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The Micromorphic Approach to Generalized Heat Equations

Abstract: In this paper, the micromorphic approach, previously developed in the mechanical context is applied to heat transfer and shown to deliver new generalized heat equations as well as the nonlocal effects. The latter are compared to existing formulations: the classical Fourier heat conduction, the hyperbolic type with relaxation time, the gradient of temperature or entropy theories, the double temperature model, the micro-temperature model or micro-entropy models. A new pair of thermodynamically-consistent micromorphic heat equations are derived from appropriate Helmholtz-free energy potentials depending on an additional micromorphic temperature and its first gradient. The additional micromorphic temperature associated with the classical local temperature is introduced as an independent degree of freedom, based on the generalized principle of virtual power. This leads to a new thermal balance equation taking into account the nonlocal thermal effects and involving an internal length scale which represents the characteristic size of the system. Several existing extended generalized heat equations could be retrieved from constrained micromorphic heat equations with suitable selections of the Helmholtz-free energy and heat flux expressions. As an example the propagation of plane thermal waves is investigated according to the various generalized heat equations. Possible applications to fast surface processes, nanostructured media and nanosystems are also discussed.

Keywords: hyperbolic heat conduction, finite propagation, micromorphic theory, nonlocal effects

1 Introduction

In the analysis of conduction heat transfer, the Fourier model [1, 2], indicating that the heat flux vector is proportional to the temperature gradient, is the most widely employed model in solving engineering problems. Such a heat conduction equation combined with the energy conservation law leads to a parabolic equation predicting an infinite propagation speed of thermal pulses [3–13]. From the experimental standpoint, this result is inaccurate for thermal conduction problems at cryogenic temperature or in the presence of strong temperature gradients during material surface processing by laser treatment or fast machining. For instance, the propagation speed of thermal signals has been measured in liquid helium [14, 15], in NaF at about 10°K [16] and in Bi at 3.4°K [17]. A thermal Mach number has also been reported for heat conduction through solids [18]. On the other side, several experiments or simulations on heat transport of materials with structure at nanometer length scale (e. g. semiconductor quantum dots and superlattices, carbon nanotubes, polymer nanocomposites, multilayer coatings, microelectronic and optoelectronic devices, and microelectromechanical sensors) [19–23] which involves high frequencies and small length scales, show significantly different results from those of the classical Fourier model. For instance, the heat flux obtained by applying a temperature gradient in nanometer-scale silicon samples is

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found to be significantly lower than that predicted by Fourier's model [24], which is suggested as the result of a reduction in the conductivity of the material. Considering the heat carriers' mean-free path l and a relevant characteristic length L of the system expressed by the Knudsen number $Kn = l/L$, the Fourier model is valid in the limit of very small Knudsen number (i. e. $l/L \ll 1$). When the Knudsen number becomes comparable to or higher than 1, due to an increase of mean-free path (as in rarefied gases and in aerospace engineering or in miniaturization/nano-technologies), the heat transport is no longer diffusive but ballistic. The nonlocal effects are especially important in the nanoscale structures [25–27]. A review of the current micro-, meso- and macroscopic methods in micro- and nanoscale heat transport has been presented in the literature [28].

To overcome the paradox of an infinite propagation speed of thermal signals, and to describe the heat transport behavior of nanoscale devices, many works have been done in the literature. They are briefly reviewed in what follows classified into four classes depending on the used theoretical framework:

1) Class I – modifications of the classical Fourier model: The original work was proposed by Cattaneo [3, 4] and Vernotte [5], which used an approximate version of kinetic theory of gases and suggested a modification of the constitutive equation of heat flux with a relaxation term. When combined with the first law of thermodynamics (energy conservation), this leads to a hyperbolic heat conduction equation, which removes the paradox of infinite temperature propagation speed and gives a solution more consistent with the experimental evidence. However, many works [6, 7, 10–12, 29] raise the question whether Cattaneo's hyperbolic heat equation is compatible with the second law of thermodynamics. They show that the modification of the Fourier model may yield a non-positive entropy production per unit volume, clearly in conflict with the second law of thermodynamics. Later, Müller [9, 30, 31], motivated by the work of Cattaneo, has tried to derive the modifications of the Fourier model of heat flux and Navier–Stokes equations from the thermodynamics of irreversible processes (TIP). He noticed that the argument of TIP could be used to derive the Cattaneo's equation from the generalized Gibbs equation. He proposed a so-called extended thermodynamics of irreversible processes based on the assumption that the generalized specific entropy is function of the heat flux. In this framework, he obtained a generalized heat equation predicting a finite speed while the second law of thermodynamics is fulfilled. Afterwards, a large number of papers have appeared, based on the Müller's methodology and mainly devoted to gases. A relatively complete review of this type of generalized irreversible thermodynamics can be found in Jou et al. [8, 32]. In the framework of the rational thermodynamics, Coleman et al. [6, 7] extended the specific internal energy to a quadratic function of the heat flux. Consequently, the specific entropy and specific Helmholtz-free energy were also extended by the heat flux, due to the relationship between internal energy and free energy. The evolution of temperature was governed by a pair of equations. Temizer et al. [33] derived a higher-order continuum formulations of linearized thermal conduction similar to the strain-gradient elasticity theory. In particular, a thermal dissipation potential that depends on the gradients of the temperature up to second-order was assumed. Accordingly, the constitutive equations associated with heat flux, derived from the dissipation potential, leads to extended Onsager-Casimir symmetry relationships. The Guyer-Krumhansl (GK) equation [34, 35], using the linearized Boltzmann equation of phonon kinetic theory in which heat transport is described by momentum and energy exchanges between colliding massless particles called phonons, emphasizes the role of nonlocal effects in heat transport. A new coefficient associated to nonlocality and higher spatial order of heat flux is introduced into the evolution equation of the heat flux. It has been successfully used to describe a heat pulse experiment at room temperature in a macroscopic, heterogeneous specimen, that cannot be modeled properly either by the Fourier model or by the Cattaneo equation [35]. In the references [32, 36, 37], the GK equation applied to model heat conduction at micro- and nanoscales has also been rigorously derived within the framework of extended irreversible thermodynamics (EIT). The other two enhanced heat fluxes: the Jeffreys type (lagging heat equation) and the Green-Naghdi equation can be also derived under the framework of EIT [36]. Similar to the Fourier model, the GK equation also predicts that thermal signals propagate at infinite velocity. In order to account for the nonlocal effects in metals

irradiated by ultrashort laser pulses inducing large temperature gradient, Sobolev [38–42] proposed an extension of Cattaneo's equation with an additional contribution from the second-order space derivative of heat flux, which can be regarded as an special case of the GK equation, and applied it to the double temperature model. In order to be satisfactory for high-frequency processes, all the higher-order fluxes must be incorporated into the formalism. The kinetic theory points out the fact that the relaxation times of higher-order fluxes are not always shorter than the collision time. When the frequency becomes comparable to the inverse of the relaxation time of the first-order flux, they will behave like independent variables [25]. This yields the Ballistic heat transport model, in which the effective thermal conductivity is expressed in terms of the classical bulk thermal conductivity, the relaxation time and the Knudsen number. The dual-phase-lag model [22, 43] evolving from the Fourier model introduces two phase lags: the phase lag of the heat flux captures the small-scale response in time and the phase lag of the temperature gradient captures the small-scale response in space, into the constitutive equation of heat flux. Combining these constitutive models with the energy balance law leads to a fourth-order partial differential heat equation as well as third-order time derivative. The Boltzmann equation being hard to solve, Chen [44] established the ballistic-diffusive heat-conduction equations, which are applicable to transient heat conduction and are derived from the Boltzmann equation under the relaxation approximation and from the splitting of the distribution function into a diffusive and a ballistic parts, as required in small structures. Computational results suggest that it is a much better alternative to the Fourier and the Cattaneo equations at scales where the mean-free path is comparable to the system size and the time is comparable to the carrier relaxation time. In fact, to obtain a finite velocity for the propagation of thermal signals it is not necessary to assume relaxation terms in heat equation. This result can be also obtained if one assumes a nonlinear diffusion equation with the thermal conductivity depending on the temperature in an appropriate way (Luikov et al. [45], Bubnov [46] and Swenson [13]).

2) Class II – considering the temperature gradient as an argument of the Helmholtz-free energy: There are essentially three ways to comply with the second law of thermodynamics when introducing the gradient of temperature or gradient of entropy:

   a. by introducing an extra entropy flux: Ireman et al. [47], based on the idea of Maugin's [48–50] work on gradients of internal variables, discuss several ways of introducing the gradient of temperature in continuum thermodynamics; the gradient theory assumes additional entropy production;

   b. modification with extra energy production: Forest et al. [51] made an attempt to incorporate temperature and temperature gradient into the second grade theory, with extra energy production (extra power) by extending the method of virtual power and continuum thermodynamics. An extra entropy production from the divergence of the generalized thermodynamic forces associated with the temperature gradient was derived. Later, Forest et al. [52] developed the entropy gradient theory and derived an enhanced heat equation which has the structure of the Cahn-Hilliard equation in mass transport theory. He also showed that the entropy gradient theory and temperature gradient theory are not equivalent. The generalized heat equation derived from the temperature gradient theory can be regarded as an approximation of the reference gradient of entropy theory;

   c. modification with additional entropy production. Nguyen [53] proposed an additional entropy production and a new relationship between internal energy and free energy to introduce the gradient of temperature in the set of state variables. The deduced generalized heat equation predicts a finite propagation of thermal pulses.

3) Class III – double temperature model: Aifantis [54–56] derived the double temperature model, from electron-phonon collisions during the ultra-short pulsed laser heating of metals, based on the mixture theory of Müller [57]. Sobolev [42] showed that the two temperature model using Fourier's model is a parabolic heat equation. He also derived an extended hyperbolic two temperature model by using the Cattaneo's equation.

4) Class IV – the micro-temperature or micro-entropy theory: The micro-temperature, that depends on the micro-coordinates of the micro-elements, was initially considered by Wozniak [58] assuming known

functions of temperature and temperature gradients. Based on the work on kinematics and dynamics of a continuum with microstructure by Eringen and Suhubi [59, 60], Grot [61] proposed a theory of thermodynamics for elastic materials with microstructure which possessed micro-deformations and micro-temperatures. The micro-temperatures were determined by a balance law. The second law of thermodynamics is modified to include micro-temperature, and the first moment of the energy equation is added to the usual balance laws of a continuum with microstructure. From the linear approximation, the temperature and micro-temperature are determined by a pair of equations. Later, Iesan [62] also deduced the same pair of governing heat equations using homogeneous and isotropic micromorphic continua [59, 60, 63–66]. The theory of micromorphic fluids with microstructures has been studied in various papers [67–70]. A study of the problem of heat conduction in micromorphic continua was presented by Riha [71] and an agreement was obtained between theoretical results and experimental data for the silicone rubber containing spherical aluminum particles and for the human body blood. Green et al. [72] illustrated in details of the thermal propagation in a rigid solid at finite speed, based on the established thermomechanical theory with the introduction of a balance of entropy and the use of energy equation as an identity for all motions and all temperature fields after the elimination of external fields [73]. Later, a thermoelasticity theory without energy dissipation for nonpolar bodies, based on the new thermomechanical theory, has been presented in Ref. [74]. It permits the transmission of heat as thermal waves at finite speed. Iesan et al. [75] used the procedure proposed by Green et al. [73] and extended it to the micromorphic continua. In the linearized theory, it predicts the thermal propagation with finite speed. In the previous theories, the micro-temperature is a vector quantity akin to a temperature gradient. In contrast, Forest et al. [56] present theories involving scalar micro-temperature or micro-entropy, by applying the micromorphic approach [76, 77] extending the local model to the micromorphic framework, to the temperature and entropy, respectively. They showed that the gradient entropy theory and gradient temperature theory can be regarded as a limit case of the micro-entropy and micro-temperature theories.

The objective of the present paper is to make a thorough discussion of existing extended heat equations and to present the generalized heat equations derived from the micromorphic approach to heat transfer, thus extending the model class IV. According to the different approaches, we summarize various heat equations into six classifications: (i) classical type using the Fourier model, (ii) hyperbolic type with relaxation time, (iii) double temperature model, (iv) temperature (or entropy) gradient theory, (v) micro-temperature model and (vi) micromorphic theory, as listed in Table 1. Section 2 reviews the classical formulations of heat equation of deformable continua under the framework of irreversible thermodynamics. Using the Fourier model of heat flux leads to the most widely used parabolic heat equation. Section 3 presents the two extended hyperbolic heat equations with relaxation time proposed by Cattaneo and Vernotte, and the works from Müller and Coleman et al. to make the hyperbolic heat equation compatible with the second law of thermodynamics, named extended thermodynamic of irreversible processes (extended TIP) and rational thermodynamics respectively. In Section 4, the temperature gradient and entropy gradient based theories are recalled. The extended internal energy and Helmholtz-free energy are proposed as functions of temperature gradient or entropy gradient, to account for the nonlocality effects. Section 5 reviews the double temperature model used extensively in ultra-short pulsed laser heating of metals, the hyperbolic two-temperature type and the extended generalized two-temperature heat equation with nonlocal effects. In Section 6, we recall the works of Iesan on the heat conduction in linear micromorphic elastic continua deduced from the Grot's theory or from Green's theory, and the work of Forest on micro-temperature and micro-entropy model by using the micromorphic approach. In Section 7, the generalized micromorphic heat equation, based on the micromorphic approach, is presented including new additional terms associated with thermal inertia. It shows that the heat equations from micro-temperature model, the double temperature model, gradient theory of temperature and hyperbolic type with relaxation time, can be retrieved from this generalized micromorphic heat equation by choosing different Helmholtz-free energies and the heat flux model. Section 8

Table 1: Various generalized heat equations.

Model	Authors	Heat equation
Classical heat equation with Fourier's model	Fourier, 1878	$\rho C_\varepsilon \dot{T} = \kappa \Delta T$
Hyperbolic type with relaxation time	Cattaneo I, 1948	$\rho C_\varepsilon \dot{T} + \tau \kappa \Delta \dot{T} = \kappa \Delta T$
	Cattaneo II, 1948	$\rho C_\varepsilon \dot{T} + \tau \rho C_\varepsilon \ddot{T} = \kappa \Delta T$
	Coleman et al., 1982	$\begin{cases} \left(\rho C_\varepsilon + \vec{q} \cdot \frac{dA}{dT} \cdot \vec{q} \right) \dot{T} + 2\vec{q} \cdot \underline{A} \cdot \dot{\vec{q}} = -div(\vec{q}) \\ \vec{q} + \underline{\tau} \cdot \dot{\vec{q}} = -\underline{\kappa} \cdot \vec{\nabla} T \end{cases}$
Temperature (or Entropy) gradient model	Nguyen, 2010	$\rho C_\varepsilon \dot{T} + M \vec{\nabla} \dot{T} \cdot \vec{\nabla} T = \kappa \Delta T$
	Forest, 2008	$\rho C_\varepsilon \dot{T} + T_0 A_T \Delta \dot{T} = \kappa \Delta T$
	Forest, 2008	$\rho C_\varepsilon \dot{T} + T_0 A_T \Delta \dot{T} + \frac{A_s^2 C_\varepsilon^2}{\rho T_0} \Delta^2 \dot{s} = \kappa \Delta T$
	Forest, 2010	$\rho C_\varepsilon \dot{T} - \alpha T_0 \ddot{T} + M T_0 \Delta \dot{T} = \kappa \Delta T$
Double temperature model	Aifantis, 1980	$\dot{T} + \alpha \ddot{T} - \beta \Delta \dot{T} + \gamma \Delta^2 T = \lambda \Delta T$
	Sobolev, 2016	$\dot{T}_i + (\tau + \tau_{12}) \ddot{T}_i + \tau \tau_{12} \dddot{T}_i - l^2 \Delta \dot{T}_i = \bar{\kappa} \Delta T_i$
		$\dot{T}_i + (\tau + \tau_{12}) \ddot{T}_i + \tau \tau_{12} \dddot{T}_i - (l^2 + l_e^2) \Delta \dot{T}_i - l_e^2 \tau_{12} \Delta^2 T_i = \bar{\kappa} \Delta T_i$
Micro-temperature model	Iesan, 2002	$\begin{cases} c\dot{T} = k\Delta T + k_1 div\left(\vec{T} \right) \\ b\dot{\vec{T}} = k_6 \Delta \vec{T} + (k_4 + k_5) \vec{\nabla} \left(div\left(\vec{T} \right) \right) - k_2 \vec{T} - k_3 \vec{\nabla} T \end{cases}$
	Iesan, 2005	$\begin{cases} a\ddot{T} + m \, div\left(\dot{\vec{T}} \right) = k\Delta T \\ b\ddot{\vec{T}} = d_2 \Delta \vec{T} + (d_1 + d_3) \vec{\nabla} \left(div\left(\vec{T} \right) \right) - m \vec{\nabla} \dot{T} \end{cases}$
Micromorphic temperature model	Micro-temperature type	$\rho T \left(\varsigma_2 \dddot{\breve{T}} + \varsigma_1 \ddot{\breve{T}} \right) - T \breve{M}^g \Delta \dot{\breve{T}}$ $- T \left[\frac{1}{2} \rho C_\varepsilon \left(\frac{\ddot{\breve{T}}}{\breve{T}} - \frac{\dot{T}}{T} \right) + 2\breve{M} \left(\dot{\breve{T}} - \dot{T} \right) \right] = \kappa \Delta T$
	Two-temperature type	$\rho C_\varepsilon \dot{T} + \rho T \left(\varsigma_2 \dddot{\breve{T}} + \varsigma_1 \ddot{\breve{T}} \right) - 2 T \breve{M} \left(\dot{\breve{T}} - \dot{T} \right) - T \breve{M}^g \Delta \dot{\breve{T}} = \kappa \Delta T$ $\rho C_\varepsilon \dot{T} + \rho (\tau C_\varepsilon + T_0 \varsigma_1) \ddot{T} + \rho T_0 (\tau \varsigma_1 + \varsigma_2) \dddot{T}$ $+ \tau \rho T_0 \varsigma_2 \ddddot{T} - T_0 \breve{M}^g \Delta \dot{\breve{T}} - T_0 \breve{M}^g \Delta \ddot{\breve{T}} = \kappa \Delta T$

examines the propagation of thermal pulses in the generalized heat equations with the help of a simple plane thermal wave problem. The presentation is limited to the small strain framework for the sake of simplicity.

2 Classical heat equation with internal variables

In the classical thermodynamics, for any material volume V, the global form of the first law (energy conservation) can be written as:

$$\frac{dE}{dt} + \frac{dK}{dt} = W + Q \tag{1}$$

where E denotes the internal energy, K is the kinetic energy, W is the mechanical power input, and Q is the heat quantity input to the system.

By using the material time derivatives of a volume integral together with the divergence theorem in order to transform surface integrals into volume integrals, and by applying the spatial localization lemma (or local action theorem) for materially simple continua, the localized (differential) form of the first principle of thermodynamics is obtained as:

$$\rho \dot{e} = \underline{\sigma} : \underline{\dot{\varepsilon}} - div(\vec{q}) + \xi \tag{2}$$

where ρ denotes the mass density of the body, e is the (specific) internal energy per unit mass, $\underline{\sigma}$ is the Cauchy stress tensor, $\underline{\dot{\varepsilon}}$ is the strain rate tensor, $\vec{q}$ indicates the heat flux vector received across the boundaries of the body and ξ is the body heat source.

The integral form of the second law (inequality of entropy) of thermodynamics is given by:

$$\frac{d}{dt} \int_V (\rho s) dV - \int_V \left(\frac{\xi}{T} \right) dV + \int_\Gamma \left(\frac{\vec{q}}{T} \cdot \vec{n} \right) dS \geq 0 \tag{3}$$

where s is the specific entropy, T is the absolute temperature and $\vec{n}$ is the outward normal to the boundary surface. The use of the divergence theorem together with the principle of space localization for materially simple continua, leads to the local (differential) form of the second principle of thermodynamics in the current configuration:

$$\rho \dot{s} + div \left(\frac{\vec{q}}{T} \right) - \frac{\xi}{T} \geq 0 \tag{4}$$

The relationship between internal energy and the Helmholtz-free energy is assumed as:

$$\psi(\underline{\varepsilon}^e, \underline{\alpha}_n, T) = e - Ts \tag{5}$$

where ψ is the specific Helmholtz-free energy, $\underline{\varepsilon}^e$ is the reversible part of the strain tensor and $\underline{\alpha}_n$ are the (n) internal state variables associated with intrinsic dissipative phenomena (i. e. kinematic hardening or isotropic hardening, ...). The dual stress-like variables associated with these internal state variables are denoted by $\underline{A}_n$.

Substituting eq. (2) and eq. (5) into eq. (4) to eliminate the body heat source ξ and the specific internal energy leads to the so-called Clausius–Duhem inequality under its local form:

$$\underline{\sigma} : \underline{\dot{\varepsilon}} - \rho \left(\dot{\psi} + s\dot{T} \right) - \frac{\vec{q}}{T} \cdot \vec{\nabla} T \geq 0 \tag{6}$$

Here, the Helmholtz-free energy is assumed to be a positive, close and convex function of the reversible strain $\underline{\varepsilon}^e$ and the overall strain-like internal state variables $\underline{\alpha}_n$, and is a concave function of temperature T. By assuming the additive decomposition of the total strain rate into elastic part and irreversible part ($\underline{\dot{\varepsilon}} = \underline{\dot{\varepsilon}}^e + \underline{\dot{\varepsilon}}^{ir}$), the Clausius–Duhem inequality eq. (6) can be expressed as:

$$\left(\underline{\sigma} - \rho \frac{\partial \psi}{\partial \underline{\varepsilon}^e} \right) : \underline{\dot{\varepsilon}}^e - \rho \left(\frac{\partial \psi}{\partial T} + s \right) \dot{T} + \underline{\sigma} : \underline{\dot{\varepsilon}}^{ir} - \frac{\partial \psi}{\partial \underline{\alpha}_n} : \underline{\alpha}_n - \frac{\vec{q}}{T} \cdot \vec{\nabla} T \geq 0 \tag{7}$$

For simplicity, we assume that the terms $\left(\underline{\sigma} - \rho \frac{\partial \psi}{\partial \underline{\varepsilon}^e} \right)$ and $\left(\frac{\partial \psi}{\partial T} + s \right)$ do not depend on the rates of the state variables respectively, then following standard arguments (Truesdell and Noll [78]), we obtain the state relations:

$$\underline{\sigma} = \rho \frac{\partial \psi}{\partial \underline{\varepsilon}^e} \quad s = -\frac{\partial \psi}{\partial T} \tag{8}$$

and the residual dissipation:

$$\varphi_V = \varphi_{in} + \varphi_{th} = \underline{\sigma} : \underline{\dot{\varepsilon}}^{ir} - \underline{A}_n : \underline{\dot{\alpha}}_n - \frac{\vec{q}}{T} \cdot \vec{\nabla} T \geq 0 \tag{9}$$

where $\varphi_{in} = \underline{\sigma} : \underline{\dot{\varepsilon}}^{ir} - \underline{A}_n : \underline{\dot{\alpha}}_n$ is the intrinsic dissipation with $\underline{A}_n = \frac{\partial \psi}{\partial \underline{\alpha}_n}$, and $\varphi_{th} = -\frac{\vec{q}}{T} \cdot \vec{\nabla} T$ is the thermal dissipation.

Using the deduced state relations eq. (8), and substituting eq. (5) into the first law of thermodynamics eq. (2), lead to the classical local form of the heat equation:

$$\rho \dot{s} T + div(\vec{q}) - \xi - \varphi_{in} = 0 \tag{10}$$

Considering the rigid body heat conduction ($\varphi_{in} = 0$) in the absence of body heat source, the above heat equation reduces to:

$$\rho C_\varepsilon \dot{T} + div(\vec{q}) = 0 \tag{11}$$

where $C_\varepsilon = T \frac{\partial s}{\partial T}$ denotes the specific heat capacity at constant volume. In particular, if the Fourier model [1] of heat flux is assumed:

$$\vec{q} = -\kappa \vec{\nabla} T \tag{12}$$

with $\kappa > 0$ is the thermal conductivity coefficient for isotropic materials, then, the equation (eq. (11)) leads to the following classical parabolic heat equation:

$$\rho C_\varepsilon \dot{T} = \kappa \Delta T \tag{13}$$

where Δ stands for the Laplacian operator.

As shown in Refs [3–5, 32], the parabolic heat equation results in a paradox of propagation of thermal signals with infinite speed. In the linear approximation, this implies that the influence of such a signal is felt immediately throughout the whole system [32]. However, from the experimental point of view, Peshkov [14] found that in liquid Helium II, the thermal wave velocity is one order of magnitude smaller than the speed of sound and is called the speed of second sound. Recent measurements of the propagation speed of thermal signals at room temperature, in inhomogeneous materials, have been performed by Kaminski [79] showing clearly that the parabolic heat model is inaccurate to describe these phenomena.

3 Hyperbolic heat equation with relaxation time

3.1 The heat equation of Cattaneo I

Cattaneo [3] addressed the question of the paradox of heat conduction in 1948. He modified the (stationary) Fourier model based on the elementary kinetic theory of gases. He focused upon a small volume element of linear dimensions of the mean-free path of the molecules in which a temperature gradient prevails, to review the Maxwell's molecular interpretation to heat conduction. After, he changed this argument slightly and argued that there is a time-lag between the start of the particles at their point of departure and the time of passing through the middle layer [9]. If the temperature changes in time, it is thus clear that the heat flux at a certain time depends on the temperature gradient at an earlier time. It therefore seemed reasonable to formulate the transient Fourier model as:

$$q_i = -\kappa \left(\frac{\partial T}{\partial x_i} - \tau \frac{d}{dt} \left(\frac{\partial T}{\partial x_i} \right) \right) \tag{14}$$

where τ is a positive scalar called thermal relaxation time of the heat conducting medium (free electrons in the case of metals).

By substituting the above modified transient Fourier model into the heat equation (eq. (11)), the following partial differential heat equation of rigid body is obtained:

$$\rho C_\varepsilon \dot{T} + \tau \kappa \Delta \dot{T} = \kappa \Delta T \tag{15}$$

Clearly, the above deduced heat equation (eq. (15)) is not a hyperbolic type, and therefore predicts infinite speed for the temperature propagation.

3.2 The heat equation of Cattaneo II

Later, Cattaneo proceeded to modify the equation (eq. (14)) by assuming that the operator $\tau(d/dt)$ is small such that:

$$\left(1 - \tau \frac{d}{dt}\right)^{-1} \approx 1 + \tau \frac{d}{dt} \tag{16}$$

By using this approximation, the transient Fourier model eq. (14) transforms into the so-called Cattaneo equation:

$$q_i + \tau \dot{q}_i = -\kappa \frac{\partial T}{\partial x_i} \tag{17}$$

Combining the above equations (eqs. (2), (10)) and the state relations eq. (8), we can easily obtain the following relationship:

$$\rho \dot{e} = \rho C_\varepsilon \dot{T} + \underline{\sigma} : \underline{\dot{\varepsilon}} - \varphi_{in} - T \left(\frac{\partial \underline{\sigma}}{\partial T} : \underline{\dot{\varepsilon}}^e + \frac{\partial A_n}{\partial T} : \underline{\dot{\alpha}}_n\right) \tag{18}$$

Substituting Cattaneo's equation eq. (17) into eq. (2) and using the above relation eq. (18), leads to the following generalized heat equation with internal variables:

$$\left[\rho C_\varepsilon \dot{T} - \varphi_{in} - T \left(\frac{\partial \underline{\sigma}}{\partial T} : \underline{\dot{\varepsilon}}^e + \frac{\partial A_n}{\partial T} : \underline{\dot{\alpha}}_n\right)\right] + \tau \frac{\partial}{\partial t}\left(\rho C_\varepsilon \dot{T} - \varphi_{in} - T \left(\frac{\partial \underline{\sigma}}{\partial T} : \underline{\dot{\varepsilon}}^e + \frac{\partial A_n}{\partial T} : \underline{\dot{\alpha}}_n\right)\right) = \kappa \Delta T + \xi + \tau \dot{\xi} \tag{19}$$

Concerning the rigid body heat conduction (with neither intrinsic dissipation nor deformations), the heat equation will transform to the often called Cattaneo's hyperbolic heat equation:

$$\rho C_\varepsilon \dot{T} + \tau \rho C_\varepsilon \ddot{T} = \kappa \Delta T \tag{20}$$

This is the telegraph equation. It is hyperbolic, if $\tau > 0$ holds.

Let us assume the plane thermal wave [32]:

$$T(x, t) = T_0 \exp[i(kx - \omega t)] \tag{21}$$

where T_0 indicates the amplitude, k is the (complex) wave number, w is the (real) frequency. The dispersion relation obtained by substituting eq. (21) into eq. (20) is:

$$-i\omega - \tau \omega^2 + \frac{\kappa}{\rho C_\varepsilon} k^2 = 0 \tag{22}$$

By solving the above equation with k a complex number, the nonlinear dispersion equation of the thermal waves and the attenuation distance α are obtained as:

$$v_p = \frac{\omega}{\mathrm{Re}(k)} = \frac{\sqrt{2\frac{\kappa}{\rho C_\varepsilon}\omega}}{\sqrt{\tau\omega + \sqrt{1 + \tau^2\omega^2}}} \tag{23}$$

$$\alpha = \frac{1}{\mathrm{Im}(k)} = 2\frac{\kappa}{\rho C_\varepsilon v_p} \tag{24}$$

For low frequencies ($\tau\omega \ll 1$), the velocity $v_p \approx \sqrt{2\kappa\omega/\rho C_\varepsilon}$ and $\alpha = \sqrt{2\kappa/\rho C_\varepsilon\omega}$, which are the results predicted by the classical heat equation using Fourier's model. In high frequency limit ($\tau\omega \gg 1$), the first-order term in eq. (20) is small compared with the two other terms, and eq. (20) becomes a wave equation

whose solution is known in the literature as second sound. The quantities v_p and α tend to the limiting values: $v_p \approx \sqrt{\frac{\kappa}{\rho C_\varepsilon \tau}}$ and $\alpha \approx 2\sqrt{\frac{\kappa \tau}{\rho C_\varepsilon}}$.

However, the most critical argument on Cattaneo's hyperbolic heat equation is its inconsistency with the second law of thermodynamics [6, 7, 10–12, 29]. Cattaneo's hyperbolic heat equation may yields negative values of the entropy production rate per unit volume.

Combing the local forms of first and second principles (eq. (2) and eq. (4)), the relation $de = Tds$ holds locally leading to the following expression of the entropy production rate [11]:

$$s_p = -\frac{\vec{q}}{T^2} \cdot \vec{\nabla} T \tag{25}$$

Substituting the generalized heat flux eq. (17) into the above entropy production rate equation leads to:

$$s_p = \frac{1}{\kappa T^2}\left(\vec{q} \cdot \vec{q} + \tau \vec{q} \cdot \frac{\partial \vec{q}}{\partial t}\right) \tag{26}$$

Whenever the heat flux $\vec{q}$ has a constant direction and decreases at some point so deeply that $|\partial \vec{q}/\partial t| > |\vec{q}|/\tau$, the right-hand side of eq. (26), becomes negative and the local equilibrium scheme cannot be applied.

3.3 The model of Müller from extended TIP

Motivated by the work of Cattaneo, Müller [9, 30, 31] tried to derive the modifications of the Fourier model of heat flux from the principle of thermodynamics of irreversible processes (TIP). He noticed that the argument of TIP could be used to derive the Cattaneo equation. All what we need is to assume that the specific entropy depends not only on the classical variables but also on the heat flux. The generalized Gibbs equation is given by:

$$\dot{s} = \frac{1}{T}\left(\dot{e} - \frac{p}{\rho^2}\dot{\rho} + 2a T\vec{q} \cdot \dot{\vec{q}} + 2bT\underline{\sigma}^{dev} : \underline{\dot{\sigma}}^{dev} + 2cT\pi\dot{\pi}\right) \tag{27}$$

where p is the pressure, $\underline{\sigma}^{dev}$ is the deviatoric part of stress tensor, $\pi = (tr(\underline{\sigma})/3 + p)$ is the dynamic pressure, and the negative coefficients a, b and c enable s to have a maximum in equilibrium, when $\vec{q}$, $\underline{\sigma}^{dev}$ and π all vanish.

By using the mass conservation law, momentum conservation and energy conservation, eq. (27) becomes the following equation of entropy balance [9]:

$$\rho \dot{s} + \vec{\nabla} \cdot \left(\frac{\vec{q}}{T}\right) = \vec{q} \cdot \left(\vec{\nabla}\left(\frac{1}{T}\right) + 2\rho a \dot{\vec{q}}\right) + \underline{\sigma}^{dev} : \left(\frac{1}{T}\left(\vec{\nabla}\vec{v}\right)^{dev} + 2\rho b \underline{\dot{\sigma}}^{dev}\right) + \pi \left(\frac{1}{T}\vec{\nabla} \cdot \vec{v} + 2\rho c \dot{\pi}\right) \tag{28}$$

Müller interpreted $\vec{q}/T$ as the entropy flux and the right side of eq. (28) as the entropy production, which is a sum of products of the thermodynamic fluxes and modified thermodynamics forces. However, because the definition of the entropy flux is ambiguous, it is possible to add the term $K\underline{\sigma}^{dev} \cdot \vec{q} + L\pi\vec{q}$ with arbitrary coefficients K and L, to the usual entropy flux without upsetting the structure of the entropy production as a sum of products of forces and fluxes:

$$\begin{aligned}\rho \dot{s} + \vec{\nabla} \cdot \vec{\phi} = \vec{q} \cdot &\left(\vec{\nabla}\left(\frac{1}{T}\right) + 2\rho a \dot{\vec{q}} + K\vec{\nabla} \cdot \underline{\sigma}^{dev} + L\vec{\nabla}\pi\right) \\ &+ \underline{\sigma}^{dev} : \left(\frac{1}{T}\left(\vec{\nabla}\vec{v}\right)^{dev} + 2\rho b \underline{\dot{\sigma}}^{dev} + K\vec{\nabla}\vec{q}\right) + \pi\left(\frac{1}{T}\vec{\nabla} \cdot \vec{v} + 2\rho c \dot{\pi} + L\vec{\nabla} \cdot \vec{q}\right)\end{aligned} \tag{29}$$

where the generalized entropy flux is $\vec{\phi} = \left(\frac{\vec{q}}{T} + K\underline{\sigma}^{dev} \cdot \vec{q} + L\pi\vec{q}\right)$. The entropy production must be non-negative. For simplification, linear phenomenological relations between the fluxes and the forces are assumed:

$$\vec{q} = -\kappa\left(\vec{\nabla}T - 2\rho aT^2\dot{\vec{q}} - KT^2\vec{\nabla}\cdot\underline{\sigma}^{dev} - LT^2\vec{\nabla}\pi\right)$$

$$\underline{\sigma}^{dev} = 2\mu\left(\left(\vec{\nabla}\vec{v}\right)^{dev} + 2\rho bT\underline{\dot{\sigma}}^{dev} + KT\vec{\nabla}\vec{q}\right)$$

$$\pi = \lambda\left(\vec{\nabla}\cdot\vec{v} + 2\rho cT\dot{\pi} + LT\vec{\nabla}\cdot\vec{q}\right)$$

It is worth noting that the heat flux in eq. (30) is equivalent to Cattaneo's equation eq. (17), if the coefficients K and L vanish and $\tau = -2\rho aT^2$ holds. And from eq. (29), it clearly shows that eq. (30) is compatible with the second law of thermodynamics.

This theory is called extended TIP, because it has extended the list of state variables on which the specific entropy and the entropy flux can depend. After this modification of ordinary TIP, a large number of papers have appeared, using the Müller's methodology and devoted in particular to gases. An exhaustive review of this type of irreversible thermodynamics is given by Jou et al. [8, 32].

3.4 The model of Coleman from rational thermodynamics

Coleman et al. [6, 7] derived the restrictions that the second law of thermodynamics imposes on constitutive equations of the type proposed by Cattaneo. For generalization, Cattaneo's equation (eq. (17)) is equivalent to:

$$\vec{q} + \underline{\tau}(T)\cdot\dot{\vec{q}} = -\underline{\kappa}(T)\cdot\vec{\nabla}T \tag{31}$$

with $\underline{\tau}(T)$ and $\underline{\kappa}(T)$ non-singular second-order tensors that, as function of temperature, depend on the material under consideration. Clearly, in the isotropic case, it will be the same as eq. (17).

The authors showed that the relation eq. (31) with the tensors $\underline{\tau}(T)$ and $\underline{\kappa}(T)$ non-singular is compatible with the thermodynamics only if $\underline{\kappa}(T)$ is positive definite, the tensor $Z(T) = \underline{\kappa}(T)^{-1}\underline{\tau}(T)$ is symmetric, and the specific internal energy e, the specific entropy s and the specific Helmholtz-free energy $\psi = e - Ts$ are not functions of T only, but are instead given by functions $\tilde{e}$, $\tilde{s}$ and $\tilde{\psi}$ of the form:

$$\rho e = \rho\tilde{e}(T,\vec{q}) = \rho e_0(T) + \vec{q}\cdot\underline{A}(T)\cdot\vec{q}$$

$$\rho s = \rho\tilde{s}(T,\vec{q}) = \rho s_0(T) + \vec{q}\cdot\underline{B}(T)\cdot\vec{q} \tag{32}$$

$$\rho\psi = \rho\tilde{\psi}(T,\vec{q}) = \rho\psi_0(T) + \frac{1}{2T}\vec{q}\cdot\underline{Z}(T)\cdot\vec{q}$$

with

$$\underline{A}(T) = -\frac{T^2}{2}\frac{d}{dT}\left(\frac{\underline{Z}(T)}{T^2}\right) \qquad \underline{B}(T) = -\frac{1}{2}\frac{d}{dT}\left(\frac{\underline{Z}(T)}{T}\right) \tag{33}$$

here, e_0, s_0 and ψ_0 are the specific internal energy, specific entropy and specific Helmholtz-free energy at equilibrium. In the absence of both deformation and a supply of heat by radiation, clearly, $\dot{e}$ is not given by the classical formula eq. (18) $\dot{e} = C_\varepsilon(T)\dot{T}$ (with $C_\varepsilon(T)$ being the equilibrium heat capacity), but is instead given by:

$$\rho\dot{e} = \left[\rho C_\varepsilon(T) + \vec{q}\cdot\frac{d}{dT}\underline{A}(T)\cdot\vec{q}\right]\dot{T} + 2\vec{q}\cdot\underline{A}(T)\cdot\dot{\vec{q}} \tag{34}$$

Thus, the evolution of the heat flux and temperature field is governed by a pair of nonlinear partial differential equations:

$$\begin{cases} \vec{q} + \underline{\tau}(T)\cdot\dot{\vec{q}} + \underline{\kappa}(T)\cdot\vec{\nabla}T = \vec{0} \\ div(\vec{q}) + \left(\rho C_\varepsilon(T) + \vec{q}\cdot\frac{d}{dT}\underline{A}(T)\cdot\vec{q}\right)\dot{T} + 2\vec{q}\cdot\underline{A}(T)\cdot\dot{\vec{q}} = 0 \end{cases} \tag{35}$$

It is worth noting that if the parameter $\underline{Z}(T)$ is isotropic and equals to $Z(T) = DT^2$ (where D is constant) [10], the general result from eq. (32) reduces to $\rho\tilde{e}(T, \vec{q}) = \rho e_0(T)$ and $\rho\tilde{s}(T, \vec{q}) = \rho s_0(T) - \frac{D}{2}\vec{q} \cdot \vec{q}$, which is similar to the relationship given by Jou et al. [8].

By comparing the two non-equilibrium thermodynamics theories, it is clear that: (i) the extended TIP theory modifies the entropy (entropy flux and entropy production) from the generalized Gibbs equation; (ii) the rational thermodynamics theory starts from the extension of the specific internal energy, and consequently, (iii) the specific entropy and specific Helmholtz-free energy are enhanced automatically due to the dual relation $\psi = e - Ts$. In summary, both theories contain Cattaneo's equation and make it compatible with the second law of thermodynamics.

4 Temperature or entropy gradient dependent free energy potential

4.1 Temperature gradient model from Nguyen

Nguyen et al. [47, 53, 80] assumed an extra entropy supply and modified the relationship between internal energy and Helmholtz-free energy, to account for the temperature gradient effects. The dual relationship is given under the following modified form:

$$e(s, \vec{s}, \vec{\nabla}\vec{u}, r) = \psi(T, \vec{\nabla}T, \vec{\nabla}\vec{u}, r) + sT + \vec{s} \cdot \vec{\nabla}T \tag{36}$$

with the introduction of a new specific entropy vector $\vec{s}$.

The first principle (energy conservation) of thermodynamics remains unchanged, while an additional entropy source is assumed in the second law, the local forms being given by:

$$\begin{cases} \rho\dot{e} = \underline{\sigma} : \underline{\dot{\varepsilon}} - div(\vec{q}) + \xi \\ \rho\left(\dot{s} + \frac{\dot{\vec{s}}}{T} \cdot \vec{\nabla}T\right) - \left(\frac{\xi}{T}\right) + div\left(\frac{\vec{q}}{T}\right) \geq 0 \end{cases} \tag{37}$$

Combining the above equations (eq. (37) and eq. (36)) leads to the following local form of the Clausius–Duhem inequality:

$$\left(\underline{\sigma} - \rho\frac{\partial\psi}{\partial\underline{\dot{\varepsilon}}^e}\right) : \underline{\dot{\varepsilon}}^e - \rho\left(\frac{\partial\psi}{\partial T} + s\right)\dot{T} - \rho\left(\frac{\partial\psi}{\partial\vec{\nabla}T} + \vec{s}\right) \cdot \vec{\nabla}\dot{T} + \underline{\sigma} : \underline{\dot{\varepsilon}}^{ir} - \frac{\partial\psi}{\partial\underline{\alpha}_n} : \underline{\dot{\alpha}}_n - \frac{\vec{q}}{T} \cdot \vec{\nabla}T \geq 0 \tag{38}$$

For simplicity, we assume that the terms $\left(\underline{\sigma} - \rho\frac{\partial\psi}{\partial\underline{\varepsilon}^e}\right)$, $\left(\frac{\partial\psi}{\partial T} + s\right)$ and $\left(\frac{\partial\psi}{\partial\vec{\nabla}T} + \vec{s}\right)$ do not depend on the rates of the state variables respectively, then following standard arguments (Truesdell and Noll [78]), we obtain the following state relations:

$$\underline{\sigma} = \rho\frac{\partial\psi}{\partial\underline{\varepsilon}^e} \quad s = -\frac{\partial\psi}{\partial T} \quad \vec{s} = -\frac{\partial\psi}{\partial\vec{\nabla}T} \tag{39}$$

and the classical residual dissipation:

$$\varphi_V = \varphi_{in} + \varphi_{th} = \underline{\sigma} : \underline{\dot{\varepsilon}}^{ir} - \underline{A}_n : \underline{\dot{\alpha}}_n - \frac{\vec{q}}{T} \cdot \vec{\nabla}T \geq 0 \tag{40}$$

Substituting the new dual relations into the first principle of thermodynamics yields the following form of the heat equation:

$$\rho T\dot{s} + \rho\dot{\vec{s}} \cdot \vec{\nabla}T - \xi + div(\vec{q}) - \varphi_{in} = 0 \tag{41}$$

In the case of rigid body conduction (no intrinsic dissipation), the free energy can be defined as follows:

$$\rho\psi\left(T, \vec{\nabla}T\right) = -\rho C_\varepsilon T\left(\ln\frac{T}{T_0} - 1\right) - \frac{M}{2}\vec{\nabla}T \cdot \vec{\nabla}T \tag{42}$$

where $C_\varepsilon > 0$ and $M > 0$ are two positive material coefficients [53]. Using the state relations eq. (39), the generalized heat equation is given by:

$$\rho C_\varepsilon \dot{T} + M\vec{\nabla}\dot{T} \cdot \vec{\nabla}T + div(\vec{q}) = 0 \tag{43}$$

Taking into account the Fourier model of heat flux, the heat equation just becomes:

$$\rho C_\varepsilon \dot{T} + M\vec{\nabla}\dot{T} \cdot \vec{\nabla}T = \kappa\Delta T \tag{44}$$

Nguyen [53] considers the example of the small perturbation of a stationary temperature $T_0(x) = T_0 + gx/L$, in a 1D rigid rod $[0, L]$.

The associated linearized heat equation, using Fourier's model of heat flux, can be expressed as:

$$\rho C_\varepsilon T_{,t} + \frac{Mg}{L}T_{,xt} = \kappa T_{,xx} \tag{45}$$

An elementary solution given in the work [53] shows that the thermal pulses propagate at finite velocity given by $-Mg\kappa L/\left(\rho^2 C_\varepsilon^2 l^2 L^2 + M^2 g^2\right)$ with damping coefficient defined by $\rho C_\varepsilon \kappa L^2/\left(\rho^2 C_\varepsilon^2 l^2 L^2 + M^2 g^2\right)$.

4.2 Hyper-temperature (entropy gradient) model

In 2008, Forest and Amestoy [52] extended the classical thermodynamics of solids to incorporate a non-trivial dependence of the internal energy density function on the gradient of entropy. The generalized heat equation derived from this theory is shown to be different from existing models including gradient of temperature effects in the free energy. In this theory, the additional contributions to the power of internal and external forces are assumed, in the form:

$$P_{int} = -\int_V \left(\underline{\sigma}:\vec{\nabla}\vec{v}^* + \sigma_s \dot{s}^* + \vec{\sigma}_s \cdot \vec{\nabla}\dot{s}^*\right)dV \quad P_a = \rho\int_V \left(\dot{\vec{v}} \cdot \vec{v}^*\right)dV$$

$$P_{ext} = \int_V \left(\rho\vec{f}^u \cdot \dot{\vec{u}}^* + f^s \dot{s}^* + \vec{f}^s \cdot \vec{\nabla}\dot{s}^*\right)dV + \int_\Gamma \left(\vec{F}^u \cdot \dot{\vec{u}}^* + F^s \dot{s}^*\right)dS \tag{46}$$

with, σ_s and $\vec{\sigma}_s$ (called hyper-temperature vector [52]) denotes micro-stresses. $\vec{f}^u$, f^s, $\vec{f}^s$ and $\vec{F}^u$, F^s are the simple and generalized body forces and contact forces respectively associated with $\vec{u}$ and s, and no micro-inertia effects are attached to entropy.

Applying the principle of virtual power results in the following balance equations and associated boundary conditions:

$$\begin{cases} \vec{\nabla} \cdot \underline{\sigma} + \rho\vec{f}^u = 0 & in\ \Omega \\ \underline{\sigma} \cdot \vec{n} = \vec{F}^u & on\ \Gamma \end{cases} \quad \begin{cases} \vec{\nabla} \cdot \left(\vec{\sigma}_s - \vec{f}^s\right) - \sigma_a + f^s = 0 & in\ \Omega \\ \left(\vec{\sigma}_s - \vec{f}^s\right) \cdot \vec{n} = F^s & on\ \Gamma \end{cases} \tag{47}$$

Consequently, using the kinetic energy theorem, the local form of the energy balance follows:

$$\rho\dot{e} = \underline{\sigma}:\dot{\underline{\varepsilon}} + \sigma_s \dot{s} + \vec{\sigma}_s \cdot \vec{\nabla}\dot{s} - div(\vec{q}) + \xi \tag{48}$$

Substituting the above extended rate of specific internal energy eq. (48) into the classical second law of thermodynamics (eq. (4)) leads to the generalized Clausius–Duhem inequality:

$$\rho(T\dot{s} - \dot{e}) + \underline{\sigma}:\dot{\underline{\varepsilon}} + \sigma_s \dot{s} + \vec{\sigma}_s \cdot \vec{\nabla}\dot{s} - \frac{\vec{q}}{T} \cdot \vec{\nabla}T \geq 0 \tag{49}$$

Considering thermoelastic solids and the set of state variables $(\underline{\varepsilon}, s, \vec{\nabla}s)$, the inequality eq. (49) becomes:

$$\rho\left(T - \frac{\partial e}{\partial s} + \frac{\sigma_s}{\rho}\right)\dot{s} + \left(\underline{\sigma} - \rho\frac{\partial e}{\partial\underline{\varepsilon}}\right):\dot{\underline{\varepsilon}} + \left(\vec{\sigma}_s - \rho\frac{\partial e}{\partial\vec{\nabla}s}\right)\cdot\vec{\nabla}\dot{s} - \frac{\vec{q}}{T}\cdot\vec{\nabla}T \geq 0 \tag{50}$$

from which the state relations are derived based on the Coleman–Noll standard arguments:

$$\underline{\sigma} = \rho\frac{\partial e}{\partial\underline{\varepsilon}} \quad T = \frac{\partial e}{\partial s} - \frac{\sigma_s}{\rho} \quad \vec{\sigma}_s = \rho\frac{\partial e}{\partial\vec{\nabla}s} \tag{51}$$

and the residual thermal dissipation:

$$\varphi_V = \varphi_{th} = -\frac{\vec{q}}{T}\cdot\vec{\nabla}T \geq 0 \tag{52}$$

Substituting the state laws eq. (51) into the local energy balance equation (eq. (48)) leads to the classical form of the heat equation:

$$\rho T\dot{s} = -div(\vec{q}) + \xi \tag{53}$$

Concerning the following internal energy density function, linearized around the reference entropy s_0 in rigid heat conduction:

$$\rho e(s, \vec{\nabla}s) = \rho s T_0 + \rho^2\frac{(s - s_0)^2}{4\beta} + \frac{1}{2}A_s\vec{\nabla}s\cdot\vec{\nabla}s \tag{54}$$

where β and A_s are two positive material parameters. Using the derived state laws in eq. (51) and the Fourier model eq. (12), the heat equation is derived as:

$$\rho T_0\dot{s} = \kappa\Delta T = \frac{\rho\kappa}{2\beta}\left(\Delta s - l_s^2\Delta^2 s\right) \tag{55}$$

where $l_s^2 = \frac{2\beta A_s}{\rho^2}$ is a characteristic length related to material parameters.

It is worth noting that this enhanced heat equation (eq. (55)) has the structure of the Cahn–Hilliard equation in mass transport theory [81, 82]. The classical heat equation is retrieved for a vanishing intrinsic length scale $l_s = 0$, or equivalently $A_s = 0$, in the absence of prescribed external micro-forces. The additional contribution in eq. (55) accounts for the size effects in heat conduction in micro-heterogeneous bodies.

By using the expressions of internal energy density function, the free energy density is given by:

$$\rho\psi(T, \vec{\nabla}T, \Delta s) = -\rho(T - T_0)s_0 - \beta(T - T_0)^2 + \frac{2A_s\beta^2}{\rho^2}\vec{\nabla}T\cdot\vec{\nabla}T$$
$$+ \frac{4A_s^2\beta^2}{\rho^3}\vec{\nabla}T\cdot\vec{\nabla}(\Delta s) + \frac{\beta A_s^2}{\rho^2}(\Delta s)^2 + \frac{2\beta^2 A_s^3}{\rho^4}\vec{\nabla}(\Delta s)\cdot\vec{\nabla}(\Delta s) \tag{56}$$

The first line of the expression corresponds to the quadratic form in a linearized theory of rigid heat conduction including the temperature gradient as state variables:

$$\rho\psi_T(T, \vec{\nabla}T) = -\rho(T - T_0)s_0 - \frac{1}{2}\frac{\rho C_\varepsilon}{T_0}(T - T_0)^2 + \frac{1}{2}A_T\vec{\nabla}T\cdot\vec{\nabla}T \tag{57}$$

with $\beta = \rho C_\varepsilon/2T_0$ and $A_T T_0^2 = A_s C_\varepsilon^2$. The second line of the expression shows that the gradient of entropy and gradient of temperature theories are not equivalent. If the parameter A_s is sufficiently small, the gradient of temperature model eq. (57) can be regarded as an approximation of the gradient of entropy theory.

The generalized heat equation derived from the gradient entropy theory can be rewritten as:

$$\rho C_\varepsilon\dot{T} + \frac{A_s C_\varepsilon^2}{T_0}\Delta\dot{T} + \frac{A_s^2 C_\varepsilon^2}{\rho T_0}\Delta^2\dot{s} = \kappa\Delta T \tag{58}$$

If one considers a sufficiently small value of A_s, the last term in the left hand of eq. (58) can be neglected, the heat equation becomes:

$$\rho C_\varepsilon \dot{T} + T_0 A_T \Delta \dot{T} = \kappa \Delta T \tag{59}$$

which is exactly the heat equation derived from the temperature gradient theory eq. (57). Note that the heat equation (eq. (59)), derived from the gradient temperature theory, has the same form as the heat equation of Cattaneo I (eq. (15)) taking $\tau = T_0 A_T / \kappa$.

5 Double temperature model

5.1 Aifantis' double temperature model

A generalized heat equation has also been phenomenologically derived by Aifantis [54–56] based on a double temperature mixture type theory. For the sake of brevity, we limit ourselves to the rigid body assumption. In the case of a heat conducting two components system (indicated by 1 and 2), the following two energy balance equations must be fulfilled:

$$\rho_1 \dot{e}_1 = -div(\vec{q}_1) + \xi_1 \quad \rho_2 \dot{e}_2 = -div(\vec{q}_2) + \xi_2 \tag{60}$$

In a mixture for which the particle densities are $\rho_1 = \rho_2 = \rho/2$, the macroscopic internal energy, heat flux and effective temperature are defined as:

$$2e = e_1 + e_2 \quad 2\vec{q} = \vec{q}_1 + \vec{q}_2 \quad 2T = T_1 + T_2 \tag{61}$$

In the absence of chemical reactions, the source terms are assumed to compensate $\xi_1 + \xi_2 = 0$. The entropy production due to each constituent is assumed to take the form:

$$\rho \dot{s}_1 + div\left(\frac{\vec{q}_1}{T_1}\right) = \pi_1 \quad \rho \dot{s}_2 + div\left(\frac{\vec{q}_2}{T_2}\right) = \pi_2 \quad 2\pi = \pi_1 + \pi_2 \geq 0 \tag{62}$$

Combining the balance equations of energy eq. (60) and the entropy inequality eq. (62), the reduced Clausius–Duhem inequality is derived:

$$\rho\left(1 - \frac{1}{T_1}\frac{\partial e_1}{\partial s_1}\right)\dot{s}_1 + \rho\left(1 - \frac{1}{T_2}\frac{\partial e_2}{\partial s_2}\right)\dot{s}_2 + \rho\left(\frac{\xi_1}{T_1} + \frac{\xi_2}{T_2}\right) - \frac{\vec{q}_1}{T_1}\cdot\frac{\vec{\nabla}T_1}{T_1} - \frac{\vec{q}_2}{T_2}\cdot\frac{\vec{\nabla}T_2}{T_2} \geq 0 \tag{63}$$

Then, the state relationships linking partial temperature and entropies are derived based on the Coleman–Noll arguments:

$$T_1 = \frac{\partial e_1}{\partial s_1} \quad T_2 = \frac{\partial e_2}{\partial s_2} \tag{64}$$

The residual dissipation:

$$\varphi_V = \varphi_{th} = \rho\left(\frac{\xi_1}{T_1} + \frac{\xi_2}{T_2}\right) - \frac{\vec{q}_1}{T_1}\cdot\frac{\vec{\nabla}T_1}{T_1} - \frac{\vec{q}_2}{T_2}\cdot\frac{\vec{\nabla}T_2}{T_2} \geq 0 \tag{65}$$

If the temperature variations remain sufficiently small, the generalized Fourier model is assumed to take the form:

$$\left\{\begin{array}{c}\vec{q}_1\\\vec{q}_2\end{array}\right\} = -\left[\begin{array}{cc}\kappa_{11} & \kappa_{12}\\\kappa_{21} & \kappa_{22}\end{array}\right]\left\{\begin{array}{c}\vec{\nabla}T_1\\\vec{\nabla}T_1\end{array}\right\} \tag{66}$$

and the generalized specific heat capacities (C_{ij}) are also introduced in the form:

$$\rho_1 e_1 = C_{11}T_1 + C_{12}T_2 \quad \rho_2 e_2 = C_{21}T_1 + C_{22}T_2 \tag{67}$$

Within the assumption of linear framework, the coupling term may be taken as proportional to the temperature gap:

$$\xi_1 = -\xi_2 = h(T_2 - T_1) \tag{68}$$

where h is a coupling parameter (microscopic heat transfer).

Substituting the above state constitutive equations into the corresponding balance equations, the following partial differential equations are obtained:

$$\begin{bmatrix} C_{11}\frac{d}{dt} - \kappa_{11}\Delta + h & C_{12}\frac{d}{dt} - \kappa_{12}\Delta - h \\ C_{21}\frac{d}{dt} - \kappa_{21}\Delta - h & C_{22}\frac{d}{dt} - \kappa_{22}\Delta + h \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{69}$$

Elimination of T_1 and T_2 leads to the following higher-order heat equation for the effective (global) temperature:

$$\dot{T} + \alpha\ddot{T} - \beta\Delta\dot{T} + \gamma\Delta^2 T = \lambda\Delta T \tag{70}$$

where,

$$\alpha = \frac{C_{11}C_{22} - C_{12}C_{21}}{h(C_{11} + C_{22} + C_{12} + C_{21})} \quad \beta = \frac{\kappa_{11}C_{22} + \kappa_{22}C_{11} + \kappa_{12}C_{21} + \kappa_{21}C_{12}}{h(C_{11} + C_{22} + C_{12} + C_{21})}$$

$$\gamma = \frac{\kappa_{11}\kappa_{22} - \kappa_{12}\kappa_{21}}{h(C_{11} + C_{22} + C_{12} + C_{21})} \quad \lambda = \frac{\kappa_{11} + \kappa_{22} + \kappa_{12} + \kappa_{21}}{(C_{11} + C_{22} + C_{12} + C_{21})} \tag{71}$$

It must be noted that eq. (70) contains two more terms, the second time derivative and fourth space derivative of temperature, compared with the heat equation of Cattaneo I (eq. (15)). The fourth-order spatial derivative term can be neglected when the thermal conductivity of electron component κ_{11} is much larger than the lattice component κ_{22}, e. g. in pure metal [42]. This equation is also similar to Barenblatt's infiltration theory for the pore pressure in double porosity media [54].

5.2 Hyperbolic two-temperature model

As summarized by Sobolev [42], the double temperature model is still a parabolic heat equation. In order to obtain hyperbolic heat equations of the two-temperature model, he adopts the Cattaneo equation eq. (17) as the extension of the heat flux model. It yields the heat conduction equation for the electron and lattice temperatures in the form:

$$\begin{cases} \dfrac{\partial T_1}{\partial t} + (\tau + \tau_{12})\dfrac{\partial^2 T_1}{\partial t^2} + \tau\tau_{12}\dfrac{\partial^3 T_1}{\partial t^3} = \bar{\kappa}\dfrac{\partial^2 T_1}{\partial x^2} + l^2\dfrac{\partial^3 T_1}{\partial x^2\partial t} + \bar{Q} + (\tau + \tau_2)\dfrac{\partial\bar{Q}}{\partial t} + \tau\tau_2\dfrac{\partial^2\bar{Q}}{\partial t^2} \\ \dfrac{\partial T_2}{\partial t} + (\tau + \tau_{12})\dfrac{\partial^2 T_2}{\partial t^2} + \tau\tau_{12}\dfrac{\partial^3 T_2}{\partial t^3} = \bar{\kappa}\dfrac{\partial^2 T_2}{\partial x^2} + l^2\dfrac{\partial^3 T_2}{\partial x^2\partial t} + \bar{Q} + \tau\dfrac{\partial\bar{Q}}{\partial t} \end{cases} \tag{72}$$

where τ is the electron relaxation time, $\tau_{12} = C_1 C_2/\alpha(C_1 + C_2)$ is the characteristic time of energy exchange between the electron and the lattice, α is the electron-phonon coupling factor, C_1 and C_2 are the electron and lattice heat capacity, respectively, $\bar{\kappa} = \kappa/(C_1 + C_2)$ with κ is the electron thermal conductivity, $l = \sqrt{\kappa C_2/\alpha(C_1 + C_2)}$ is the characteristic (internal) length scale, $\bar{Q} = Q/(C_1 + C_2)$ with Q is a heat source due to laser irradiation, $\tau_2 = C_2/\alpha$ is a characteristic warm-up time of the lattice.

If the heat source is omitted, the heat conduction equation can be written as:

$$\dot{T}_i + (\tau + \tau_{12})\ddot{T}_i + \tau\tau_{12}\,\dddot{T}_i - l^2\Delta\dot{T}_i = \bar{\kappa}\Delta T_i \tag{73}$$

In the work [42] a plane thermal wave $T(x,t) = T_0\exp[i(\omega t - kx)]$ is considered to examine the properties of the heat equations. The dispersion relation for eq. (72) takes the form:

$$i\omega^3\tau\tau_{12} + \omega^2(\tau + \tau_{12}) - i\omega(1 + \tau_2\bar{\kappa}k^2) - \bar{\kappa}k^2 = 0 \tag{74}$$

At the high frequency limit, eq. (74) implies that the temperature discontinuities propagate in the system with finite velocity $V_p = \sqrt{\kappa/C_1\tau}$, which depends only on the physical parameters of electron gas and is of the order of Fermi velocity. In the intermediate frequency, the local equilibrium in the electron gas is reached and energy is transferred into the bulk metal by diffusive transport of thermalized electrons and from the electron gas to the initially cold lattice by electron-phonon coupling. In the low frequency, the thermal equilibrium between electron and the lattice is reached, i. e. the energy exchange between the electron gas and the lattice is negligible and energy transfer in the system is described by classical heat conduction equation of parabolic type.

5.3 Generalized two-temperature model with nonlocal effects

To account for the spatial nonlocal effects (when the characteristic length scale of the process becomes comparable with the mean-free path of heat carriers) due to large temperature gradients, Sobolevs [42] extended Cattaneo's equation with additional second-order space derivatives:

$$\vec{q} + \tau\dot{\vec{q}} = -\kappa\vec{\nabla}T + l_e^2\Delta\vec{q} \tag{75}$$

where l_e is the space nonlocality of the heat transfer process (correlation length), which for metal is of the order of the mean-free path of electrons. It is worthy to note that eq. (75) is similar to the Guyer-Krumhansl equation which can be derived from the linearized Boltzmann equation [34], the kinetic theory with more general assumptions [83] or the framework of the non-equilibrium thermodynamics [36].

For the sake of simplicity, the material parameters are assumed to be constant and the extended heat equation takes the form:

$$\dot{T}_i + (\tau + \tau_{12})\ddot{T}_i + \tau\tau_{12}\,\dddot{T}_i - (l^2 + l_e^2)\Delta\dot{T}_i - l_e^2\tau_{12}\Delta\ddot{T}_i = \bar{\kappa}\Delta T_i + Q_i^{eff} \tag{76}$$

where, the index i refers to the two temperatures, Q_i^{eff} is the effective energy source, which is expressed as:

$$\begin{aligned}
Q_1^{eff} &= \bar{Q} + (\tau + \tau_2)\dot{\bar{Q}} - l_e^2\Delta\bar{Q} + \tau\tau_2\ddot{\bar{Q}} - \tau_2 l_e^2\Delta\dot{\bar{Q}} \\
Q_2^{eff} &= \bar{Q} + \tau\dot{\bar{Q}} - l_e^2\Delta\bar{Q}
\end{aligned} \tag{77}$$

Comparing with the hyperbolic two-temperature model eq. (72), the extended second-order space derivatives of heat flux result in an additional fourth-order derivative of the temperature $l_e^2\tau_{12}\Delta\ddot{T}$, and additional terms in the effective energy source. If $l_e = 0$, eq. (76) reduces to eq. (73), which implies a hyperbolic type with finite propagation of thermal waves. When $l_e > 0$, it is a parabolic function and discontinuities are smoothed by diffusion associated with the effective thermal diffusivity.

6 Micro-temperature models

6.1 Continuum with micro-temperature vector

6.1.1 Heat equation of linear micro-stretch thermoelastic body using Grot's theory

Based on the theory, established by Grot [61] dealing with thermodynamics of elastic bodies with micro-structure whose microelements possess a micro-temperature vector, Iesan et al. [62, 75, 84–86] derived a linear theory of micro-stretch thermo-elastic bodies with micro-temperatures. The second moment of the stress tensor and the average micro-stress moment are neglected in the balance laws since these functions

appear only nonlinearly in the field equations. The local forms of balance of energy and the balance of first moment of energy can be expressed as:

$$\rho\dot{e} = \underline{\sigma} : \vec{\nabla}\vec{v} + \left(\breve{\underline{\sigma}} - \underline{\sigma}\right) : \underline{v} + \vec{\underline{\sigma}}_m \therefore \vec{\nabla}\underline{v} + div(\vec{q}) + \xi \tag{78}$$

and

$$\rho\dot{\vec{e}}_m = \vec{\underline{\sigma}}_m : \left(\vec{\nabla}v - \underline{v}\right) + div\left(\underline{q}\right) + \vec{q} - \vec{Q} + \vec{\xi} \tag{79}$$

with, $\vec{v}$ is the velocity vector, $\breve{\underline{\sigma}}$ is the micro-stress tensor, $\underline{v}$ denotes the micro-gyration tensor, $\vec{\underline{\sigma}}_m$ is the first stress moment third-rank tensor, $\vec{e}_m$ represents the first moment of energy vector, $\underline{q}$ is the first heat flux moment second-rank tensor, $\vec{Q}$ is the micro-heat flux average, and $\vec{\xi}$ is the first heat supply moment tensor.

The local form of the second law of thermodynamics is modified to include the micro-temperature:

$$\rho\dot{s} - div\left(\frac{\vec{q}}{T} + \frac{q}{T} \cdot \vec{T}\right) - \frac{1}{T}\left(\xi + \vec{\xi} \cdot \vec{T}\right) \geq 0 \tag{80}$$

where $\vec{T}$ is the micro-temperature vector.

Concerning the linear micro-stretch elastic solid (see the work of Eringen [66]), the above entropy inequality can be rewritten as:

$$\rho\left(T\dot{s} - \dot{e} - \vec{T} \cdot \dot{\vec{e}}_m\right) + \underline{\sigma} : \dot{\underline{\varepsilon}} + \underline{\sigma}_c : \dot{k} + \vec{h} \cdot \dot{\vec{\zeta}} - g\dot{\varphi} + \vec{\underline{\sigma}}_m : (\dot{\underline{\varepsilon}} - \dot{\varphi}\underline{I}) \cdot \vec{T} + \frac{\vec{q}}{T} \cdot \vec{\nabla}T$$
$$+ \frac{\vec{\nabla}T}{T} \cdot q \cdot \vec{T} - q : \vec{\nabla}\vec{T} + \left(\vec{q} - \vec{Q}\right) \cdot \vec{T} \geq 0 \tag{81}$$

with, $\underline{\varepsilon} = \vec{\nabla}\vec{u} + \vec{\underline{\alpha}} \cdot \vec{\varphi}$, $\underline{k} = \vec{\nabla}\vec{\varphi}$ and $\vec{\zeta} = \vec{\nabla}\varphi$ indicate the generalized strain tensors in the linear theory of micro-stretch continua, $\vec{\underline{\alpha}}$ denotes the alternating symbol, $\vec{\varphi}$ is the micro-rotation vector and φ denotes the micro-dilatation function, $\underline{\sigma}_c$ is the coupled stress tensor, $\vec{h}$ indicates the micro-stretch vector, g is the internal body force, $\underline{I}$ denotes the identity tensor. Considering the following form of free energy:

$$\psi = e - Ts + \vec{T} \cdot \vec{e}_m \tag{82}$$

the inequality just becomes:

$$-\rho\left(\dot{\psi} + \dot{T}s - \dot{\vec{T}} \cdot \vec{e}_m\right) + \left(\underline{\sigma} + \vec{\underline{\sigma}}_m \cdot \vec{T}\right) : \dot{\underline{\varepsilon}} + \underline{\sigma}_c : \dot{k} + \vec{h} \cdot \dot{\vec{\zeta}} - \left(g + \vec{\underline{\sigma}}_m : \underline{I} \cdot \vec{T}\right)\dot{\varphi} + \frac{\vec{q}}{T} \cdot \vec{\nabla}T$$
$$+ \vec{\nabla}TT \cdot \underline{q} \cdot \vec{T} - \underline{q} : \vec{\nabla}\vec{T} + \left(\vec{q} - \vec{Q}\right) \cdot \vec{T} \geq 0 \tag{83}$$

In the linear theory of thermodynamic materials with micro-temperature, the following constitutive equations [84] are obtained:

$$\underline{\sigma} = \rho\frac{\partial\psi}{\partial\underline{\varepsilon}} \quad \underline{\sigma}_c = \rho\frac{\partial\psi}{\partial\underline{k}} \quad \vec{h} = \rho\frac{\partial\psi}{\partial\vec{\zeta}}$$
$$g = -\rho\frac{\partial\psi}{\partial\varphi} \quad s = -\frac{\partial\psi}{\partial T} \quad \vec{e}_m = \frac{\partial\psi}{\partial\vec{T}} \tag{84}$$

and residual thermal dissipation:

$$\varphi_V = \varphi_{th} = \vec{q} \cdot \vec{\nabla}T - T_0 q : \vec{\nabla}\vec{T} + T_0\left(\vec{q} - \vec{Q}\right) \cdot \vec{T} \geq 0 \tag{85}$$

The linear approximations for the generalized heat fluxes are given by:

$$\vec{q} = \underline{\kappa} \cdot \vec{\nabla}T + \underline{H} \cdot \vec{T} \quad \underline{q} = -\underline{P} : \vec{\nabla}\vec{T} \quad \vec{Q} = (\underline{\kappa} - \underline{K}) \cdot \vec{\nabla}T + (\underline{H} - \underline{\Lambda}) \cdot \vec{T} \tag{86}$$

with, the constitutive coefficients $\underline{\kappa}$, $\underline{H}$, $\underline{P}$, $\underline{K}$ and $\underline{\Lambda}$ satisfy the inequality equation (eq. (85)).

By using the equations (eqs. (78), (79), (82) and (84)), and in the framework of the linear theory [84], the following field equations of temperature and micro-temperature are obtained:

$$\begin{cases} k\Delta T + k_1 div\left(\vec{T}\right) - \beta_0 T_0 div(\vec{v}) - \beta_1 T_0 \dot{\varphi} - c\dot{T} = -\xi \\ k_6\Delta\vec{T} + (k_4 + k_5)\vec{\nabla}\left(div\left(\vec{T}\right)\right) + \mu_1\underline{\dot{a}} : \vec{\nabla}\dot{\varphi} - \mu_2\vec{\nabla}\dot{\varphi} - b\dot{\vec{T}} - k_2\vec{T} - k_3\vec{\nabla}T = \vec{\xi} \end{cases} \tag{87}$$

where k_i, β_i, c and b are material constants.

Concerning the rigid body heat conduction (without body heat sources), the above generalized heat equations become:

$$\begin{cases} c\dot{T} = k\Delta T + k_1 div\left(\vec{T}\right) \\ b\dot{\vec{T}} = k_6\Delta\vec{T} + (k_4 + k_5)\vec{\nabla}\left(div\left(\vec{T}\right)\right) - k_2\vec{T} - k_3\vec{\nabla}T \end{cases} \tag{88}$$

Combining the above two heat equations and considering the internal constraint that $\vec{T} = \vec{\nabla}T$, the generalized heat equation becomes:

$$c\dot{T} = (k + k_1 - k_2 - k_3)\Delta T - b\Delta\dot{T} + (k_4 + k_5)\Delta^2 T \tag{89}$$

If $\Delta^2 T$ is small enough, the above heat equation (eq. (89)) is equivalent to the one derived from the hyper-temperature model (eq. (59)).

6.1.2 Heat equation in micromorphic continua using Green's theory

Ieşan and Nappa [75] also deduced the heat equation in micromorphic continua, while using the thermo-mechanical theory established by Green et al. [73]. In particular, he showed that in the linearized theory according to this approach, heat can be transferred as thermal waves with finite speed. The heat equations of linear theory for temperature and micro-temperature are given by:

$$\begin{cases} a\ddot{T} + m\, div\left(\dot{\vec{T}}\right) = k\Delta T \\ b\ddot{\vec{T}} = d_2\Delta\vec{T} + (d_1 + d_3)\vec{\nabla}\left(div\left(\vec{T}\right)\right) - m\vec{\nabla}\dot{T} \end{cases} \tag{90}$$

where a, m, k, b and d_i are material constants.

Combining the above two heat equations and assuming that $\vec{T} = \vec{\nabla}T$, the generalized heat equation just becomes:

$$a\ddot{T} + 2m\Delta\dot{T} + b\Delta\ddot{T} - (d_1 + d_2 + d_3)\Delta^2 T = k\Delta T \tag{91}$$

Compared with the heat equations (eqs. (88)) from Grot's theory, eq. (90) predicts a finite propagation. For a hypothetical medium in which $m = 0$, eqs. (90) are uncoupled in the sense that the temperature is independent of micro-temperatures. In this case the temperature satisfies the classical wave equation. Moreover, it is interesting to note the similarity between the equations for micro-temperatures and the Navier's equations of motion.

6.2 Scalar micro-temperature and micro-entropy

Forest and Aifantis [56] proposed theories based on scalar micro-temperature and micro-entropy model, by applying the micromorphic approach [76] to the temperature and entropy. They showed that the gradient of entropy theory and gradient temperature theories can be regarded as a limit case of the micro-entropy and micro-temperature theories. The formulations of micro-entropy model are now briefly reviewed in the

purely thermal case. It assumes that there exist additional independent power of internal and external generalized forces due to the introduced micro-entropy variable $\breve{s}$ and $\vec{\nabla}\breve{s}$. The virtual power of the generalized internal forces [87] is enhanced:

$$P_{\text{int}} = -\int_V \left(\breve{\sigma}_a \dot{\breve{s}}^\star + \vec{\breve{\sigma}}_b \cdot \vec{\nabla}\dot{\breve{s}}^\star \right) dV \tag{92}$$

where $\breve{\sigma}_a$ and $\vec{\breve{\sigma}}_b$ are generalized stresses or micro-forces according to Gurtin's [82] terminology.

The virtual power of external generalized forces is expressed as:

$$P_{ext} = \int_\Gamma \left(F^{\breve{s}} \dot{\breve{s}}^\star \right) dS \tag{93}$$

where $F^{\breve{s}}$ is the generalized contact forces.

In the quasi-static case, applying the generalized principle of virtual power results in the following balance equation and the associated Neumann-type boundary condition:

$$\begin{cases} \vec{\nabla} \cdot \vec{\breve{\sigma}}_b - \breve{\sigma}_a = 0 & in\ \Omega \\ \vec{\breve{\sigma}}_b \cdot \vec{n} = F^{\breve{s}} & on\ \Gamma \end{cases} \tag{94}$$

Accordingly, the first principle of thermodynamics is extended with the micro-entropy variable. The local forms of the principles of thermodynamics are given by:

$$\begin{cases} \rho\dot{e} = \breve{\sigma}_a \dot{\breve{s}} + \vec{\breve{\sigma}}_b \cdot \vec{\nabla}\dot{\breve{s}} - div(\vec{q}) + \xi \\ \rho\dot{s} - \left(\frac{\xi}{T}\right) + div\left(\frac{\vec{q}}{T}\right) \geq 0 \end{cases} \tag{95}$$

Using the classical relationship between internal energy and free energy (eq. (5)) yields the following extended Clausius–Duhem inequality:

$$\rho\left(T - \frac{\partial e}{\partial s}\right)\dot{s} + \left(\breve{\sigma}_a - \rho\frac{\partial e}{\partial \breve{s}}\right)\dot{\breve{s}} + \left(\vec{\breve{\sigma}}_b - \rho\frac{\partial e}{\partial \vec{\nabla}\breve{s}}\right) \cdot \vec{\nabla}\dot{\breve{s}} - \frac{\vec{q}}{T}\vec{\nabla}T \geq 0 \tag{96}$$

The following state relations are derived based on the Coleman–Noll arguments:

$$T = \frac{\partial e}{\partial s} \quad \breve{\sigma}_a = \rho\frac{\partial e}{\partial \breve{s}} \quad \vec{\breve{\sigma}}_b = \rho\frac{\partial e}{\partial \vec{\nabla}\breve{s}} \tag{97}$$

and the residual dissipation:

$$\varphi_V = \varphi_{th} = -\frac{\vec{q}}{T} \cdot \vec{\nabla}T \geq 0 \tag{98}$$

Substituting the state laws eq. (97) into the local energy balance equation (eq. (95)) leads to the classical form of the heat equation:

$$\rho T\dot{s} = -div(\vec{q}) + \xi \tag{99}$$

Within a linear context, the following quadratic energy potential [56] is defined:

$$\rho e\left(s, \breve{s}, \vec{\nabla}\breve{s}\right) = \rho s T_0 + \rho^2\frac{(s - s_0)^2}{4\beta} + \frac{1}{2}\breve{H}\left(s - \breve{s}\right)^2 + \frac{1}{2}A_s\vec{\nabla}\breve{s} \cdot \vec{\nabla}\breve{s} \tag{100}$$

where β, $\breve{H}$ and A_s are material parameters. The state relations are obtained:

$$T = T_0 + \rho \frac{(s - s_0)}{2\beta} + \frac{\breve{H}(s - \breve{s})}{\rho} \quad \breve{\sigma}_a = -\breve{H}(s - \breve{s}) \quad \vec{\sigma}_b = A_s \vec{\nabla} \breve{s} \tag{101}$$

Combining the Fourier model (eq. (12)), the heat equation (eq. (99)), the relevant state relations (eq. (101)) and the balance equation (eq. (94)), the following generalized heat equation is obtained:

$$\rho T_0 \dot{s} = \kappa \left(\frac{\rho}{2\beta} \vec{\nabla}^2 s - \frac{A_s}{\rho} \vec{\nabla}^4 \breve{s} \right) \tag{102}$$

It should be noticed that this equation coincides with the heat equation (eq. (55)) derived from the gradient of entropy theory, specialized for $\breve{\sigma}_a = 0$, when the following internal constraint is enforced:

$$\breve{s} = s \tag{103}$$

As a result, the gradient of entropy theory can be regarded as a limit case of the micro-entropy model.

7 Heat equation from the micromorphic approach

7.1 Balance equations

In this section, the micromorphic temperature $\breve{T}$ is introduced, based on the micromorphic approach [76], into the constitutive equations as an additional degree of freedom (dof). It assumes that there exist extra contributions to the virtual power from the micromorphic temperature. By using the generalized principle of virtual power [87], the virtual power of internal forces is extended as (only the first gradient of micromorphic temperature is considered, for the sake of simplicity):

$$P_{int} = -\int_V \left(\underline{\sigma} : \dot{\underline{\varepsilon}}^* + \zeta \dot{\breve{T}}^* + \vec{\zeta} \cdot \vec{\nabla} \dot{\breve{T}}^* \right) dV \tag{104}$$

where $\vec{\nabla} \breve{T}$ is the first gradient of the micromorphic temperature; ζ and $\vec{\zeta}$ denote the generalized stress-like variables with respect to micromorphic temperature and its first gradient respectively.

The virtual power of external forces (body forces and contact forces) is also enriched by the micromorphic thermal contributions:

$$P_{ext} = \int_V \rho \left(\vec{f}^u \cdot \dot{\vec{u}}^* + f^{\breve{T}} \cdot \dot{\breve{T}}^* + \vec{f}^{g\breve{T}} \cdot \vec{\nabla} \dot{\breve{T}}^* \right) dV + \int_\Gamma \left(\vec{F}^u \cdot \dot{\vec{u}}^* + F^{\breve{T}} \cdot \dot{\breve{T}}^* \right) dS \tag{105}$$

where $\vec{f}^u, f^{\breve{T}}, \vec{f}^{g\breve{T}}$ and $\vec{F}^u, F^{\breve{T}}$ are the simple and generalized body forces and contact forces associated with displacement, micromorphic temperature and its first gradient, respectively.

The virtual power of the generalized inertia terms (acceleration forces) is assumed as:

$$P_a = \int_V \rho \left(\ddot{\vec{u}} \cdot \dot{\vec{u}}^* + \varsigma_1 \dot{\breve{T}} \dot{\breve{T}}^* + \varsigma_2 \ddot{\breve{T}} \dot{\breve{T}}^* \right) dV \tag{106}$$

with ς_i represents the generalized mass of micromorphic temperature. Two thermal contributions are introduced for the sake of generality. The first one mimics the mechanical acceleration and introduces the second time derivative of the micromorphic temperature. The second one involves only the first time derivative of the micromorphic temperature based on the idea of micro-kinetic energy in the spirit of the kinetic theory of gas. Clearly, if $\varsigma_i = 0$, the theory reduces to a micromorphic temperature model similar to the micro-entropy model derived by Forest and Aifantis [56].

It should be noticed that there are two kinds of inertia contributions (first and second time derivatives) from micromorphic temperature. The first inertia term (second time derivative) is similar to the inertia of displacement. The second inertia term (the first time derivative) is based on some justifications from kinetic theory of gas, where the kinetic energy is proportional to temperature. In Ref. [88] it is recalled that Maxwell and Boltzmann have developed a kinetic theory that yields a fundamental understanding of temperature in gases. The kinetic theory assumes that pressure is caused by the force associated with individual atoms striking the walls, and that this energy is translational kinetic energy. Using a sophisticated symmetry argument, Boltzmann deduced what is now called the Maxwell–Boltzmann probability distribution function for the velocity of particles in an ideal gas [89]. From that probability distribution function, the average kinetic energy, E_k (per particle), of a monatomic ideal gas is:

$$E_k = \frac{1}{2} m v_{rms}^2 = \frac{3}{2} k_B T \tag{107}$$

where k_B is the Boltzmann constant, and v_{rms} is the root-mean-square speed. It is reasonable to assume that the temperature is function of velocity of particles. Then the first time derivative of temperature will result in an acceleration term, which is a kind of generalized force. That is why we postulate that the generalized thermal inertia [90, 91] (inertia of temperature) is proportional to the first and second time derivatives of temperature.

The generalized principle of virtual power stipulates that the sum of the virtual powers of internal and external forces balances the power of acceleration forces for all virtual velocity and microtemperature rate fields (K.A. stands for kinematically admissible fields in the usual sense):

$$P_{\text{int}}\left(\overset{\cdot\,*}{\vec{u}}, \overset{\cdot\,*}{\breve{T}} \right) + P_{ext}\left(\overset{\cdot\,*}{\vec{u}}, \overset{\cdot\,*}{\breve{T}} \right) = P_a\left(\overset{\cdot\,*}{\vec{u}}, \overset{\cdot\,*}{\breve{T}} \right) \qquad \forall\left(\overset{\cdot\,*}{\vec{u}}, \overset{\cdot\,*}{\breve{T}} \right) K.A. \tag{108}$$

The consequences are the following balance field equations and Neumann boundary conditions:

$$\begin{cases} \vec{\nabla} \cdot \underline{\sigma} + \rho \vec{f}^u = \rho \ddot{\vec{u}} & in\ \Omega \\ \underline{\sigma} \cdot \vec{n} = \vec{F}^u & on\ \Gamma \end{cases} \quad and \quad \begin{cases} \left(\vec{\nabla} \cdot \vec{\zeta} - \zeta \right) + \rho \left(f^{\breve{T}} - \vec{\nabla} \cdot \vec{f}^{g\breve{T}} \right) = \rho \left(\varsigma_1 \dot{\breve{T}} + \varsigma_2 \ddot{\breve{T}} \right) & in\ \Omega \\ \left(\vec{\zeta} - \rho \vec{f}^{g\breve{T}} \right) \cdot \vec{n} = F^{\breve{T}} & on\ \Gamma \end{cases} \tag{109}$$

7.2 Thermodynamics consistency and heat equation

Consequently, using the kinetic energy theorem, the first law of thermodynamics is extended because of the introduction of the micromorphic temperature, while the second principle of thermodynamics remains unchanged. The local forms of the two principles of thermodynamics can be expressed as:

$$\begin{cases} \rho \dot{e} = \left(\underline{\sigma} : \underline{\dot{\varepsilon}} + \zeta \dot{\breve{T}} + \vec{\zeta} \cdot \vec{\nabla}\dot{\breve{T}} \right) - div(\vec{q}) + \xi \\ \rho \dot{s} + div\left(\frac{\vec{q}}{T} \right) - \rho \frac{\varsigma}{T} \geq 0 \end{cases} \tag{110}$$

Substituting the Helmholtz-free energy ($\psi = e - Ts$) into the above local forms eq. (110) leads to the following Clausius–Duhem inequality:

$$\left(\underline{\sigma} - \rho \frac{\partial \psi}{\partial \underline{\varepsilon}^e} \right) : \underline{\dot{\varepsilon}}^e - \rho \left(s + \frac{\partial \psi}{\partial T} \right) \dot{T} + \left(\zeta - \rho \frac{\partial \psi}{\partial T} \right) \dot{\breve{T}} + \left(\vec{\zeta} - \rho \frac{\partial \psi}{\partial \vec{\nabla}\breve{T}} \right) \cdot \vec{\nabla}\dot{\breve{T}}$$

$$+ \underline{\sigma} : \underline{\dot{\varepsilon}}^{ir} - \frac{\partial \psi}{\partial \underline{\alpha}_n} : \underline{\dot{\alpha}}_n - \frac{\vec{q}}{T} \vec{\nabla} T \geq 0 \tag{111}$$

For simplicity, we assume that the micromorphic temperature doesn't dissipate and the terms $\left(\underline{\sigma} - \rho \frac{\partial \psi}{\partial \underline{\varepsilon}^e}\right), \left(s + \frac{\partial \psi}{\partial T}\right), \left(\zeta - \rho \frac{\partial \psi}{\partial \breve{T}}\right)$ and $\left(\vec{\zeta} - \rho \frac{\partial \psi}{\partial \vec{\nabla} \breve{T}}\right)$ do not depend on the rates of the state variables respectively, then following standard arguments (Truesdell and Noll [78]), we obtain the following state relations:

$$\underline{\sigma} = \rho \frac{\partial \psi}{\partial \underline{\varepsilon}^e} \quad s = -\frac{\partial \psi}{\partial T} \quad \zeta = \rho \frac{\partial \psi}{\partial \breve{T}} \quad \vec{\zeta} = \rho \frac{\partial \psi}{\partial \vec{\nabla} \breve{T}} \tag{112}$$

and the residual dissipation:

$$\varphi_V = \varphi_{in} + \varphi_{th} = \underline{\sigma} : \underline{\dot{\varepsilon}}^{ir} - \frac{\partial \psi}{\partial \underline{\alpha}_n} : \underline{\dot{\alpha}}_n - \frac{\vec{q}}{T} \cdot \vec{\nabla} T \geq 0 \tag{113}$$

Substituting the state relations (eq. (112)) and free energy into the local form of the first principle of thermodynamics yields the following form of the heat equation:

$$\rho \dot{s} T + div(\vec{q}) - \xi - \varphi_{in} = 0 \tag{114}$$

For the sake of simplifications, in the next we focus on the rigid body heat transfer ($\varphi_{in} = 0$). Using the state relations, the above classical form of heat equation becomes:

$$\rho T \frac{\partial s}{\partial T} \dot{T} - T \left(\frac{\partial \zeta}{\partial T} \dot{\breve{T}} + \frac{\partial \vec{\zeta}}{\partial T} \cdot \vec{\nabla} \dot{\breve{T}} \right) + div(\vec{q}) - \xi = 0 \tag{115}$$

or

$$-T \left(\rho \frac{\partial^2 \psi}{\partial T^2} \dot{T} + \rho \frac{\partial^2 \psi}{\partial \breve{T} \partial T} \dot{\breve{T}} + \rho \frac{\partial^2 \psi}{\partial \vec{\nabla} \breve{T} \partial T} \cdot \vec{\nabla} \dot{\breve{T}} \right) + div(\vec{q}) - \xi = 0 \tag{116}$$

After substituting the balance equation of micromorphic temperature and neglecting the external forces, the temperature and micromorphic temperature fields are found to be governed by a pair of equations:

$$\begin{cases} \rho T \frac{\partial s}{\partial T} \dot{T} - T \left(\frac{\partial \zeta}{\partial T} \dot{\breve{T}} + \frac{\partial \vec{\zeta}}{\partial T} \cdot \vec{\nabla} \dot{\breve{T}} \right) + div(\vec{q}) - \xi = 0 \\ \vec{\nabla} \cdot \left(\rho \frac{\partial \psi}{\partial \vec{\nabla} \breve{T}} \right) - \rho \frac{\partial \psi}{\partial \breve{T}} = \rho \left(\varsigma_1 \dot{\breve{T}} + \varsigma_2 \ddot{\breve{T}} \right) \end{cases} \tag{117}$$

Combining the two equations in eq. (117) leads to the generalized partial differential micromorphic heat equation associated to the Helmholtz-free energy:

$$\rho T \left(\varsigma_1 \ddot{\breve{T}} + \varsigma_2 \dddot{\breve{T}} \right) - \rho T \left[\frac{\partial^2 \psi}{\partial T^2} \dot{T} + \frac{\partial^2 \psi}{\partial \breve{T} \partial T} \left(\dot{\breve{T}} - \dot{T} \right) - \frac{\partial^2 \psi}{\partial \breve{T}^2} \dot{\breve{T}} + \left(\frac{\partial^2 \psi}{\partial T \partial \vec{\nabla} \breve{T}} - \frac{\partial^2 \psi}{\partial \vec{\nabla} \breve{T} \partial \breve{T}} \right) \cdot \vec{\nabla} \dot{\breve{T}} \right]$$

$$- T div \left(\rho \frac{\partial^2 \psi}{\partial T \partial \vec{\nabla} \breve{T}} \dot{T} + \rho \frac{\partial^2 \psi}{\partial \breve{T} \partial \vec{\nabla} \breve{T}} \dot{\breve{T}} + \rho \frac{\partial^2 \psi}{\partial \vec{\nabla} \breve{T}^2} \cdot \vec{\nabla} \dot{\breve{T}} \right) + div(\vec{q}) - \xi = 0 \tag{118}$$

where the temperature and micro-temperature variables are both present.

7.3 Special cases of the micromorphic heat equation

In this section specific constitutive expressions of the free energy and dissipation potentials are chosen in order to derive explicit generalized heat equations within the micromorphic framework.

7.3.1 Wave type heat equation

If the Helmholtz-free energy is defined as:

$$\rho\psi\left(T,\breve{T},\vec{\nabla}\breve{T}\right) = -\rho CT\left(\ln\frac{T}{T_0}-1\right) - \rho\breve{C}\breve{T}\left(\ln\frac{\breve{T}}{T_0}-1\right) - \frac{1}{2}\breve{M}\left(T-\breve{T}\right)^2 + \frac{1}{2}\breve{M}^g\,\vec{\nabla}\breve{T}\cdot\vec{\nabla}\breve{T} \tag{119}$$

where C and $\breve{C}$ are the coefficients for local and micromorphic temperature, respectively. $\breve{M}$ is the micromorphic temperature modulus coupling both the local and the micromorphic temperature. $\breve{M}^g$ denotes the micromorphic modulus related to the first gradient of the micromorphic temperature (for simplicity, the moduli are assumed to be independent of local temperature and the isotropic case only is considered).

According to the deduced state laws eq. (112), the following stress-like variables are given by:

$$\begin{aligned}
s &= -\frac{\partial\psi}{\partial T} = C\ln\frac{T}{T_0} + \frac{\breve{M}}{\rho}\left(T-\breve{T}\right)\\
\zeta &= \rho\frac{\partial\psi}{\partial\breve{T}} = -\rho\breve{C}\ln\frac{\breve{T}}{T_0} + \breve{M}\left(T-\breve{T}\right)\\
\vec{\zeta} &= \rho\frac{\partial\psi}{\partial\vec{\nabla}\breve{T}} = \breve{M}^g\,\vec{\nabla}\breve{T}
\end{aligned} \tag{120}$$

Substituting the above free energy eq. (119) into the generalized heat equation eq. (118) leads to:

$$\rho T\left(\varsigma_1\ddot{\breve{T}}+\varsigma_2\dddot{\breve{T}}\right)\left[\rho\breve{C}\frac{\dot{\breve{T}}}{\breve{T}}-\rho C\frac{\dot{T}}{T}+2\breve{M}\left(\dot{\breve{T}}-\dot{T}\right)\right] - T\breve{M}^g\Delta\dot{\breve{T}} + div(\vec{q}) - \xi = 0 \tag{121}$$

The heat equation can also be directly obtained from eq. (114) which provides an equivalent but shorter expression of the heat equation (eq. (121)):

$$\rho C\dot{T} + T\breve{M}\left(\dot{T}-\dot{\breve{T}}\right) + div(\vec{q}) - \xi = 0 \tag{122}$$

This equation can also be obtained from eq. (121) after noting that:

$$\breve{M}^g\Delta\breve{T} + \rho\breve{C}\ln\frac{\breve{T}}{T_0} - \breve{M}\left(T-\breve{T}\right) = \rho\left(\varsigma_1\dot{\breve{T}}+\varsigma_2\ddot{\breve{T}}\right) \tag{123}$$

which follows from the combination of the micromorphic balance equation eq. (109) and the state laws eq. (120).

If the Fourier model of heat flux vector is chosen and neglecting the body heat source, the above generalized heat equation will become to:

$$\rho T\left(\varsigma_1\ddot{\breve{T}}+\varsigma_2\dddot{\breve{T}}\right)\left[\rho\breve{C}\frac{\dot{\breve{T}}}{\breve{T}}-\rho C\frac{\dot{T}}{T}+2\breve{M}\left(\dot{\breve{T}}-\dot{T}\right)\right] - T\breve{M}^g\Delta\dot{\breve{T}} = \kappa\Delta T \tag{124}$$

It is interesting to consider the internal constraint which consists in forcing the difference between temperature and micromorphic temperature to be close to 0. The micromorphic temperature model then degenerates into a gradient of temperature model. After enforcing this constraint, the following generalized heat equation is obtained:

$$\rho T(\varsigma_2\dddot{T}+\varsigma_1\ddot{T}) + \rho(C-\breve{C})\dot{T} - T\breve{M}^g\Delta\dot{T} = \kappa\Delta T \tag{125}$$

We may note that:

a) If $\varsigma_2 = 0$ (implying that the inertia is only proportional to the first time derivative of temperature), the linearization of the heat equation (eq. (125)) leads to:

$$\rho T_0 \varsigma_1 \ddot{T} + \rho \left(C - \breve{C} \right) \dot{T} - T_0 \breve{M}^g \Delta \dot{T} = \kappa \Delta T \tag{126}$$

which has the same form as the generalized heat equations derived from the double temperature model in eq. (70), provided that the fourth-order term in eq. (70) is small enough to be neglected (e. g. in pure metal, the thermal conductivity of electron component is much larger than that of lattice component), if $C - \breve{C} = C_\varepsilon$ holds.

b) If the two generalized heat capacities are equivalent $C = \breve{C}$, the heat equation becomes:

$$\rho T_0 \varsigma_1 \ddot{T} - T_0 \breve{M}^g \Delta \dot{T} = \kappa \Delta T \tag{127}$$

It is similar to the heat equation (eq. (91)) derived by Iesan and Nappa [75] in the absence of the fourth-order derivatives, based on the balance laws of micromorphic continua established by Eringen [64, 65]. It involves $\rho T_0 \varsigma_1 = a$ and $\breve{M}^g = -m/T_0$.

c) In eq. (127), if $\breve{M}^g = 0$ for a hypothetical medium or if $\Delta \dot{T}$ is relatively small compared with other two terms, it indicates that the temperature satisfies the classical wave equation:

$$\ddot{T} = \frac{\kappa}{\rho T_0 \varsigma_1} \Delta T \tag{128}$$

This wave equation eq. (128) indicates a finite propagation velocity $\sqrt{\frac{\kappa}{\rho T_0 \varsigma_1}}$.

7.3.2 Double temperature type heat equation

If the coefficient of micromorphic temperature vanishes $\breve{C} = 0$ is assumed for eq. (119), then the generalized heat equation (eq. (125)) becomes:

$$\rho C_\varepsilon \dot{T} + \rho T_0 \left(\varsigma_2 \dddot{T} + \varsigma_1 \ddot{T} \right) - T_0 \breve{M}^g \Delta \dot{T} = \kappa \Delta T \tag{129}$$

The following special cases are derived:

a) If $\varsigma_2 = 0$, the linearized heat equation (eq. (129)) just becomes:

$$\rho C_\varepsilon \dot{T} + \rho T_0 \varsigma_1 \ddot{T} - T_0 \breve{M}^g \Delta \dot{T} = \kappa \Delta T \tag{130}$$

which is the same with the classical two temperature model in eq. (70), if $\alpha = T_0 \varsigma_1 / C_\varepsilon$, $\beta = T_0 \breve{M}^g / \rho C_\varepsilon$, $\gamma = 0$ and $\lambda = \kappa / \rho C_\varepsilon$ hold.

b) If $\varsigma_1 = 0$ (implying that the inertia is proportional to the second time derivative of temperature), the heat equation (eq. (129)) is expressed as:

$$\rho C_\varepsilon \dot{T} + \rho T_0 \varsigma_2 \dddot{T} - T_0 \breve{M}^g \Delta \dot{T} = \kappa \Delta T \tag{131}$$

which is a third-order partial differential equation.

7.3.3 Hyper-temperature type heat equation

Considering the constrained micromorphic heat equation (eq. (129)), if we postulate that the inertia of temperature is sufficiently small compared with other terms, the generalized heat equation becomes:

$$\rho C_\varepsilon \dot{T} - T_0 \breve{M}^g \Delta \dot{T} = \kappa \Delta T \tag{132}$$

which is exactly the heat equation derived from the temperature gradient theory (eq. (57)). It can also be regarded as an approximation of the heat equation derived from the entropy gradient theory (eq. (58)), if $\breve{M}^g = -A_T$ holds.

7.3.4 Hyperbolic type heat equation

Let us now focus on the linearized heat equation (eq. (130)), for cases where the term $\Delta \dot{T}$ is sufficiently small compared with other terms. The heat equation then is the same as the hyperbolic heat equation with relaxation time:

$$\rho C_\varepsilon \dot{T} + \rho T_0 \varsigma_1 \ddot{T} = \kappa \Delta T \tag{133}$$

by taking $\varsigma_1 = \tau C_\varepsilon / T_0$ holds.

8 Comparison of generalized heat equations based on a simple example

In this section, the simple plane thermal wave eq. (21), which is a solution of the hyperbolic heat equation with relaxation time, is adopted to examine the generalized heat equations derived above. Since the heat equations, resulting from the Cattaneo I model (eq. (15)), the hyper-temperature (entropy gradient) model (eq. (58) and eq. (59)), the microtemperature model using Grot's theory (eq. (89)), and the micro-temperature/micro-entropy model (eq. (102)), can be regarded as special cases of the heat equations from classical two-temperature model of Aifantis (eq. (70)), as well as the hyperbolic and generalized two-temperature heat equations have been analyzed by Sobolev [42], only the following three types of generalized heat equations will be analyzed:

$$\begin{cases} F_N(T) = T_{,t} + A_N T_{,xt} - B_N T_{,xx} = 0 \\ F_D(T) = T_{,t} + A_D T_{,tt} - B_D T_{,xx} - C_D T_{,xxt} = 0 \\ F_W(T) = A_W T_{,tt} - B_W T_{,xx} - C_W T_{,xxt} = 0 \end{cases} \tag{134}$$

where $F_N(T)$ is the generalized heat equation derived from Nguyen's gradient temperature model (eq. (45)), $F_D(T)$ is the heat equation from the constrained micromorphic theory (eq. (130)) in double temperature type with omitted fourth-order derivative, $F_W(T)$ is the heat equation deduced from the micromorphic theory which has the same structure as the micro-temperature model using Green's theory (eq.). These coefficients are given by: $A_N = \frac{Mg}{\rho C_\varepsilon L}$ and $B_N = \frac{\kappa}{\rho C_\varepsilon} > 0$, $A_D = \frac{T_0 \varsigma_1}{C_\varepsilon}$, $B_D = \frac{\kappa}{\rho C_\varepsilon} > 0$ and $C_D = \frac{T_0 \breve{M}^g}{\rho C_\varepsilon}$, $A_W = \rho T_0 \varsigma_1$, $B_W = \kappa > 0$ and $C_W = T_0 \breve{M}^g$.

Substituting the assumed plane thermal wave solution, $T(x, t) = T_0 \exp[i(kx - \omega t)]$(eq.) with k is the (complex) wave number indicating the attenuation of the amplitude of the propagating wave, w is the (real) frequency, into the above nonlocal heat equations, leads to the following dispersion relationships:

$$\begin{cases} -i\omega + A_N k\omega + B_N k^2 = 0 \\ -i\omega - A_D \omega^2 + B_D k^2 - iC_D \omega k^2 = 0 \\ -A_W \omega^2 + B_W k^2 - iC_W \omega k^2 = 0 \end{cases} \tag{135}$$

8.1 Solutions of Nguyen's generalized heat equation

Let us write $k = a + ib$ and substitute it into the dispersion relation (eq. (135)a), leading to:

$$(-\omega + A_N b\omega + 2abB_N)i + (A_N a\omega + B_N(a^2 - b^2)) = 0 \tag{136}$$

In order to satisfy the above equation, it requires:

$$\begin{cases} \omega = A_N b\omega + 2abB_N \\ A_N a\omega + B_N(a^2 - b^2) = 0 \end{cases} \tag{137}$$

Solving the above nonlinear dispersion equations leads to the following velocity V_p and the attenuation coefficient α:

$$V_p = \frac{\omega}{\mathrm{Re}k} = \left[-\frac{1}{2}\frac{A_N}{B_N} + \frac{2}{\sqrt{-2\omega\left(A_N^2\omega - \sqrt{A_N^4\omega^2 + 16B_N^2}\right)}} \right] \tag{138}$$

$$\alpha = \frac{1}{Im(k)} = \frac{1}{(A_N + 2B_N V_p)\omega} \tag{139}$$

for lower frequencies $A_N^2\omega \ll 4B_N$, the velocity is infinite; in high frequency $A_N^2\omega \gg 4B_N$, the velocity approaches the limit $V_p \approx -2Mg/(\kappa L)$ and the attenuation distance is infinite.

Clearly, the heat equation provides a finite propagation speed of thermal waves in high frequency, while requiring the coefficient g to be a negative number in order to obtain a positive velocity V_p, due to the thermal conductivity coefficient $\kappa > 0$, material parameter $M > 0$ and the length $L > 0$ of the rod, are all positive defined parameters in the literature [53]. Consequently, $A_N = Mg/(\rho C_\varepsilon L) < 0$ holds. It should be noticed that it predicts an infinite velocity of thermal propagation at low frequency, in the case of existence the attenuation of amplitude of thermal wave (k is complex), which differs from the classical Fourier model and the hyperbolic heat equation with relaxation time. However, as shown in literature [53], the heat equation gives a finite speed propagation of thermal wave with the help of $k = i/l$ and ω are both complex numbers.

8.2 Solutions of the heat equation in the double temperature theory

Let us take $k = a + ib$ and substitute it into the dispersion relation (eq. (135)b), leading to:

$$\left(-\omega + 2abB_D - C_D\omega(a^2 - b^2)\right)i + \left(-A_D\omega^2 + B_D(a^2 - b^2) + 2abC_D\omega\right) = 0 \tag{140}$$

In order to satisfy the above equation, it is required that:

$$\begin{cases} \omega = 2abB_D - C_D\omega(a^2 - b^2) \\ A_D\omega^2 = B_D(a^2 - b^2) + 2abC_D\omega \end{cases} \tag{141}$$

Solving the above nonlinear dispersion equations results in the following velocity V_p and attenuation distance α:

$$V_p = \frac{\omega}{\mathrm{Re}k} = \sqrt{\frac{2\left(B_D^2 + C_D^2\omega^2\right)}{(A_D B_D - C_D) + \sqrt{(A_D B_D - C_D)^2 + \left(\frac{B_D}{\omega} + A_D C_D\omega\right)^2}}} \tag{142}$$

$$\alpha = \frac{1}{Im(k)} = \frac{2\left(B_D^2 + C_D^2\omega^2\right)}{(B_D + A_D C_D\omega^2)V_p} \tag{143}$$

a) At low frequency $C_D^2\omega^2 \ll B_D^2$, the velocity $V_p \approx \sqrt{2\kappa\omega/(\rho C_\varepsilon)}$ and $\alpha \approx \sqrt{2\kappa/(\rho C_\varepsilon\omega)}$ has the same form as predicted by the classical theory based on Fourier's model [32]. In the high-frequency limit $C_D^2\omega^2 \gg B_D^2$, the velocity approaches $V_p \approx \sqrt{2\breve{M}^g\omega/(\rho\varsigma_1)}$ and $\alpha \approx 2C_D/(A_D V_p) \approx \sqrt{2\breve{M}^g/(\rho\varsigma_1\omega)}$. Clearly, the factor $\left(\breve{M}^g/\varsigma_1\right)$ should be positive ($A_D C_D > 0$) to have an infinite real velocity V_p of the propagation.

b) If $A_D = 0$, it yields the hyper-temperature heat equation (eq. (132)), or the heat equation from Cattaneo I (eq. (15)), or the heat equation (eq. (89)) derived by Iesan [84] in the absence of the spatial fourth-order term. The velocity and attenuation distance are given by:

$$V_p = \frac{\omega}{\text{Re}k} = \sqrt{\frac{2\left(B_D^2 + C_D^2\omega^2\right)}{-C_D + \sqrt{C_D^2 + \left(\frac{B_D}{\omega}\right)^2}}} \tag{144}$$

$$\alpha = \frac{1}{Im(k)} = \frac{2\left(B_D^2 + C_D^2\omega^2\right)}{B_D V_p} \tag{145}$$

Clearly, for low frequency $C_D^2\omega^2 \ll B_D^2$, the velocity $V_p \approx \sqrt{2\kappa\omega/(\rho C_\varepsilon)}$ and $\alpha \approx \sqrt{2\kappa/(\rho C_\varepsilon\omega)}$ in the high frequency limit $C_D^2\omega^2 \gg B_D^2$, the velocity approaches $V_p \approx \sqrt{-2T_0\breve{M}^g\omega/(\rho C_\varepsilon)}$ and $\alpha \approx \sqrt{-2\left(T_0\breve{M}^g\right)^3\omega^3/(\rho C_\varepsilon\kappa^2)}$, by taking $\breve{M}^g < 0$. this type of heat equations also predicts propagation at infinite velocity.

c) If $C_D = 0$, it yields the hyperbolic heat equation with relaxation time (eq. (133)). The velocity and attenuation distance are obtained:

$$V_p = \frac{\omega}{\text{Re}k} = \sqrt{\frac{2B_D^2}{A_D B_D + \sqrt{(A_D B_D)^2 + \left(\frac{B_D}{\omega}\right)^2}}} \tag{146}$$

$$\alpha = \frac{1}{Im(k)} = \frac{2B_D}{V_p} \tag{147}$$

which are exactly the solutions given by the hyperbolic heat equation of Cattaneo II in eq. (20), corresponding to the finite propagation of thermal waves: in low frequencies ($A_D\omega \ll 1$), the velocity $v_p \approx \sqrt{2\kappa\omega/\rho C_\varepsilon}$ and $\alpha = \sqrt{2\kappa/\rho C_\varepsilon\omega}$, in high frequency limit ($A_D\omega \gg 1$), the velocity $v_p \approx \sqrt{\kappa/(\rho T_0\varsigma_1)}$ and $\alpha \approx 2\sqrt{T_0\varsigma_1\kappa/(\rho C_\varepsilon^2)}$, by taking $\varsigma_1 = \tau C_\varepsilon/T_0$.

8.3 Solutions of the heat equations in wave type

Taking $k = a + ib$ and substitute it into the dispersion relation (eq. (135c) gives:

$$\left(2abB_W - C_W\omega\left(a^2 - b^2\right)\right)i + \left(-A_W\omega^2 + B_W\left(a^2 - b^2\right) + 2abC_W\omega\right) = 0 \tag{148}$$

In order to satisfy the above equation, it is required that:

$$\begin{cases} -A_W\omega^2 + B_W(a^2 - b^2) + 2abC_W\omega = 0 \\ 2abB_W - C_W\omega(a^2 - b^2) = 0 \end{cases} \tag{149}$$

Solving the above nonlinear dispersion equations, yields the velocity V_p:

$$V_p = \frac{\omega}{\text{Re}k} = \sqrt{\frac{2\left(B_W^2 + C_W^2\omega^2\right)}{A_W B_W + A_W\sqrt{B_W^2 + C_W^2\omega^2}}} \tag{150}$$

and the attenuation distance α:

$$\alpha = \frac{2\left(B_W^2 + C_W^2\omega^2\right)}{\left(A_W C_W\omega^2\right)V_p} \tag{151}$$

for low frequencies ($C_W^2\omega^2 \ll B_W^2$), the finite velocity $v_p \approx \sqrt{\kappa/(\rho T_0\varsigma_1)}$ and the attenuation distance is infinite; in high-frequency limit ($C_W^2\omega^2 \gg B_W^2$), the velocity approaches $v_p \approx \sqrt{2T_0\breve{M}^g\omega}$ and $\alpha \approx \sqrt{2T_0\breve{M}^g/\omega}\Big/\rho T_0\varsigma_1$, by taking the two parameters ς_1 and $\breve{M}^g$ being positive.

Clearly, the velocity given by eq. (150) also predicts the infinite speed of thermal propagation, and this infinite propagation is caused by the term $\Delta\dot{T}$ in eq. (126). If $C_W = 0$ for a hypothetical medium, or if the term $\Delta\dot{T}$ is sufficiently small compared with other two terms to be neglected, the velocity is finite:

$$V_p = \frac{\omega}{\mathrm{Re}\,k} = \sqrt{\frac{B_W}{A_W}} = \sqrt{\frac{\kappa}{\rho T_0 \varsigma_1}} \tag{152}$$

which is exactly the velocity of the wave equation eq. (128). The attenuation distance is infinite.

9 Conclusions

In this work, the micromorphic approach, initially developed by Mindlin and Eringen in the mechanical context, has been applied to the problem of heat transfer by means of a generalized principle of virtual power. For that purpose, an independent microtemperature field is introduced and its gradient is assumed to contribute to the free energy density function. The proposed phenomenological method has been shown to lead to a pair of coupled generalized heat equations. The model identically satisfies the local condition of positive dissipation rate. The constrained case for which the microtemperature coincides with the temperature itself is of particular interest because it can be compared to the various extensions of the heat equation available in the literature. The proposed equation is enhanced by essentially three new contributions which were illustrated in the linearized case. The three terms affect the transient thermal behavior of the material only. The first new term is related to the Laplacian of the temperature rate, a contribution which was proposed first by Cattaneo in an early version of his theory. A second contribution is proportional to the second time derivative of the temperature which changes the usual heat equation into a hyperbolic equation with a positive characteristic time. The last term associated with the third time derivative of temperature remains rather unexplored even though it is present in some generalized theories found in the literature.

The proposed theory was shown to differ from existing thermomechanical extensions of Eringen's micromorphic model that rely on the introduction of a microtemperature vector akin to a relaxed temperature gradient. It has also been compared to the effective heat equation arising from the double temperature model involving two coupled heat equations and leading also to fourth-order spatial derivatives:

- The hyperbolic heat equation of Cattaneo II with relaxation time can be obtained from the thermodynamically consistent micromorphic heat equation by neglecting the nonlocal effects of the temperature.
- The classical, hyperbolic and generalized two-temperature models can be also retrieved from the micromorphic heat equations.
- Both the micro-temperature heat equation using Green's theory and the micromorphic heat equation provide a pair of governing equations for local and micro/micromorphic temperature. And the classical wave equation can be obtained from their constrained heat equations.

The proposed theory allows for a direct coupling with the mechanics of materials and is thought to be useful for the simulation of fast heat and mechanical treatments of materials including metal forming at high speeds, laser surface treatments of materials, etc. The nonlocality in the proposed theory resulting from the introduction of the gradient of micromorphic temperature could be an alternative to the analysis of heat transport based on enhanced heat fluxes. Applications are expected in nano-structured media and nanosystems, but also for fast material processing and surface treatments. The work performed by Saanouni and Hamed [77] for finite time-independent plasticity will be extended to the thermomechanical processes (finite viscoplasticity) using the generalized heat equation based on the present model of micromorphic temperature.

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