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The role of the fluctuation field in higher order homogenization

Duy-Khanh Trinh[1,*] and **Samuel Forest**[1]

[1] MINES ParisTech, Centre des matériaux, CNRS UMR 7633, BP 87 91003 Evry Cedex, France

Homogenization methods aim at replacing a composite material by a homogeneous equivalent medium endowed by effective properties [1]. When the size of the heterogeneities (inclusions, grains...) is much smaller than the wave length of the variation of the macroscopic stress and strain fields, i.e. under the assumption of separation of scales, the effective properties of a composite with periodic microstructures can be determined by considering a unit cell V and applying the following loading conditions:

$$u_i = E_{ij} x_j + v_i, \quad \forall \boldsymbol{x} \in V \quad \text{with} \quad v_i(\boldsymbol{x}^+) = v_i(\boldsymbol{x}^-), \quad \boldsymbol{t}(\boldsymbol{x}^+) = -\boldsymbol{t}(\boldsymbol{x}^-) \tag{1}$$

The displacement vector inside V is $\boldsymbol{u}$. The constant tensor E_{ij} is the average strain applied to the unit cell and $\boldsymbol{v}$ is a periodic fluctuation, meaning that it takes the same value at homologous points $(\boldsymbol{x}^-, \boldsymbol{x}^+)$ of the boundary ∂V of the unit cell. The traction vector $\boldsymbol{t} = \boldsymbol{\sigma} \cdot \boldsymbol{n}$ takes opposite values at homologous points of the boundary with normal vector $\boldsymbol{n}$.

When the macroscopic medium is subjected to slowly–varying mean fields so that strain gradients may influence the material response, the previous homogenization scheme must be extended. For that purpose, several authors have proposed to replace the composite material by a generalized continuum, like a Cosserat medium in [2], a couple stress continuum in [3], a strain gradient model in [4] and a micromorphic model in [5].

1 Non–homogeneous loading conditions of the unit cell

The identification of the generalized continuum model in the mentioned contributions goes through the extension of the classical loading conditions (1) to non–homogeneous conditions:

$$u_i = E_{ij} + D_{ijk} x_j x_k + v_i, \quad \forall \boldsymbol{x} \in V \tag{2}$$

where D_{ijk} is a constant third rank tensor, symmetric with respect to the two last indices. The coefficients of the quadratic polynomial can be related to the generalized strain measures of the effective continuum but, in this note, the attention is focused on the properties of the fluctuation $\boldsymbol{v}$. In [5], the fluctuation is assumed to vanish, which may lead to the prediction of too stiff effective elastic properties. In [2], the fluctuation was considered periodic as in Eq. (1), which is generally not a valid assumption for such quadratic boundary conditions as discussed in [6]. In [4], the periodicity requirement for $\boldsymbol{v}$ is relaxed by an heuristic integral condition but the traction vector is still assumed to be anti–periodic, which cannot be expected in general because of the existence of stress gradient induced by the quadratic conditions.

The objective of this note is to provide a precise characterization of the real fluctuation field under quadratic boundary conditions by means of a computational homogenization approach.

2 Determination of the fluctuation field

For that purpose, numerical simulations have been performed for a composite material made of an elastic grid with soft square inclusions under plane strain conditions, see the unit cell of figure 1(a). The elastic properties of the grid and inclusion materials respectively are ($E = 200000$ MPa, $\nu = 0.3$) and ($E = 20000$ MPa, $\nu = 0.3$). The edge length of the unit cell is 1 mm. The quadratic boundary conditions (2) where applied to the boundary of one single unit cell, but also to on ensemble of cells containing 3×3 (see 1(b)), 5×5, up to 9×9 cells. Fixed values of the coefficients D_{ijk} were chosen and the fluctuation $\boldsymbol{v}$ was set to zero at the outer boundary of the $N \times N$ ensemble of cells.

The normalized elastic energy fields $\boldsymbol{\sigma}(\boldsymbol{x}) : \boldsymbol{\varepsilon}(\boldsymbol{x})/W$ with $W =< \boldsymbol{\sigma} : \boldsymbol{\varepsilon} >_{V_{9\times9}}$ are drawn in figure 1, where $< \cdot >_{V_{N\times N}}$ denotes spatial averaging over the central unit cell of the $N \times N$ set of cells. The only non–vanishing component of the polynomial was $D_{112} = 1$ mm^{-1}. Strong boundary layer effects arise close to the outer boundary where the Dirichlet conditions are applied. In the central cell a deviation from the polynomial field appears that is such that its boundaries do not remain straight lines during deformation. We find that the elastic energy distribution in the central unit cell and the fluctuation $\boldsymbol{v}$ at its boundary converge toward fixed fields when the number of cells N increases. This means that there exists a representative volume element size for quadratic boundary conditions. For the considered material, convergence was reached

* Corresponding author e-mail: duy-khanh.trinh@ensmp.fr, Phone: +33 1 60 76 30 28, Fax: +33 1 60 76 31 50

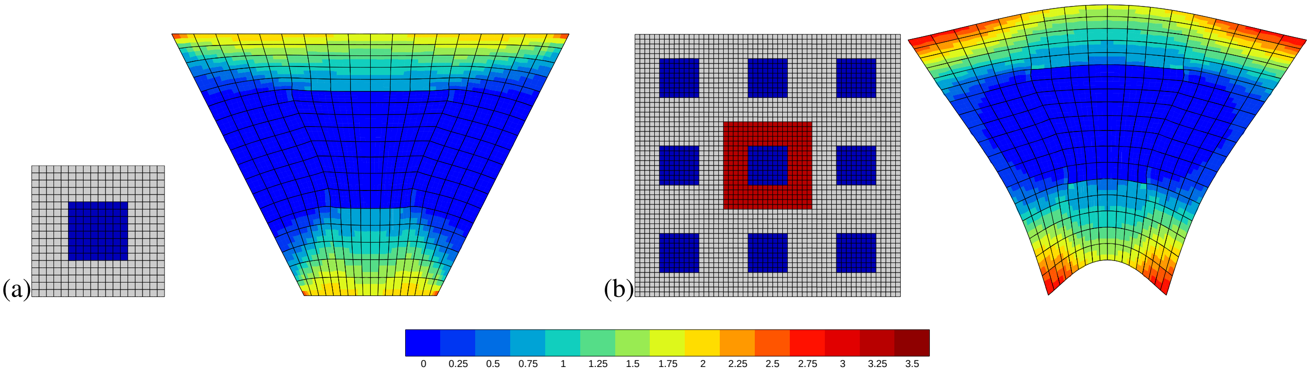

Fig. 1 Unit cell of a composite material (a, left) and 3×3 ensemble of cells (b, right, the central cell is coloured). Both meshes were subjected to the quadratic Dirichlet boundary conditions $D_{112} = 1\ \mathrm{mm}^{-1}$. The normalized energy field is displayed for the unit cell (a, right) and for the central cell (b, right).

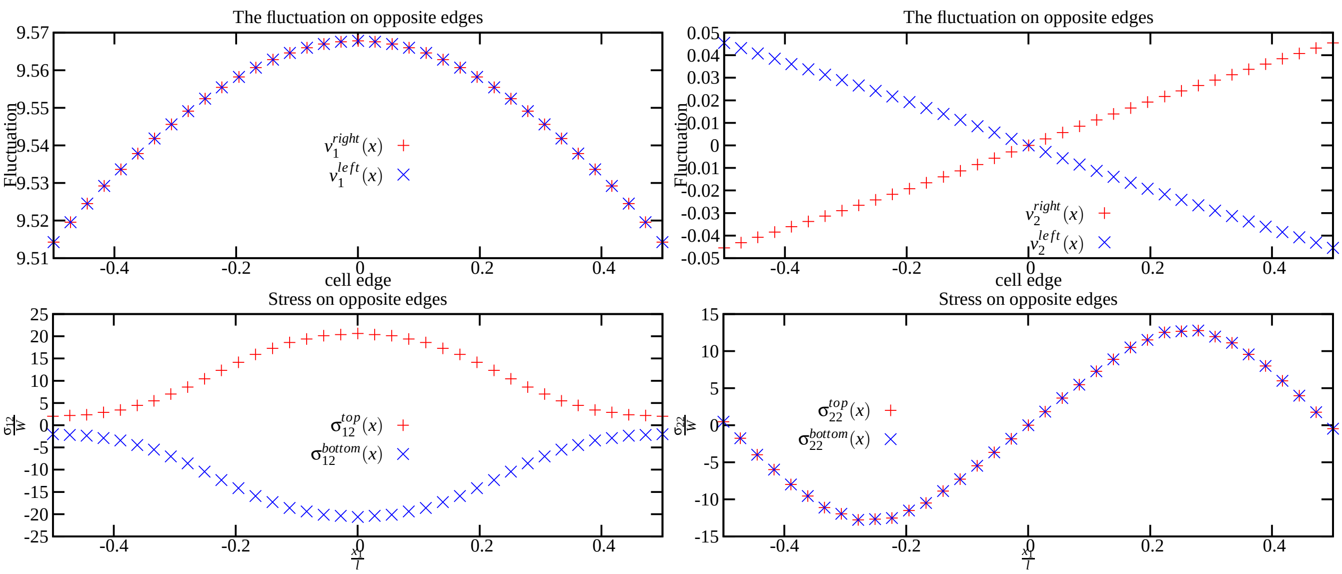

Fig. 2 Fuctuation $\boldsymbol{v}$ and normalized stress components on opposite edges of the central unit cell of a 9×9 assembly for prescribed $D_{111} = 1\ \mathrm{mm}^{-1}$. Top: v_1 and v_2 on the left and right edges. Bottom: σ_{12} and σ_{22} on the top and bottom edges.

with the 5×5 cell assembly. The figure 1(b,right) shows that the loading D_{112} corresponds to a sort of bending. All other quadratic terms were tested but they cannot be presented in this note.

From these converged results, the actual fluctuation field $v_i(\boldsymbol{x})$ can be computed as the difference between the actual displacement field $u_i(\boldsymbol{x})$ provided by the Finite Element analysis and the polynomial function $D_{ijk}x_jx_k$. The found values of v_1 and v_2 are compared on the right and left sides of the central unit cell in figure 2. The component v_1 is found to be periodic whereas v_2 turns out to be anti–periodic for the considered loading condition. Similarly some of the traction vector components on the top and bottom edges of the central unit cell are compared in figure 2. The stress component σ_{22} is found to be periodic whereas σ_{12} is anti-periodic.

As a conclusion, the fluctuation field in the central unit cell under quadratic conditions prescribed on a remote boundary is generally not symmetric as anticipated in [6]. Similarly, the traction vector is generally not anti–periodic. We propose to use the elastic energy on the central unit cell to derive the effective higher order elastic properties.

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