_{i,j} \tag{47}$$

which means that the microdeformation $\underset{\sim}{\boldsymbol{\chi}}$ is no longer independent of the displacement field. Accordingly, the microdeformation gradient $\underset{\sim}{\boldsymbol{\chi}} \otimes \nabla$ coincides with the second gradient of the displacement field $\underline{\boldsymbol{u}} \otimes \nabla \otimes \nabla$. Mindlin's second gradient theory is retrieved in that way. The simple force stress tensor then is $\underset{\sim}{\boldsymbol{\sigma}} + \overset{1}{\underset{\sim}{\boldsymbol{s}}}$ and Mindlin's hyperstress tensor is $\overset{1}{\underset{\approx}{\boldsymbol{S}}}$.

2 *Second order micromorphic model with static internal constraint.* In the spirit of Eringen and Germain, $\underset{\sim}{\boldsymbol{\chi}}$ and $\underset{\approx}{\boldsymbol{\Phi}}$ have the physical

[2] For a scalar field ϕ, we have $D^n_j \phi = \phi_{,k} n_k n_j$, $D_n \phi = \phi_{,k} n_k = \nabla \phi \cdot \underline{\boldsymbol{n}}$, $D^t_j \phi = \phi_{,j} - D^n_j \phi$. For the derivation of boundary conditions, the Stokes theorem for a smooth closed surface S is used:

$$\int_S \nabla^t \phi \, da = \int_S (\nabla^t \cdot \underline{\boldsymbol{n}}) \phi \underline{\boldsymbol{n}} \, da, \quad \int_S D^t_i \phi \, da = \int_S (D^t_k n_k) \phi n_i \, da$$

Table 1
Comparison between the strain gradient and stress gradient theories.

	Strain gradient continuum	Stress gradient continuum
DOF	3 displacements u_i	3 displacements u_i +15 deviatoric micro-displacements $\Phi_{ijk} = \Phi_{jik}$
Stress tensors	Simple stress tensor (Pa) 6 components $\sigma_{ij} = \sigma_{ji}$ Double stress tensor (Pa m) 18 components $S_{ijk} = S_{ikj}$	Simple stress tensor (Pa) 6 components $\sigma_{ij} = \sigma_{ji}$ Deviatoric stress gradient (Pa m^{-1}) 15 components $R_{ijk} = R_{jik}$
Essential boundary conditions	3 prescribed displacements u_i 3 prescribed normal Displacement gradient $D_n \underline{\boldsymbol{u}}$	6 prescribed components $u_{(i}n_{j)} + \Phi_{ijk}n_k$
Natural boundary conditions	3 simple tractions (Pa) t_i 3 double tractions (Pa m) m_i	6 stress tensor components (Pa) σ_{ij}
Body forces	Simple forces f_i (N m^{-3}) Double forces F_{ij} (N m^{-2}) Triple forces F_{ijk} (N m^{-1})	Simple forces f_i (N m^{-3}) Third rank simple forces F_{ijk} (N m^{-3})
Isotropic linear elastic moduli	2 Lamé constants (Pa) +5 higher order moduli (Pa m^2)	2 Lamé constants (Pa) +3 higher order modulus (Pa m^{-2}) (2D)

dimensions of strain and strain gradient, respectively. But more general physical types of degrees of freedom can be imagined. For instance we can enforce the following static internal constraint for the highest order stress tensor:

$$\overset{2}{\underset{\approx}{\boldsymbol{S}}} \equiv \underset{\sim}{\boldsymbol{\sigma}} \otimes \underset{\sim}{\mathbf{1}}, \quad \overset{2}{S}_{ijkl} \equiv \sigma_{ij}\delta_{kl} \tag{48}$$

which means that the second order stress tensor is not independent from the simple stress tensor. From the third balance equations in (46) it follows that

$$\overset{2}{\underset{\approx}{\boldsymbol{s}}} = \overset{2}{\underset{\approx}{\boldsymbol{S}}} \cdot \nabla = \underset{\sim}{\boldsymbol{\sigma}} \otimes \nabla, \quad \overset{2}{s}_{ijk} = \overset{2}{S}_{ijkl,l} = (\sigma_{ij}\delta_{kl})_{,l} = \sigma_{ij,k} \tag{49}$$

Under these conditions, and considering the second order micromorphic degrees of freedom only, the power density function (45) reduces to the expression (29) postulated for the stress gradient medium.

4. Two-dimensional stress gradient isotropic linear elasticity

Constitutive equations are derived for a linear elastic isotropic stress gradient material. The elastic potential is an isotropic function of the second order symmetric tensor $\underset{\sim}{\boldsymbol{e}}$ and of the third order tensor $\underset{\approx}{\boldsymbol{\Phi}} \in \mathcal{D}$. The form of isotropic scalar functions depending on a symmetric second rank tensor and on a third rank tensor $\in \mathcal{R}$ has been worked out by Mindlin for linear elastic isotropic second gradient materials. This function can be used in the present context:

$$w(\underset{\sim}{\boldsymbol{e}}, \underset{\approx}{\boldsymbol{\Phi}}) = \frac{1}{2}\lambda e_{ii}e_{jj} + \mu e_{ij}e_{ij} + a_1\Phi_{iik}\Phi_{kjj} + a_2\Phi_{ijj}\Phi_{ikk} + a_3\Phi_{iik}\Phi_{jjk} + a_4\Phi_{ijk}\Phi_{ijk} + a_5\Phi_{ijk}\Phi_{kji} \tag{50}$$

where λ, μ are generalized Lamé constants and $a_i (i = 1, 5)$ are higher order elastic moduli (Mindlin and Eshel, 1968). The stress tensors are obtained by means of the state laws:

$$\underset{\sim}{\boldsymbol{\sigma}} = \frac{\partial w}{\partial \underset{\sim}{\boldsymbol{e}}} = \lambda (\mathrm{trace}\, \underset{\sim}{\boldsymbol{e}}) \underset{\sim}{\mathbf{1}} + 2\mu \underset{\sim}{\boldsymbol{e}}, \quad \underset{\approx}{\boldsymbol{R}} = \frac{\partial w}{\partial \underset{\approx}{\boldsymbol{\Phi}}} \tag{51}$$

The remainder of this work is limited to a two-dimensional isotropic elastic continuum, for the sake of simplicity. Note that plane strain conditions in a three-dimensional space will generally induce non-vanishing stress gradient components $R_{331} = \sigma_{33,1}$ and $R_{332} = \sigma_{33,2}$. Conversely, plane stress conditions will in general be associated with non-vanishing micro-displacements Φ_{331} and Φ_{332} induced by the generalized elasticity law. That is why the analysis is now limited to a two-dimensional space, like for an in-plane loaded membrane continuum. In that case, all indices of the considered tensors take the values 1 or 2.

Under these conditions, the elastic potential (50) leads to the following constitutive equations written in a matrix form:

$$\begin{bmatrix} R_{111} \\ 2R_{122} \\ R_{221} \end{bmatrix} = \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} \Phi_{111} \\ \Phi_{122} \\ \Phi_{221} \end{bmatrix} \quad \begin{bmatrix} R_{222} \\ 2R_{211} \\ R_{112} \end{bmatrix} = \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} \Phi_{222} \\ \Phi_{211} \\ \Phi_{112} \end{bmatrix} \tag{52}$$

where the matrix $[A]$ is

$$\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} 2a & a_1 + 2a_2 & a_1 + 2a_3 \\ a_1 + 2a_2 & 2a_2 + 4a_4 + 2a_5 & a_1 + 2a_5 \\ a_1 + 2a_3 & a_1 + 2a_5 & 2a_3 + 2a_4 \end{bmatrix} \tag{53}$$

with $a = a_1 + a_2 + a_3 + a_4 + a_5$. The factor two appearing in front of the components R_{122} and R_{211} ensures that the scalar product of the 6-dimensional vectors R_I and Φ_I coincides with the work of internal forces $\underset{\approx}{\boldsymbol{R}} \therefore \underset{\approx}{\boldsymbol{\Phi}}$.

Due to the fact that the tensors $\underset{\approx}{\boldsymbol{R}}$ and $\underset{\approx}{\boldsymbol{\Phi}}$ are deviatoric, there exist some linear relationships between some of the higher order elastic moduli. We enforce the condition that, when applied to a deviatoric third rank tensor, the relation (53) should deliver a deviatoric third rank tensor, which means that

$$\Phi_{111} + \Phi_{122} = 0, \quad \Phi_{211} + \Phi_{222} = 0,$$
$$R_{111} + R_{122} = 0, \quad R_{211} + R_{222} = 0 \tag{54}$$

It can be shown that the enforced condition then requires that:

$$3a_1 + 4a_3 + 2a_5 = 0 \tag{55}$$

Taking this linear relationship into account, the generalized Hooke matrix in (53) becomes

$$\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} -a_1 + 2a_2 - 2a_3 + 2a_4 & a_1 + 2a_2 & a_1 + 2a_3 \\ a_1 + 2a_2 & -3a_1 + 2a_2 - 4a_3 + 4a_4 & -2a_1 - 4a_3 \\ a_1 + 2a_3 & -2a_1 - 4a_3 & 2a_3 + 2a_4 \end{bmatrix} \tag{56}$$

It can be checked that a spherical tensor $\underset{\approx}{\boldsymbol{\Phi}}$, when inserted in (56), leads to a spherical tensor $\underset{\approx}{\boldsymbol{R}}$.

Since the stress gradient theory only considers deviatoric tensors $\underset{\sim}{\boldsymbol{\Phi}}$ and $\underset{\sim}{\boldsymbol{R}}$, the components $\Phi_{122} = -\Phi_{111}$ and $\Phi_{211} = -\Phi_{222}$ can be eliminated from the constitutive law (56). We obtain the following reduced matrix form for the isotropic linear elastic stress gradient constitutive relation:

$$\begin{bmatrix} R_{111} \\ R_{221} \end{bmatrix} = \begin{bmatrix} \hat{A} \end{bmatrix} \begin{bmatrix} 3\Phi_{111} \\ \Phi_{221} \end{bmatrix} \qquad \begin{bmatrix} R_{222} \\ R_{112} \end{bmatrix} = \begin{bmatrix} \hat{A} \end{bmatrix} \begin{bmatrix} 3\Phi_{222} \\ \Phi_{112} \end{bmatrix} \tag{57}$$

with

$$\begin{bmatrix} \hat{A} \end{bmatrix} = \begin{bmatrix} \frac{2}{3}(a_4 - a_1 - a_3) & a_1 + 2a_3 \\ a_1 + 2a_3 & 2(a_3 + a_4) \end{bmatrix} \tag{58}$$

As a result, two-dimensional linear isotropic elasticity involves 3 independent elasticity moduli. The factor 3 is introduced in front of Φ_{111} and Φ_{222} in (58) so that the work of internal forces can still be computed by the scalar product of the R_I and Φ_I vectors. Definite positivity of the quadratic form is equivalent to $a_4 - a_1 - a_3 > 0$, $a_3 + a_4 > 0$ and $4/3(a_4 - a_1 - a_3)(a_3 + a_4) - (a_1 + 2a_3)^2 > 0$.

Generalized elastic compliances can be defined by inverting the generalized Hooke law (58) in the form:

$$\begin{bmatrix} 3\Phi_{111} \\ \Phi_{221} \end{bmatrix} = \begin{bmatrix} B \end{bmatrix} \begin{bmatrix} R_{111} \\ R_{221} \end{bmatrix} \qquad \begin{bmatrix} 3\Phi_{222} \\ \Phi_{112} \end{bmatrix} = \begin{bmatrix} B \end{bmatrix} \begin{bmatrix} R_{222} \\ R_{112} \end{bmatrix} \tag{59}$$

with

$$\begin{bmatrix} B \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix} \tag{60}$$

where the coefficients b_{ij} are directly related to the a_i. Definite positivity of the elastic potential is equivalent to $b_{11} > 0$, $b_{22} > 0$ and $b_{11}b_{22} - b_{12}^2 > 0$.

At this stage, some comments on the physical dimension of the introduced quantities are necessary. We already mentioned that the units of micro-displacements and stress gradient respectively are m and $\mathrm{Pa\,m^{-1}} \equiv \mathrm{N\,m^{-3}}$. The physical dimension of the generalized moduli a_i is $\mathrm{Pa\,m^{-2}} \equiv \mathrm{N\,m^{-4}}$. The inverse of the matrix of generalized elasticity moduli in (53) provides the generalized compliances with unit $\mathrm{m^2\,Pa^{-1}}$.

5. Resolution of a boundary value problem for stress and strain gradient media

We consider a generic problem of an elastic medium subjected to spatially periodic volume forces that we solve successively for Cauchy, stress gradient and strain gradient continua in the static case. The objective is to point out the main differences in the response of gradient continua to loading in the bulk.

5.1. Position of the problem and reference Cauchy solutions

We consider a system of body forces of the form

$$\underline{\boldsymbol{f}} = (f_1 \underline{\boldsymbol{e}}_1 + f_2 \underline{\boldsymbol{e}}_2) \sin \omega x_2 \tag{61}$$

which display a periodicity in the x_2 direction, with given wave length $2\pi/\omega$. The effect of each component of the body force is investigated for a two-dimensional homogeneous medium assuming that no additional loading is applied at infinity. We seek for the solution of the problem exhibiting the same symmetry properties as the loading, namely it is periodic with respect to direction 2 and invariant with respect to direction 1.

5.1.1. Horizontal body force in a Cauchy medium

The displacement and stress fields in the bulk of the body are given first when $f_1 \neq 0$ and $f_2 = 0$. A displacement of the following form is searched for

$$u_1 = u_1(x_2), \quad u_2 = u_3 = 0 \quad \Rightarrow \quad \varepsilon_{11} = \varepsilon_{22} = 0,$$
$$\varepsilon_{12} = \frac{u_{1,2}}{2}, \quad \sigma_{12} = \mu u_{1,2} \tag{62}$$

The balance equation implies that

$$\sigma_{12,2} + f_1 \sin \omega x_2 = \mu u_{1,22} + f_1 \sin \omega x_2 = 0 \tag{63}$$

which provides

$$u_1 = \frac{f_1}{\mu\omega^2} \sin \omega x_2 + c x_2 + d, \quad \sigma_{12} = \frac{f_1}{\mu\omega} \cos \omega x_2 + \mu c \tag{64}$$

where c, d are integration constants. The overall stress must vanish so that $c = 0$.

This loading condition therefore induces a periodic shear stress gradient $R_{122} = \sigma_{12,2}$.

5.1.2. Vertical body force in a Cauchy medium

The displacement and stress fields in the bulk of the body are now given when $f_1 = 0$ and $f_2 \neq 0$. The following form of the displacement field is searched for

$$u_1 = 0, \quad u_2 = u_2(x_2), \quad u_3 = 0 \quad \Rightarrow \quad \varepsilon_{11} = \varepsilon_{12} = 0, \quad \varepsilon_{22} = u_{2,2}$$

$$\sigma_{11} = \lambda u_{2,2}, \quad \sigma_{22} = (\lambda + 2\mu) u_{2,2}$$

The balance of momentum equation implies that

$$\sigma_{22,2} + f_2 \sin \omega x_2 = 0 \quad \Rightarrow \quad (\lambda + 2\mu) u_{2,22} + f_2 \sin \omega x_2 = 0 \tag{65}$$

which provides

$$u_2 = \frac{f_2}{(\lambda + 2\mu)\omega^2} \sin \omega x_2 + c x_2 + d \tag{66}$$

$$\sigma_{11} = \frac{\lambda f_2}{(\lambda + 2\mu)\omega} \cos \omega x_2 + \lambda c, \quad \sigma_{22} = \frac{f_2}{\omega} \cos \omega x_2 + (\lambda + 2\mu) c \tag{67}$$

where the constant c must vanish in the absence of mean stress. This loading condition therefore induces the stress gradients $R_{112} = \sigma_{11,2}$ and $R_{222} = \sigma_{22,2}$.

5.2. Periodic body forces in a stress gradient medium

The two previous elementary problems are reconsidered for an isotropic linear elastic two-dimensional stress gradient medium.

5.2.1. Horizontal body force

The balance of momentum equations still require that the shear stress component takes the form

$$\sigma_{12} = \frac{f_1}{\omega} \cos \omega x_2 \tag{68}$$

It follows that the eligible stress gradient components are *a priori* $\sigma_{12,2}$ and $\sigma_{11,2}$. According to the relations (14), the active components of the deviatoric part of the stress gradient are

$$R_{111} = -\frac{2}{3}\sigma_{12,2}, \quad R_{122} = \frac{2}{3}\sigma_{12,2}, \quad R_{112} = \sigma_{11,2}$$

These stress gradient components activate all micro-displacements components that can be computed by means of Eq. (59):

$$3\Phi_{111} = b_{11} R_{111}, \quad \Phi_{221} = b_{12} R_{111} \tag{69}$$

$$3\Phi_{222} = b_{12} R_{112}, \quad \Phi_{112} = b_{22} R_{112} \tag{70}$$

We also have $\Phi_{122} = -\Phi_{111}$ and $\Phi_{211} = -\Phi_{222}$.

We look for a displacement field of the form $u_1 = u_1(x_2)$, $u_2 = 0$. The non-vanishing components of the generalized strain measure $\underset{\sim}{\boldsymbol{e}}$ are then

$$e_{11} = \Phi_{112,2}, \quad e_{22} = \Phi_{222,2}, \quad e_{12} = \frac{u_{1,2}}{2} + \Phi_{122,2} \tag{71}$$

The associated stress components are

$$\sigma_{11} = (\lambda + 2\mu)e_{11} + 2\mu e_{11}, \quad \sigma_{22} = \lambda e_{11} + (\lambda + 2\mu)e_{22}, \quad \sigma_{12} = 2\mu e_{12} \tag{72}$$

The stress components σ_{22} must vanish for equilibrium to be fulfilled. This implies that

$$\Phi_{222,2} = -\frac{\lambda}{\lambda + 2\mu}\Phi_{112,2} \tag{73}$$

The first equation in (72) then leads to the differential equation

$$\sigma_{11} = \frac{4\mu(\lambda + 2\mu)}{\lambda + 2\mu} b_{22}\sigma_{11,22} \tag{74}$$

Note that the coefficients $\mu > 0$, $\lambda + 2\mu > 0$, $\lambda + \mu > 0$ and $b_{22} > 0$. It follows that the general solution is a combination of cosh and sinh functions of the variable x_2. Since these functions are not periodic with respect to x_2, the corresponding coefficients must vanish. As a result, we get $\sigma_{11} = 0$.

The last equation in (72) is now used in the form

$$\sigma_{12} = \mu\left(u_{1,2} + \frac{4}{9}b_{11}\sigma_{12,22}\right) \tag{75}$$

from which the displacement is deduced as

$$u_1 = \left(\frac{1}{\omega^2\mu} + \frac{4}{9}b_{11}\right) f_1 \sin\omega x_2 \tag{76}$$

As a result, the shear stress component induced by the body force is the same for the Cauchy and stress gradient theories. However, the corresponding displacements differ by a term proportional to the generalized compliance b_{11}.

The stress gradient medium degenerates into the classical continuum if the micro-displacements Φ_{ijk} vanish which happens when the generalized compliance b_{ij} tends to zero. In that case, the previous solution (76) reduces to (64).

5.2.2. Vertical body force

The balance of momentum equation still implies that $\sigma_{22} = (f_2/\omega)\cos\omega x_2$.

It follows that the eligible stress gradient components are *a priori* $\sigma_{22,2}$ and $\sigma_{11,2}$. According to the relations (14), the active components of the deviatoric part of the stress gradient are

$$R_{222} = \frac{1}{3}\sigma_{22,2}, \quad R_{211} = -\frac{1}{3}\sigma_{22,2}, \quad R_{112} = \sigma_{11,2}$$

The inverse Hooke law (59) then provides the active micro-displacements:

$$3\Phi_{222} = b_{11}R_{222} + b_{12}R_{112}, \quad \Phi_{112} = b_{12}R_{222} + b_{22}R_{112} \tag{77}$$

We also have $\Phi_{211} = -\Phi_{222}$. We look for displacement components $u_1 = 0$, $u_2(x_2)$. The non-vanishing components of the generalized strain measure $\underset{\sim}{\boldsymbol{e}}$ are then

$$e_{11} = \Phi_{112,2}, \quad e_{22} = u_{2,2} + \Phi_{222,2} \tag{78}$$

The associated stress components are

$$\sigma_{11} = \lambda(u_{2,2} + \Phi_{222,2} + \Phi_{112,2}) + 2\mu\Phi_{112,2}, \quad \sigma_{22} = \lambda(u_{2,2} + \Phi_{222,2} + \Phi_{112,2}) + 2\mu(u_{2,2} + \Phi_{222,2}) \tag{79}$$

from which the following relationship can be derived

$$\frac{\sigma_{11}}{\lambda} - \frac{\sigma_{22}}{\lambda + 2\mu} = \Phi_{112,2}\left(\frac{\lambda + 2\mu}{\lambda} - \frac{\lambda}{\lambda + 2\mu}\right) \tag{80}$$

This relation combined with the generalized elasticity law provides the following differential equation for the stress component σ_{11}:

$$\sigma_{11} - \frac{4\mu(\lambda + \mu)}{\lambda + 2\mu} b_{22}\sigma_{11,22} = \left(\frac{\lambda}{\lambda + 2\mu} - \frac{4\mu(\lambda + \mu)}{3(\lambda + 2\mu)} b_{12}\omega^2\right)\frac{f_2}{\omega}\cos\omega x_2 \tag{81}$$

Since $\mu(\lambda + \mu)/(\lambda + 2\mu)b_{22}$ is positive due to the positivity of the quadratic elastic potential, the general solution of the homogeneous part of the previous differential equation is a combination of non periodic functions which coefficients must therefore vanish. As a result, the solution reduces to the particular one of the form

$$\sigma_{11} = \alpha\frac{f_2}{\omega}\cos\omega x_2 \tag{82}$$

The coefficient α must take the following value:

$$\alpha(\lambda + 2\mu + 4\mu(\lambda + \mu)b_{22}\omega^2) = \lambda - \frac{4}{3}\mu(\lambda + \mu)b_{12}\omega^2 \tag{83}$$

The displacement field is finally obtained by integrating the equation

$$u_{2,2} = \frac{\sigma_{22}}{\lambda + 2\mu} - \Phi_{222,2} - \frac{\lambda}{\lambda + 2\mu}\Phi_{112,2} \tag{84}$$

which delivers

$$u_2 = \left(\frac{1}{(\lambda + 2\mu)\omega^2} + \frac{b_{11}}{9} + \alpha\frac{b_{12}}{3} + \frac{\lambda}{\lambda + 2\mu}\left(\frac{b_{12}}{3} + \alpha b_{22}\right)\right) f_2 \sin\omega x_2 \tag{85}$$

As a result, the stress component σ_{22} is the same according to both Cauchy and stress gradient theories. In contrast, the component σ_{11} and the displacement u_2 differ by terms that involve the generalized compliance b_{ij}. When the generalized compliance tends to zero, α tends toward the finite value $\lambda/(\lambda + 2\mu)$ and the Cauchy solution (66) is retrieved, as expected.

5.3. Solution for a strain gradient medium

The previous problems can also be considered for Mindlin's strain gradient medium.

5.3.1. Horizontal body force in a strain gradient medium

We search for periodic solutions in a bulk strain gradient material subjected to the periodic horizontal body force $f_1 \sin\omega x_2$. The single non-vanishing displacement component $u_1(x_2)$ induces the second gradient component $u_{1,22}$. In the second gradient theory, the isotropic linear elasticity law formally has the same form as (53):

$$\begin{bmatrix} S_{111} \\ 2S_{221} \\ S_{122} \end{bmatrix} = [A] \begin{bmatrix} u_{1,11} \\ u_{2,21} \\ u_{1,22} \end{bmatrix} \qquad \begin{bmatrix} S_{222} \\ 2S_{112} \\ S_{211} \end{bmatrix} = [A] \begin{bmatrix} u_{2,22} \\ u_{1,12} \\ u_{2,11} \end{bmatrix} \tag{86}$$

where

$$\left[A\right] = \begin{bmatrix} 2a & a_1+2a_2 & a_1+2a_3 \\ a_1+2a_2 & 2a_2+4a_4+2a_5 & a_1+2a_5 \\ a_1+2a_3 & a_1+2a_5 & 2a_3+2a_4 \end{bmatrix} \quad (87)$$

has the same form as the matrix (53). The five higher order elasticity moduli a_i are independent. In the considered boundary value problem, the following components of the hyper-stress tensor are activated:

$$S_{111} = (a_1+2a_3)u_{1,22}, \quad S_{122} = 2(a_3+a_4)u_{1,22},$$
$$2S_{221} = (a_1+2a_5)u_{1,22} \quad (88)$$

Note that the physical dimensions of the moduli a_i differ in the stress and strain gradient theories (see Table 1) and that the hyper-stress tensor is symmetric with respect to its two last indices. The shear stress and the hyper-stress component S_{122} arise in the following balance of momentum equation

$$\sigma_{12,2} - S_{122,22} + f_1 \sin\omega x_2 = 0 \quad (89)$$

from which the following partial differential equation for the displacement u_2 is derived:

$$\mu u_{1,22} - 2(a_3+a_4)u_{1,2222} + f_1 \sin\omega x_2 = 0 \quad (90)$$

Since $(a_3+a_4)/\mu > 0$ due to the positivity of the elasticity strain gradient potential, the general solution of the homogeneous part of the previous differential equation is not periodic. The hyperbolic functions cosh, sinh characterize boundary layers effects that are well-known in strain gradient elasticity and plasticity (Forest and Sedláček, 2003). They are not relevant for the bulk behavior considered here, in the absence of applied load at infinity. The solution of the boundary value problem therefore reduces to the particular solution of (90) of the form

$$u_1 = \frac{f_1 \sin\omega x_2}{\omega^2(\mu + 2(a_3+a_4)\omega^2)} \quad (91)$$

and the associated shear stress

$$\sigma_{12} = \frac{f_1 \cos\omega x_2}{\omega(1 + 2(a_3+a_4)\omega^2/\mu)} \quad (92)$$

It appears that both the shear stress and displacement fields differ from the Cauchy solution. This is in contrast to the stress gradient solution for which only the displacement is affected by the generalized moduli.

The Cauchy solution is recovered for vanishingly small moduli a_i, as it should be.

5.3.2. Vertical body force in a strain gradient medium

We search for periodic solutions in a bulk strain gradient material subjected to the periodic vertical body force $f_2\sin\omega x_2$. The single non-vanishing displacement component $u_2(x_2)$ induces the second gradient component $u_{2,22}$ which triggers the following components of the hyper-stress tensor according to the isotropic linear elasticity (87):

$$S_{222} = 2au_{2,22}, \quad 2S_{112} = (a_1+2a_2)u_{2,22}, \quad S_{211} = (a_1+2a_3)u_{2,22} \quad (93)$$

The stress σ_{22} and the hyper-stress S_{222} intervene in the following balance of momentum equation

$$\sigma_{22,2} - S_{222,22} + f_2 \sin\omega x_2 = 0 \quad (94)$$

from which the following partial differential equation for the displacement u_2 is derived:

$$(\lambda+2\mu)u_{2,22} - 2au_{2,2222} + f_2 \sin\omega x_2 = 0 \quad (95)$$

Since $a/(\lambda+2\mu) > 0$, the general solution of the homogeneous part of the previous differential equation is not periodic. As a result, the solution of the boundary value problem reduces to the particular solution of (95) of the form

$$u_2 = \frac{f_2}{\omega^2(\lambda+2\mu+2a\omega^2)} \sin\omega x_2 \quad (96)$$

and the associated stress components

$$\sigma_{22} = \frac{f_2}{\omega(1+2a\omega^2/(\lambda+2\mu))} \cos\omega x_2,$$
$$\sigma_{11} = \frac{\lambda f_2}{\omega(\lambda+2\mu+2a\omega^2)} \cos\omega x_2 \quad (97)$$

It appears that both the stress and displacement fields differ from the Cauchy solution. This is in contrast to the stress gradient solution for which only the displacement and the component σ_{11} are affected by the generalized moduli.

The Cauchy solution (66) and (67) is recovered for vanishingly small moduli a_i, as expected.

6. Discussion and final remarks

A stress gradient continuum theory has been proposed in this work, that incorporates the effect of the deviatoric part of the gradient of the Cauchy stress tensor into the constitutive framework. It requires the introduction of additional independent micro-displacement degrees of freedom that are the dual quantities to the deviatoric stress gradient in the generalized work of internal forces. In that sense the theory is closer to Germain's second order micromorphic continuum that also includes a third rank tensor as new degrees of freedom (Germain, 1973b), than to Mindlin's strain gradient model, entirely based on the usual displacement degrees of freedom. It has been explained how the strain and stress gradient models arise as limit cases of Germain's general micromorphic model. The strain gradient model can be regarded as a first order micromorphic medium endowed with a kinematic internal constraint. In contrast, the stress gradient model can be seen as a second order micromorphic medium with a static constraint on the higher order stresses. The fact that the strain and stress gradient models correspond to different internal constraints shows again that both theories are essentially distinct models. Balance and boundary conditions have been shown to drastically differ according to the strain and stress gradient theories. It was found that the theory displays the unusual feature that the whole stress tensor can be prescribed at a boundary and not only the traction vector. The elasticity theory has been formulated for the stress gradient continuum. In the purely two-dimensional case, isotropic linear elasticity was shown to contain three new higher order moduli.

A generic boundary value problem involving periodic body forces has been solved for both the stress and strain gradient linear isotropic models. Solutions were found to differ significantly. The size-dependence of the solutions is reflected by the fact that the additional contributions to the classical Cauchy solutions involve the ratio of the higher order moduli and of the square of the wave length of the applied body force field.

The main finding of the present work is that the stress gradient and strain gradient models represent two distinct theories which can account for different features of size-dependent material behavior. This has been shown by formulating rigorously the stress gradient model, which had not been done before. The introduction of higher order gradients breaks the usual duality between the classical stress based and strain based constitutive models. In particular, the theory shows that Mindlin's third order hyperstress tensor cannot be identified with the stress gradient tensor, as assumed in a constitutive way in some published literature. A similar situation has been encountered by Forest and Amestoy (2008) to develop generalized heat equations in rigid heat-conducting solids. A gradient of entropy effect was introduced in the internal energy density resulting in a higher order heat equation. It was shown that, even though temperature and entropy are dual quantities in classical thermodynamics, a gradient of temperature model and a gradient of entropy model lead to different generalized heat equations and represent therefore different theories. In general, the use of higher order gradient breaks the duality of the original primal and conjugate variables.

Future work is also needed to assess the relevance of the new theory to account for size effects in material behavior in a better or complementary way than the strain gradient model. For that purpose, we intend to show how the higher order moduli of the stress gradient elasticity theory can be determined by homogenization of composite materials. Such an approach already exists for the derivation of strain gradient properties, as done by Boutin (1996), Forest and Sab (1998), Geers et al. (2001), and Forest and Trinh (2011). An alternative homogenization scheme must be designed for the stress gradient model. We will therefore have to address the following question: When is an overall stress gradient medium preferable to an effective strain gradient material? From the micromechanical point of view, there has been a constant endeavour in the past 15 years to derive the properties of an effective higher order continuum from the knowledge of the microstructure of composite materials subjected to non-homogeneous boundary conditions (see Forest and Trinh, 2011). Most attempts aimed at determining the higher order moduli of an effective strain gradient medium by applying polynomial displacement boundary conditions on a representative volume element. These generalized homogenization procedures are recognized in the literature to be still not completely satisfactory. In contrast, some contributions by Rodin (2007) and Li (2011a,b); Lebée and Sab (2011a) suggest that static boundary conditions may be better suited to derive higher order overall properties. In the references by Li, affine stress boundary conditions are prescribed at the boundaries of the unit cell. The affine conditions are related to the macroscopic mean stress and its gradient. However, no comment is made about the nature of the overall generalized continuum which incorporates the stress gradient instead of the strain gradient. The homogenization procedure proposed by Li (2011a,b) could be used to identify the higher order elastic moduli introduced in our theory.

From the computational point of view, the stress gradient model is easier to implement in a finite element program, especially regarding the order of shape functions and the boundary conditions that are significantly more simple than in the strain gradient model. On the other hand, it has the disadvantage that it requires the introduction of 15 additional degrees of freedom per node in the three-dimensional case.

The dynamics of stress gradient media also is of interest because of the new dispersion relations that the theory will deliver compared to Mindlin's existing ones (Mindlin and Eshel, 1968).

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