

Strain localisation modes in the mechanics of materials

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Outline

- 1 Bifurcation modes in elastoplastic solids
 - Linear incremental formulation
 - Loss of uniqueness
- 2 Compatible bifurcation modes
 - Hadamard jump conditions
 - Orientation of a strain localization band
- 3 Stability / ellipticity of the boundary value problem
 - Mandel–Rice criterion
 - Orientation of localization bands in 3D and 2D
- 4 Summary of strain localization criteria
- 5 Regularization methods
 - Mesh dependence of the results
 - Mechanics of generalized continua
 - Application to the case of metal foams

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Loi de comportement élastoplastique

- Elasticity $\dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^p, \quad \sigma = \mathbf{E} : \varepsilon^e$
- Yield function $g(\sigma, \alpha), \quad \mathbf{N} = \frac{\partial g}{\partial \sigma}$
- Plastic flow $\dot{\varepsilon}^p = \lambda \mathbf{P}, \quad \mathbf{P} \neq \mathbf{N}$
- Hardening modulus $\dot{\alpha} = \lambda h, \quad H = -\frac{\partial g}{\partial \alpha}.h$
- Tangent operator (non symmetrical if non associative) $\dot{\sigma} = \mathbf{L} : \dot{\varepsilon}$

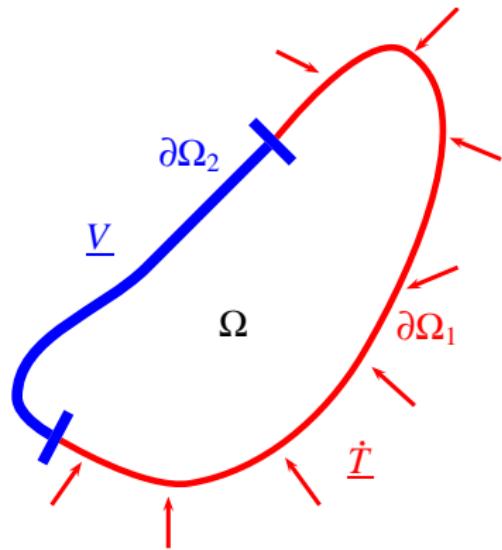
$$\mathbf{L} = \mathbf{E} \quad \text{si } g < 0 \quad \text{ou} \quad (g = 0 \quad \text{et} \quad \mathbf{N} : \mathbf{E} : \dot{\varepsilon} \leq 0)$$

$$\mathbf{L} = \mathbf{D} \quad \text{si } g = 0 \quad \text{et} \quad \mathbf{N} : \mathbf{E} : \dot{\varepsilon} > 0$$

with

$$\mathbf{D} = \mathbf{E} - \frac{(\mathbf{E} : \mathbf{P}) \otimes (\mathbf{N} : \mathbf{E})}{H + \mathbf{N} : \mathbf{E} : \mathbf{P}}$$

Rate form of the boundary value problem



$$\begin{cases} \dot{\tilde{\varepsilon}} = \frac{1}{2}(\nabla \underline{v} + \nabla^t \underline{v}) \\ \operatorname{div} \dot{\tilde{\sigma}} + \underline{f} = 0, \quad \dot{\tilde{\sigma}} = \underline{L} : \dot{\tilde{\varepsilon}} \\ \dot{\tilde{\sigma}} \cdot \underline{n} = \underline{T} \quad \text{in} \quad \partial\Omega_1 \\ \underline{v} = \underline{V} \quad \text{in} \quad \partial\Omega_2 \end{cases}$$

- Linear comparison solid: take $\underline{L} = \underline{D}$
- Variational formulation (virtual power theorem)

$$\int_{\Omega} \dot{\tilde{\sigma}} : \dot{\tilde{\varepsilon}}^* dV = \int_{\Omega} \underline{f} \cdot \underline{v}^* dV + \int_{\partial\Omega_1} \underline{T} \cdot \underline{v}^* dS + \int_{\partial\Omega_2} (\dot{\tilde{\sigma}} \cdot \underline{n}) \cdot \underline{V} dS$$

for any field \underline{v}^* such that $\underline{v}^* = \underline{V}$ in $\partial\Omega_2$

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Conditions of uniqueness of the solution

Let $(\dot{\underline{\sigma}}_A, \dot{\underline{\varepsilon}}_A)$ and $(\dot{\underline{\sigma}}_B, \dot{\underline{\varepsilon}}_B)$ be two solutions on Ω at time t

$$\int_{\Omega} \dot{\underline{\sigma}}_A : \dot{\underline{\varepsilon}}_A \, dV = \int_{\Omega} \dot{\underline{f}} \cdot \underline{v}_A \, dV + \int_{\partial\Omega_1} \dot{\underline{T}} \cdot \underline{v}_A \, dS + \int_{\partial\Omega_2} (\dot{\underline{\sigma}}_A \cdot \underline{n}) \cdot \underline{V} \, dS$$

Conditions of uniqueness of the solution

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$$\int_{\Omega} \dot{\underline{\sigma}}_A : \dot{\underline{\varepsilon}}_B dV = \int_{\Omega} \dot{\underline{f}} \cdot \underline{v}_B dV + \int_{\partial\Omega_1} \dot{\underline{T}} \cdot \underline{v}_B dS + \int_{\partial\Omega_2} (\dot{\underline{\sigma}}_A \cdot \underline{n}) \cdot \underline{V} dS$$

Conditions of uniqueness of the solution

Let $(\dot{\underline{\sigma}}_A, \dot{\underline{\varepsilon}}_A)$ and $(\dot{\underline{\sigma}}_B, \dot{\underline{\varepsilon}}_B)$ be two solutions on Ω at time t

$$\int_{\Omega} \dot{\underline{\sigma}}_A : \dot{\underline{\varepsilon}}_A dV = \int_{\Omega} \dot{\underline{f}} \cdot \underline{v}_A dV + \int_{\partial\Omega_1} \dot{\underline{T}} \cdot \underline{v}_A dS + \int_{\partial\Omega_2} (\dot{\underline{\sigma}}_A \cdot \underline{n}) \cdot \underline{V} dS$$

$$- \int_{\Omega} \dot{\underline{\sigma}}_A : \dot{\underline{\varepsilon}}_B dV = \int_{\Omega} \dot{\underline{f}} \cdot \underline{v}_B dV + \int_{\partial\Omega_1} \dot{\underline{T}} \cdot \underline{v}_B dS + \int_{\partial\Omega_2} (\dot{\underline{\sigma}}_A \cdot \underline{n}) \cdot \underline{V} dS$$

$$\int_{\Omega} \dot{\underline{\sigma}}_A : \Delta \dot{\underline{\varepsilon}} dV = \int_{\Omega} \dot{\underline{f}} \cdot \Delta \underline{v} dV + \int_{\partial\Omega_1} \dot{\underline{T}} \cdot \Delta \underline{v} dS$$

Conditions of uniqueness of the solution

Let $(\dot{\sigma}_A, \dot{\varepsilon}_A)$ and $(\dot{\sigma}_B, \dot{\varepsilon}_B)$ be two solutions on Ω at time t

$$\int_{\Omega} \dot{\sigma}_A : \Delta \dot{\varepsilon} dV = \int_{\Omega} \dot{\mathbf{f}} \cdot \Delta \mathbf{v} dV + \int_{\partial\Omega_1} \dot{\mathbf{T}} \cdot \Delta \mathbf{v} dS$$
$$-\int_{\Omega} \dot{\sigma}_B : \Delta \dot{\varepsilon} dV = \int_{\Omega} \dot{\mathbf{f}} \cdot \Delta \mathbf{v} dV + \int_{\partial\Omega_1} \dot{\mathbf{T}} \cdot \Delta \mathbf{v} dS$$

$$\int_{\Omega} \Delta \dot{\sigma} : \Delta \dot{\varepsilon} dV = 0$$

Conditions of uniqueness of the solution

Let $(\dot{\underline{\sigma}}_A, \dot{\underline{\varepsilon}}_A)$ and $(\dot{\underline{\sigma}}_B, \dot{\underline{\varepsilon}}_B)$ be two solutions on Ω at time t

$$\int_{\Omega} \Delta \dot{\underline{\sigma}} : \Delta \dot{\underline{\varepsilon}} dV = 0$$

If the solutions $(\underline{\sigma}_A, \underline{\varepsilon}_A)$ and $(\underline{\sigma}_B, \underline{\varepsilon}_B)$ have coincided until time t , one can assume

$$\dot{\underline{\sigma}}_A = \mathbf{D} : \dot{\underline{\varepsilon}}_A, \quad \dot{\underline{\sigma}}_B = \mathbf{D} : \dot{\underline{\varepsilon}}_B$$

A necessary condition for the existence of several solutions then is

$$\int_{\Omega} \Delta \dot{\underline{\varepsilon}} : \mathbf{D} : \Delta \dot{\underline{\varepsilon}} dV = 0$$

A sufficient condition for uniqueness

A sufficient condition ensuring the uniqueness of the solution ($\Delta \dot{\varepsilon} = 0$) is the positivity of *second order power* :

$$\dot{\varepsilon} : \mathbf{D}_{\approx} : \dot{\varepsilon} > 0, \quad \forall \dot{\varepsilon} \neq 0$$

$$\dot{\varepsilon} : \mathbf{D}_{\approx}^s : \dot{\varepsilon} > 0, \quad \text{avec} \quad \mathbf{D}_{\approx}^s_{ijkl} = \frac{1}{2}(D_{ijkl} + D_{klji})$$

Hill's condition for uniqueness: \mathbf{D}_{\approx}^s positive definite (Hill, 1958)

Uniqueness may be lost (possible bifurcation) when

$$\det \mathbf{D}_{\approx}^s = 0$$

$$\text{which gives } H^u = \frac{1}{2}(\sqrt{\mathbf{N}_{\approx} : \mathbf{E}_{\approx} : \mathbf{N}_{\approx}} \sqrt{\mathbf{P}_{\approx} : \mathbf{E}_{\approx} : \mathbf{P}_{\approx}} - \mathbf{N}_{\approx} : \mathbf{E}_{\approx} : \mathbf{P}_{\approx})$$

$H > H^u$: uniqueness ensured
softening!

associative plasticity: $H^u = 0$

uniaxial + Large Strain = Considère criterion...

Example : von Mises plasticity

- Von Mises plasticity:

$$g(\tilde{\sigma}, R) = J_2(\tilde{\sigma}) - R, \quad J_2(\tilde{\sigma}) = \sqrt{\frac{3}{2} \tilde{\sigma}^{\text{dev}} : \tilde{\sigma}^{\text{dev}}}$$

- Associative plasticity: $\tilde{\mathbf{N}} = \tilde{\mathbf{P}} = \frac{3}{2} \frac{\tilde{\sigma}^{\text{dev}}}{J_2}$

- The tangent operator can be computed easily!

$$\tilde{\mathbf{N}} : \tilde{\mathbf{E}} : \tilde{\mathbf{N}} = \tilde{\mathbf{N}} : (2\mu \tilde{\mathbf{N}}) = 3\mu$$

$$\tilde{\mathbf{D}} = \tilde{\mathbf{E}} - \frac{3\mu^2}{H/3 + \mu} \frac{\tilde{\sigma}^{\text{dev}} \otimes \tilde{\sigma}^{\text{dev}}}{J_2^2}$$

- Principal or eigen-tensors of $\tilde{\mathbf{D}} = \tilde{\mathbf{D}}^s$: one tries

$$\tilde{\mathbf{d}}_1 = \frac{\tilde{\sigma}^{\text{dev}}}{J_2}, \quad \tilde{\mathbf{D}} : \tilde{\mathbf{d}}_1 = \frac{2\mu H}{H + 3\mu} \tilde{\mathbf{d}}_1$$

- Principal or eigen-values:

$$k, 2\mu, \frac{2\mu H}{H + 3\mu}$$

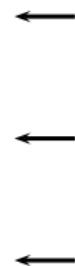
When $H = 0$ (perfect plasticity, begining of a softening regime)

Example : necking mode

Tensile test,

$$[\tilde{\mathbf{d}}_1] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

possible perturbations when $H = 0\dots$



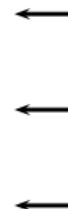
$$\dot{\tilde{\boldsymbol{\sigma}}}_A = \tilde{\mathbf{D}} : \dot{\tilde{\boldsymbol{\varepsilon}}}_A = \tilde{\mathbf{D}} : (\dot{\tilde{\boldsymbol{\varepsilon}}}_A + \lambda \tilde{\mathbf{d}}_1)$$

Example : necking mode

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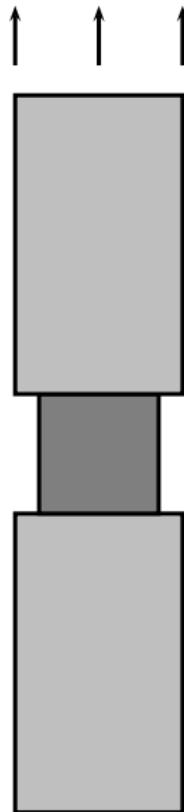
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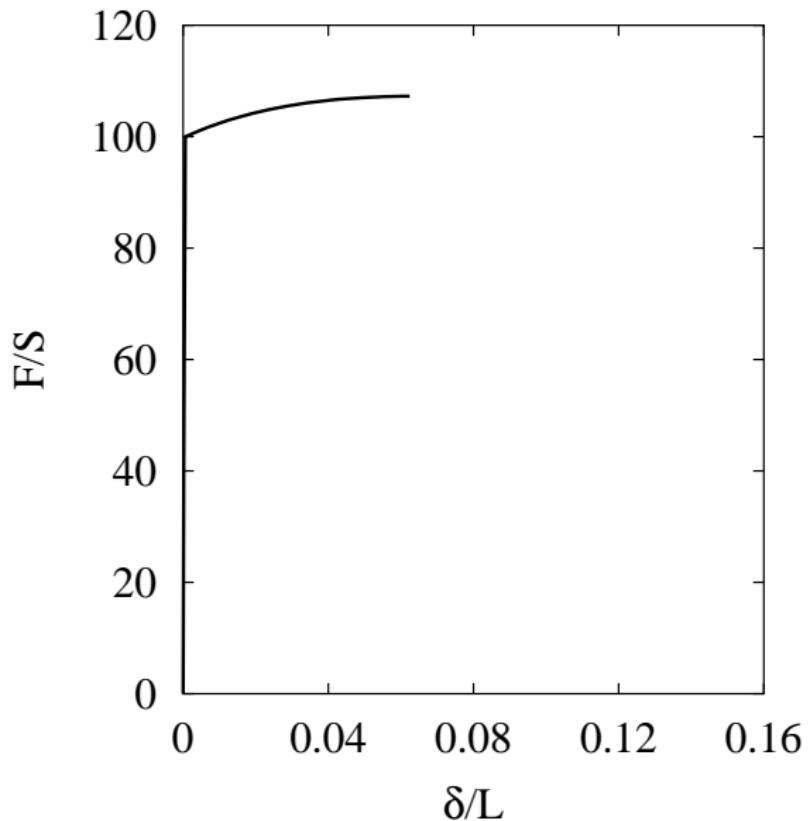
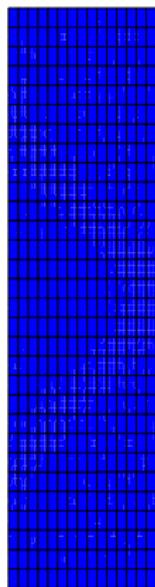
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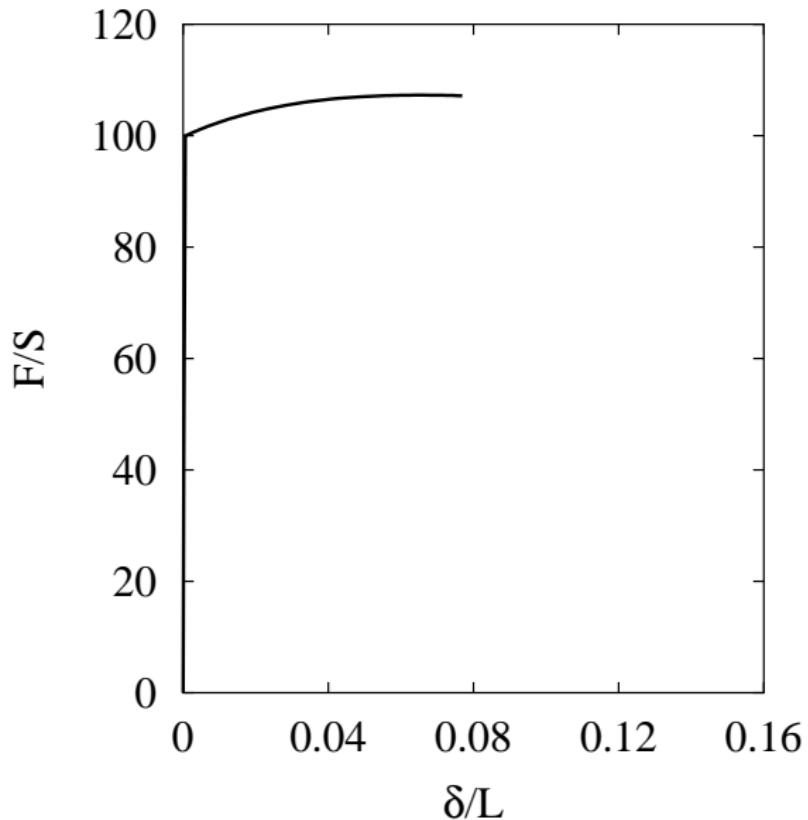
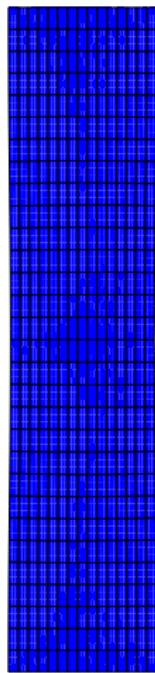
possible perturbations when $H = 0\dots$

$$\dot{\tilde{\boldsymbol{\sigma}}}_A = \tilde{\mathbf{D}} : \dot{\tilde{\boldsymbol{\varepsilon}}}_A = \tilde{\mathbf{D}} : (\dot{\tilde{\boldsymbol{\varepsilon}}}_A + \lambda \tilde{\mathbf{d}}_1)$$

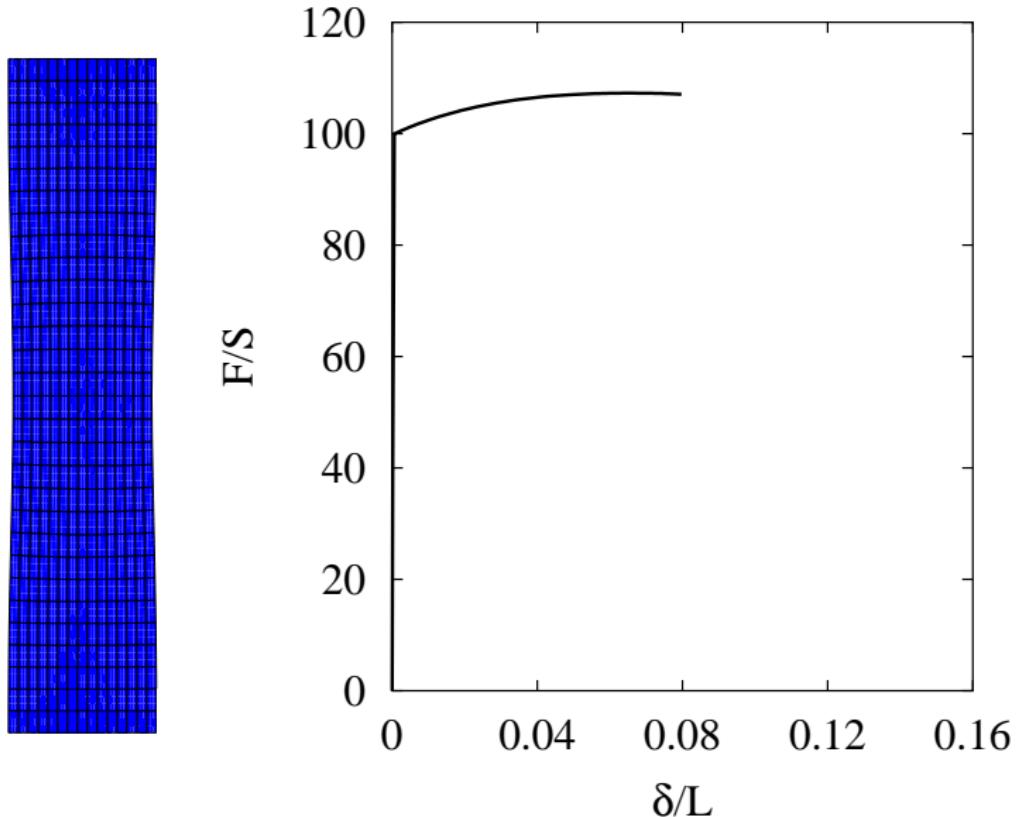
Example : necking mode



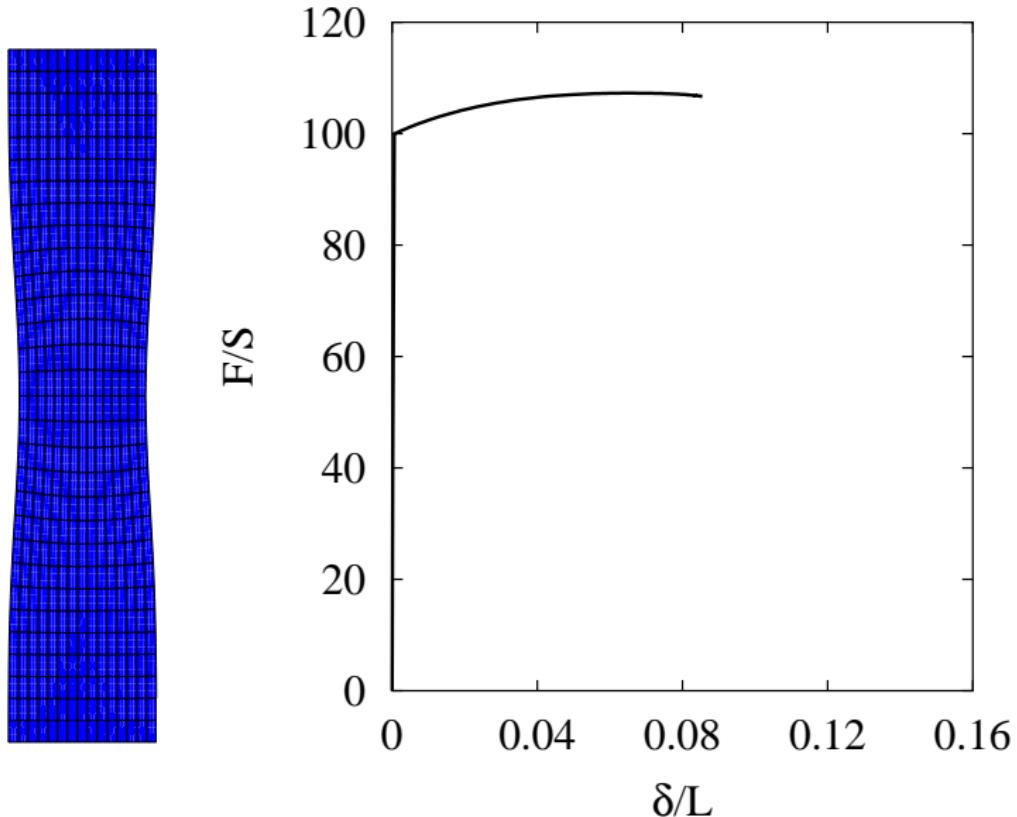
Example : necking mode



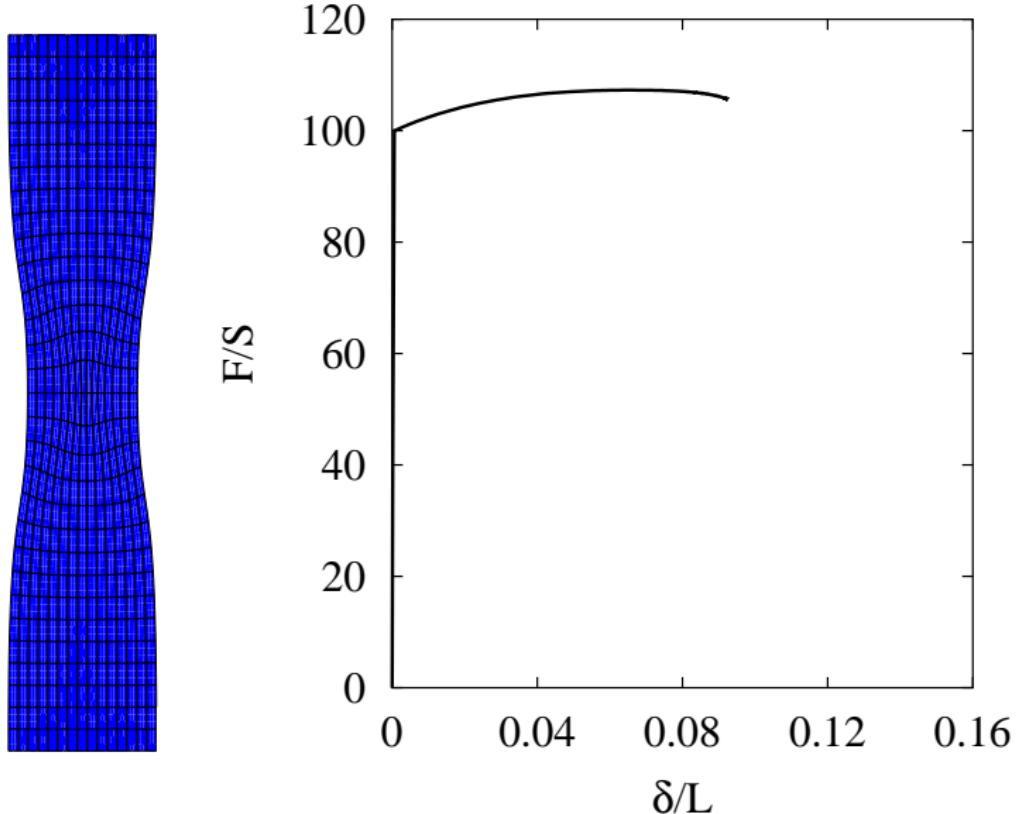
Example : necking mode



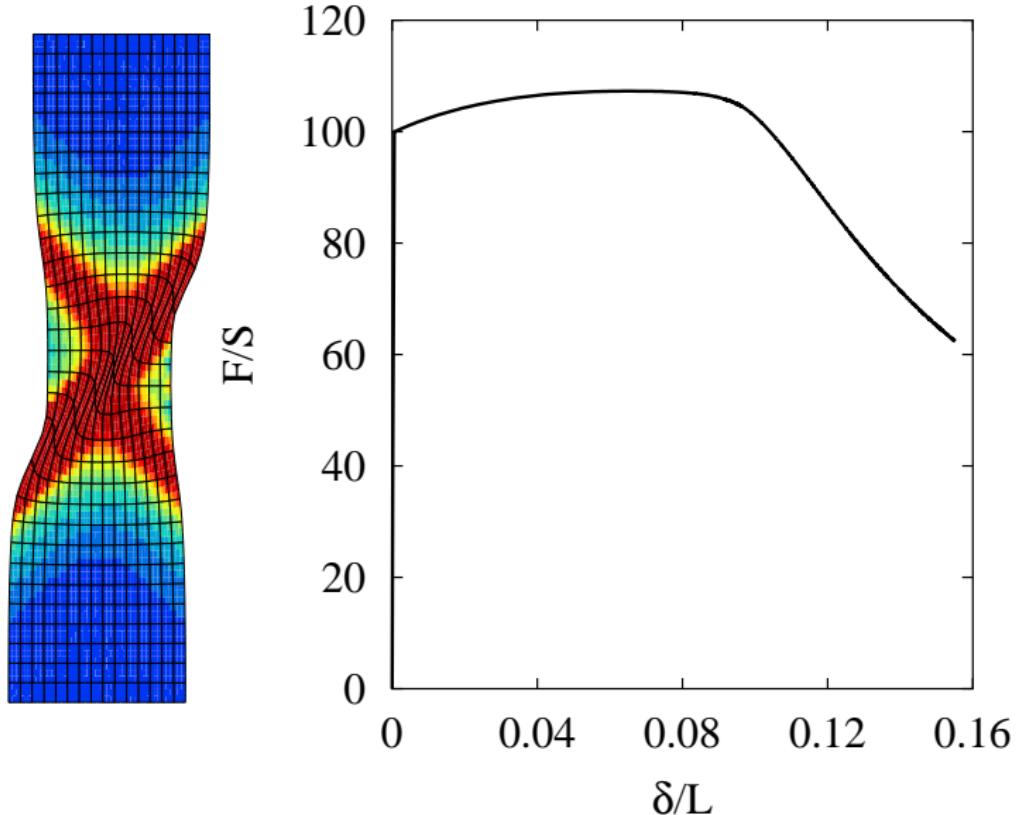
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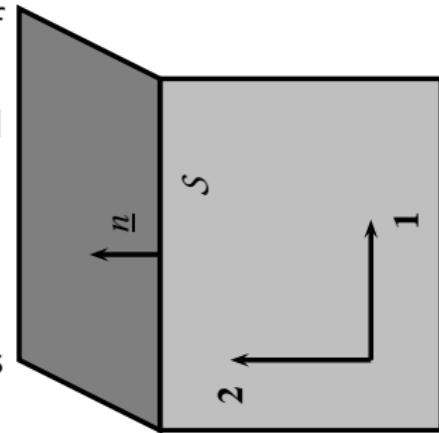
Compatible discontinuous bifurcation modes : Hadamard jump conditions

Strain localization is the precursor of fracture.

Continuity of displacement is assumed through a surface \mathcal{S}

$$[\underline{\mathbf{u}}] = [\dot{\underline{\mathbf{u}}}] = 0$$

but discontinuities of some components of its gradient $\nabla \underline{\mathbf{u}}$ are considered



Compatible discontinuous bifurcation modes : Hadamard jump conditions

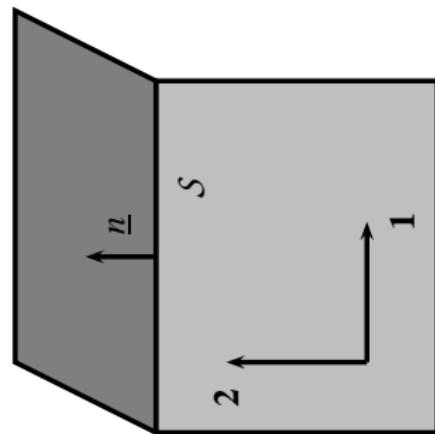
Let us work in the frame attached to the surface of discontinuity:

$$[u_1] = 0 \implies [u_{1,1}] = 0$$

$$[u_2] = 0 \implies [u_{2,1}] = 0$$

but, in general, $\exists \underline{g}$ such that

$$[u_{1,2}] = g_1 \neq 0, \quad [u_{2,2}] = g_2 \neq 0$$



Compatible discontinuous bifurcation modes : Hadamard jump conditions

Let us work in the frame attached to the surface of discontinuity:

$$[\![u_1]\!] = 0 \implies [\![u_{1,1}]\!] = 0$$

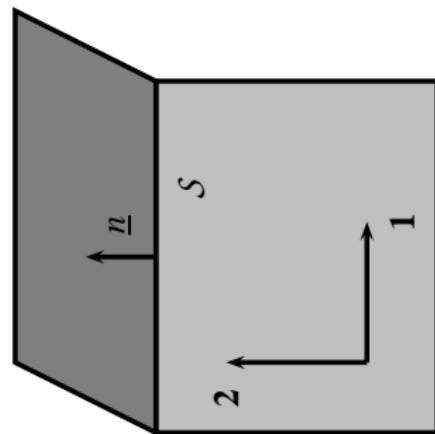
$$[\![u_2]\!] = 0 \implies [\![u_{2,1}]\!] = 0$$

but, in general,

$$[\![u_{1,2}]\!] = g_1 \neq 0, \quad [\![u_{2,2}]\!] = g_2 \neq 0$$

$$[\![\nabla \underline{\mathbf{u}}]\!] = \begin{bmatrix} 0 & g_1 \\ 0 & g_2 \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$[\![\nabla \underline{\mathbf{u}}]\!] = \underline{\mathbf{g}} \otimes \underline{\mathbf{n}}$$



Compatible discontinuous bifurcation modes : Hadamard jump conditions

$$[\![\nabla \dot{\mathbf{u}}]\!] = \underline{\mathbf{g}} \otimes \underline{\mathbf{n}}$$

$$[\![\nabla \dot{\varepsilon}]\!] = \frac{1}{2}(\underline{\mathbf{g}} \otimes \underline{\mathbf{n}} + \underline{\mathbf{n}} \otimes \underline{\mathbf{g}})$$

in general $\underline{\mathbf{g}} \cdot \underline{\mathbf{n}} \neq 0$

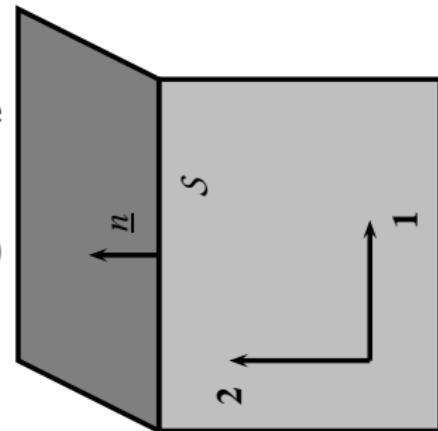
Criterion for the occurrence of compatible bifurcation modes:

$$\Delta \dot{\varepsilon} : \underline{\mathbf{D}} : \Delta \dot{\varepsilon} = (\underline{\mathbf{g}} \otimes \underline{\mathbf{n}}) : \underline{\mathbf{D}} : (\underline{\mathbf{g}} \otimes \underline{\mathbf{n}}) = 0$$

The **acoustic tensor** $\underline{\mathbf{Q}}$ is introduced:

$$\underline{\mathbf{g}} \cdot \underline{\mathbf{Q}} \cdot \underline{\mathbf{g}} = \underline{\mathbf{g}} \cdot \underline{\mathbf{Q}}^{\text{sym}} \cdot \underline{\mathbf{g}} = 0, \quad \underline{\mathbf{Q}} = \underline{\mathbf{n}} \cdot \underline{\mathbf{D}} \cdot \underline{\mathbf{n}}$$

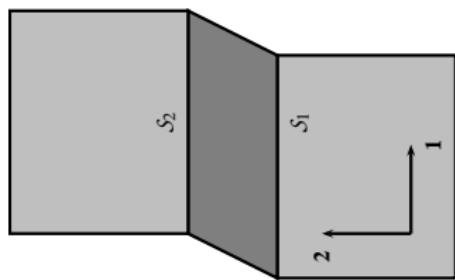
$$\Rightarrow \boxed{\det \underline{\mathbf{Q}}^{\text{sym}} = 0}$$



Compatible discontinuous bifurcation modes : Hadamard jump conditions

One can imagine two parallel surfaces of discontinuity: it is a **strain localization band** or **shear band** in a wide sense. Indeed, the band does generally not undergo pure shear...

- $\underline{g} \cdot \underline{n} = 0$: shear band
- $\underline{g} \parallel \underline{n}$: opening mode



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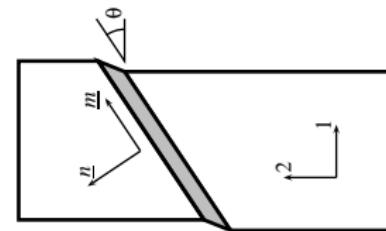
Example: compatible strain localization mode under plane stress conditions

Consider the mode

$$[\underline{d}_1] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

but write it with respect to the frame $(\underline{m}, \underline{n})$

$$[\underline{d}_1] = \begin{bmatrix} 2\sin^2\theta - \cos^2\theta & 3\cos\theta\sin\theta & 0 \\ 3\cos\theta\sin\theta & 2\cos^2\theta - \sin^2\theta & 0 \\ 0 & 0 & -1 \end{bmatrix}$$



to be identified with $\frac{1}{2}(\underline{g} \otimes \underline{n} + \underline{n} \otimes \underline{g}) = \begin{bmatrix} 0 & g_1/2 \\ g_1/2 & g_2 \end{bmatrix}$

$$\Rightarrow \tan^2\theta = \frac{1}{2}, \quad \text{ou} \quad \sin^2\theta = \frac{1}{3}$$

$$\theta = 35, 26^\circ, \quad 90^\circ - \theta = 54.74^\circ$$

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Mandel–Rice criterion

Add the equilibrium conditions at the interface which were not taken into account until now:

$$[\dot{\sigma}] \cdot \underline{n} = 0, \quad [\tilde{D} : \dot{\varepsilon}] \cdot \underline{n} = 0$$

If the solutions on each side of S are identical until time t , one has $[\tilde{D}] = 0$

$$[\tilde{D} : [\dot{\varepsilon}] \cdot \underline{n}] = 0$$

Let us consider the compatible modes:

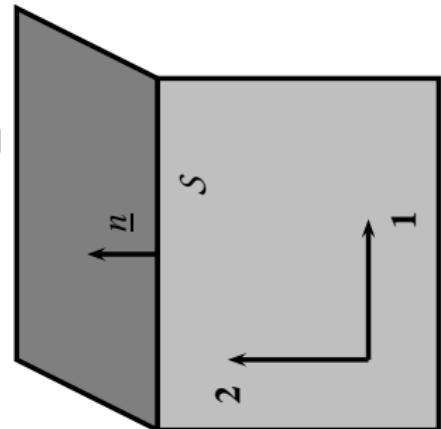
$$[\tilde{D} : (\underline{g} \otimes \underline{n}) \cdot \underline{n}] = 0$$

$$D_{ijkl} g_k n_l n_j = n_j D_{jikl} n_l g_k = 0$$

$$[\tilde{Q} \cdot \underline{g}] = 0$$

This is the **acoustic tensor**: $\tilde{Q} = \underline{n} \cdot \tilde{D} \cdot \underline{n}$

stable propagation of perturbations [Mandel, 1966]



Mandel–Rice criterion:

$$\det \tilde{Q}(\underline{n}) = 0$$

[Rice, 1976]

Vocabulary

- Plastic waves [Mandel, 1962]
the previous situation corresponds to stationary plastic waves
- **Stability:** analysis if the growth of perturbations
- **Loss of ellipticity** of the boundary value problem:
when the incremental constitutive equations are inserted in the equations of equilibrium (static case), one gets a system of partial differential equations of order 2 in \dot{u}_i . The system is said to be *elliptic* when the differential operator has definite positive eigen-values. It can be shown that the problem then is **well-posed** (meaning that the solution continuously depends on the data).

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Orientation of strain localization bands

The resolution of the equation $\det \underline{\underline{n}} \cdot \tilde{\underline{\underline{D}}} \cdot \underline{\underline{n}} = 0$ provides us with the plastic modulus $H(\underline{\underline{n}})$ for which a surface of discontinuity of normal $\underline{\underline{n}}$ can arise. To determine the first possible band, one must find H^{cr} and $\underline{\underline{n}}$ such that

$$H^{cr} = \max_{\|\underline{\underline{n}}\|=1} H(\underline{\underline{n}})$$

In the case of isotropic elasticity and incompressible associative plasticity, one finds

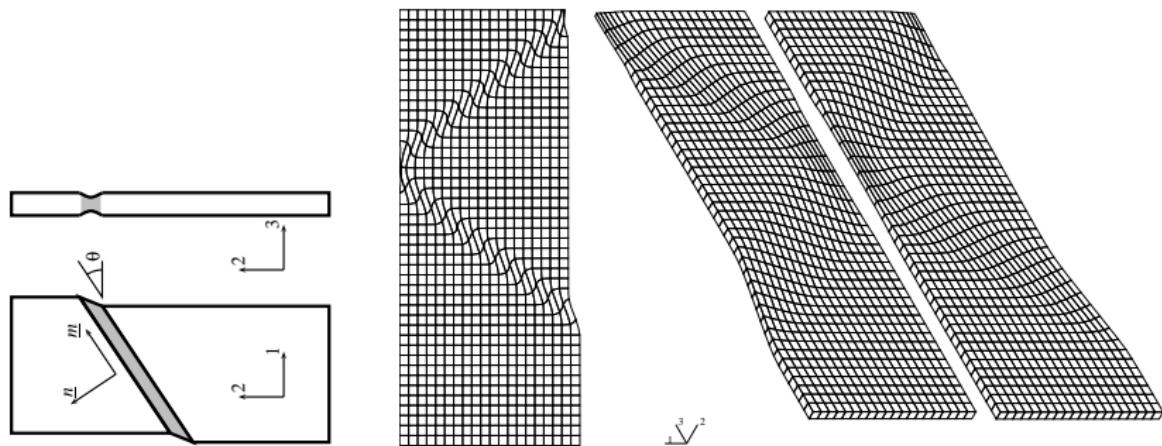
dimension of the problem	critical modulus H^{cr}	orientation $\underline{\underline{n}}$
plane stress	0	$n_1^2 = \frac{P_1}{P_1 - P_2}$ $n_2^2 = 1 - n_1^2$
3D	$-EP_k^2$	$n_i^2 = \frac{P_i + \nu P_k}{P_i - P_j}$ $n_k = 0, n_j^2 = 1 - n_i^2$
plane strain	0	$n_1^2 = n_2^2 = \frac{1}{2}$

The P_i are the eigenvalues of the plastic flow direction $\tilde{\underline{\underline{P}}} (\dot{\underline{\underline{\varepsilon}}}^p = \lambda \tilde{\underline{\underline{P}}})$.

Why is the 3D case less restrictive than the 2D case?

Tensile test, von Mises plasticity: $H_{\text{plane stress}}^{\text{cr}} = 0$, $H_{\text{3D}}^{\text{cr}} = -\frac{E}{4}$

Answer: the modes found under plane stress are incompatible with respect to the third direction...



Example: the elliptic criterion

- Plasticity criterion

$$g(\tilde{\boldsymbol{\sigma}}) = \sigma_* - R, \quad \sigma_*^2 = \frac{3}{2} C \tilde{\boldsymbol{\sigma}}^{\text{dev}} : \tilde{\boldsymbol{\sigma}}^{\text{dev}} + F(\text{trace } \tilde{\boldsymbol{\sigma}})^2$$

- Normality rule

$$\tilde{\mathbf{N}} = \frac{\partial g}{\partial \tilde{\boldsymbol{\sigma}}} = \frac{1}{\sigma_*} \left(\frac{3}{2} C \tilde{\boldsymbol{\sigma}}^{\text{dev}} + F(\text{trace } \tilde{\boldsymbol{\sigma}}) \tilde{\mathbf{1}} \right)$$

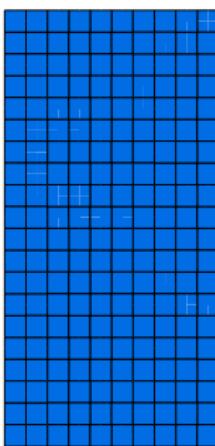
- Tensile test

$$[P] = \frac{\text{signe } \sigma_2}{\sqrt{C+F}} \begin{bmatrix} F - \frac{C}{2} & 0 & 0 \\ 0 & C+F & 0 \\ 0 & 0 & F - \frac{C}{2} \end{bmatrix}$$

Example: the elliptic criterion

Bifurcation analysis under plane stress conditions:

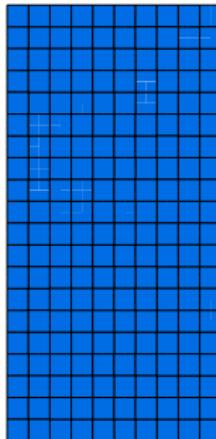
$$H^{cr} = 0, \quad n_1^2 = \frac{1}{3} \left(1 - \frac{2F}{C} \right)$$



$F = 0$ von Mises

$\theta = 35.26^\circ$ with
respect to the

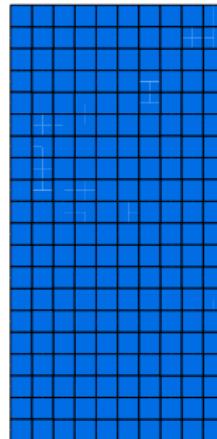
Stability / ellipticity of the boundary value problem



$$F = \frac{C}{4}$$

$\theta = 25^\circ$ with
respect to the

horizontal axis



$$F = \frac{C}{2}$$

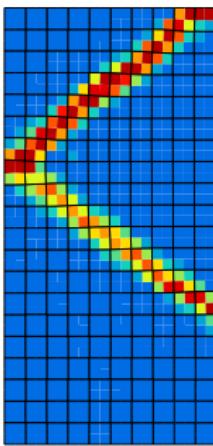
$\theta = 0^\circ$ with respect
to the horizontal

axis

Example: the elliptic criterion

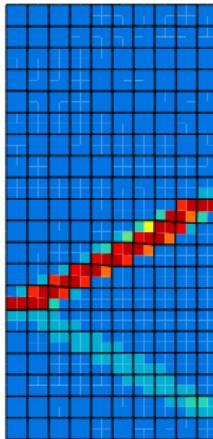
Bifurcation analysis under plane stress conditions:

$$H^{cr} = 0, \quad n_1^2 = \frac{1}{3} \left(1 - \frac{2F}{C} \right)$$



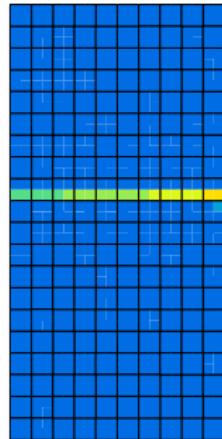
$F = 0$ von Mises

$\theta = 35.26^\circ$ with
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$$F = \frac{C}{4}$$

$\theta = 25^\circ$ with
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$$F = \frac{C}{2}$$

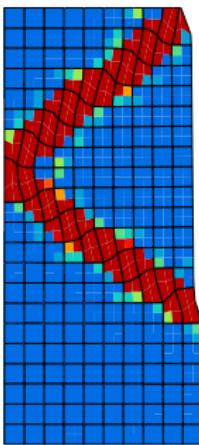
$\theta = 0^\circ$ with respect
to the horizontal
axis

Stability / bifurcations of the boundary value problem

Example: the elliptic criterion

Bifurcation analysis under plane stress conditions:

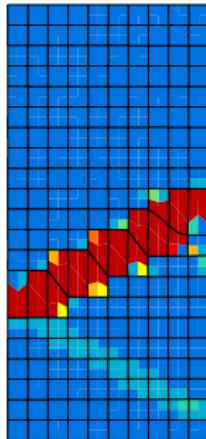
$$H^{cr} = 0, \quad n_1^2 = \frac{1}{3} \left(1 - \frac{2F}{C} \right)$$



$F = 0$ von Mises

$\theta = 35.26^\circ$ with
respect to the

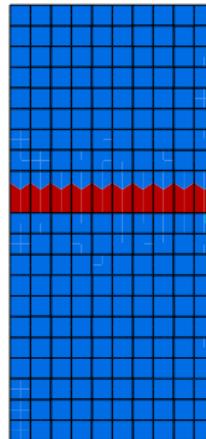
Stability / ellipticity axis of the boundary value problem



$$F = \frac{C}{4}$$

$\theta = 25^\circ$ with
respect to the

horizontal axis

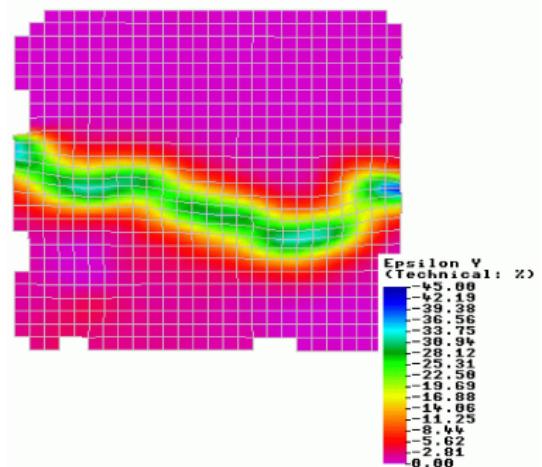
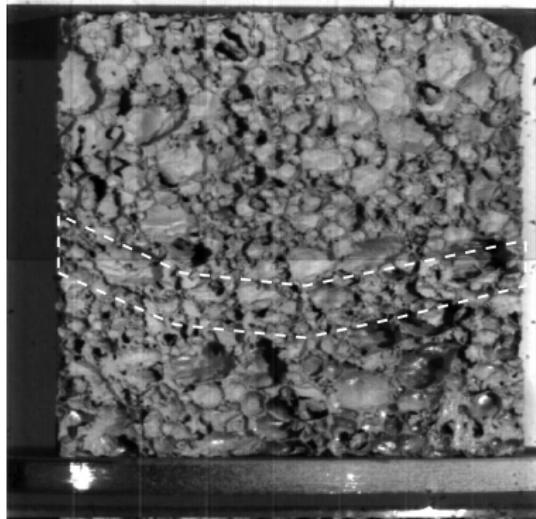


$$F = \frac{C}{2}$$

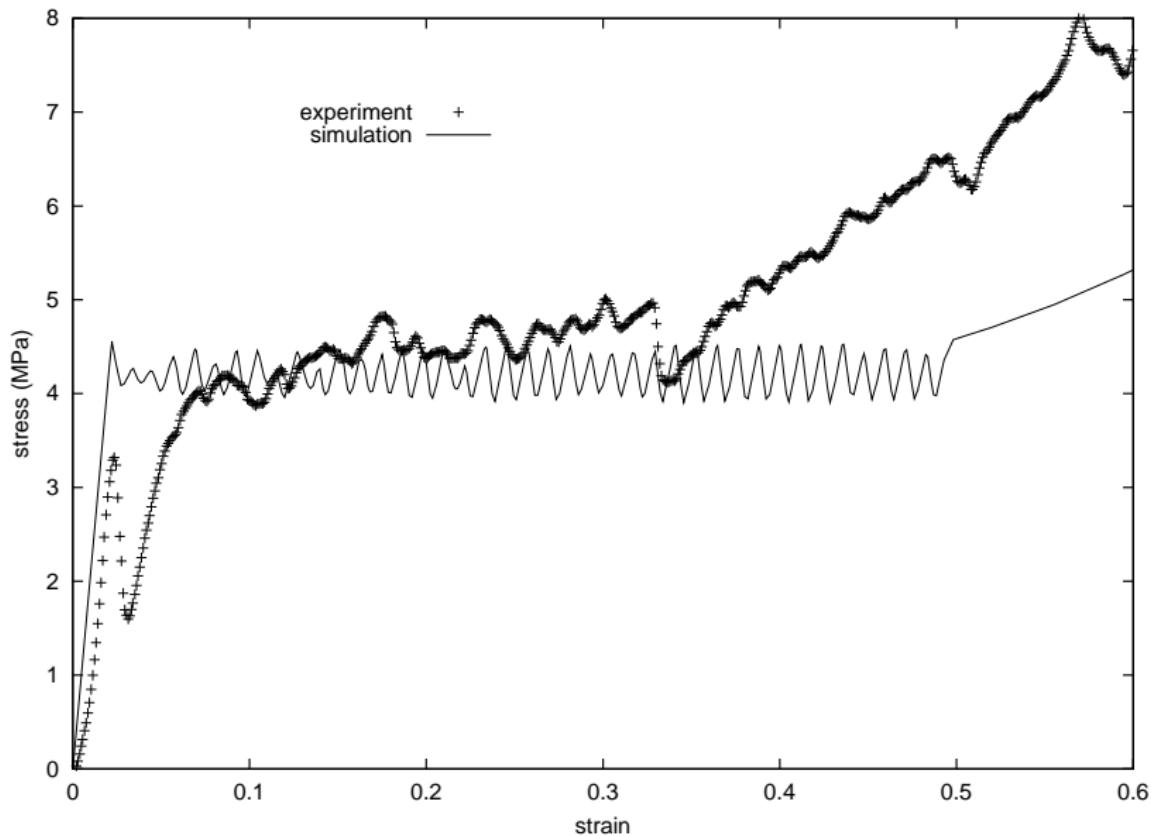
$\theta = 0^\circ$ with respect
to the horizontal

axis

Example: compression of an aluminium foam



Example: compression of an aluminium foam

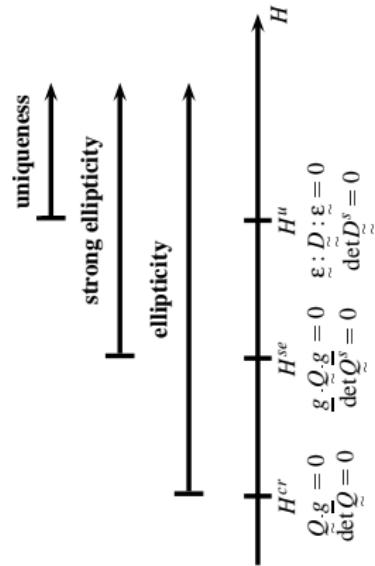


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Strain localisation criteria

- An open question!
- In 2D, Rice's criterion is plausible (test it in compressible and non associative plasticity!)
- In 3D, the criterion for occurrence of compatible modes is more plausible (strong ellipticity; Rice's criterion is too optimistic!)



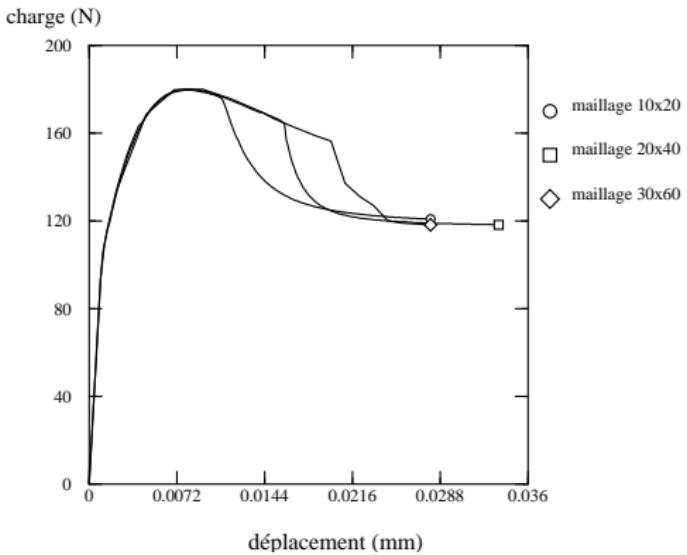
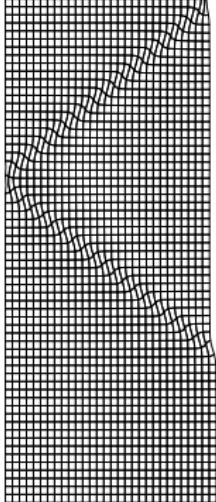
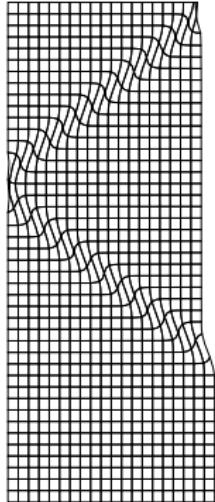
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Spurious / pathological mesh-dependence of finite element results



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Mécanique des milieux continus généralisés

Principe de l'*action locale*: seule compte l'*histoire d'un voisinage arbitrairement petit de la particule X*

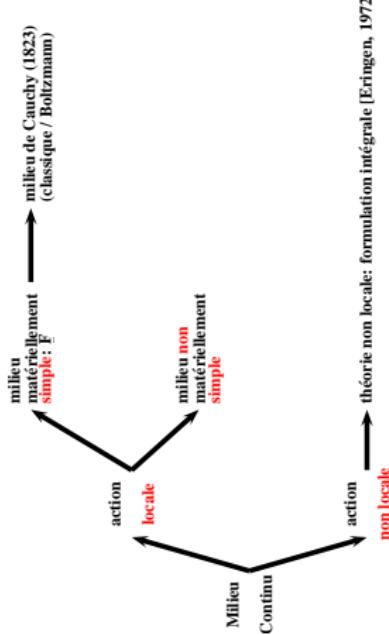
[Truesdell, Toupin, 1960] [Truesdell, Noll, 1965]



Mécanique des milieux continus généralisés

Principe de l'**action locale**: seule compte l'histoire d'un voisinage arbitrairement petit de la particule **X**

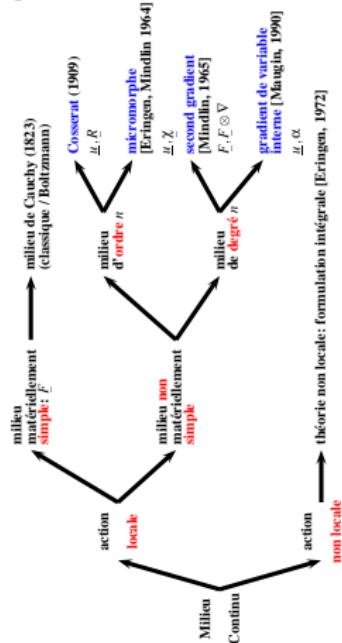
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Mécanique des milieux continus généralisés

Principe de l'**action locale**: seule compte l'histoire d'un voisinage arbitrairement petit de la particule X

[Truesdell, Toupin, 1960] [Truesdell, Noll, 1965]



Le milieu micromorphe: cinématique

- Degrés de liberté et raffinement de la théorie

$$DOF := \{\underline{\mathbf{u}}, \quad \tilde{\boldsymbol{\chi}}\}$$

$$\tilde{\boldsymbol{\chi}} = \tilde{\boldsymbol{\chi}}^s + \tilde{\boldsymbol{\chi}}^a$$

$$MODEL := \{\underline{\mathbf{u}} \otimes \nabla, \quad \tilde{\boldsymbol{\chi}} \otimes \nabla\}$$

Cas particuliers:

- ★ Milieu de Cosserat
- ★ Milieu *microstrain*
- ★ Théorie du second gradient

$$\begin{aligned}\tilde{\boldsymbol{\chi}} &\equiv \tilde{\boldsymbol{\chi}}^a = -\underline{\varepsilon} \cdot \underline{\Phi} \\ \tilde{\boldsymbol{\chi}} &\equiv \tilde{\boldsymbol{\chi}}^s \\ \tilde{\boldsymbol{\chi}} &\equiv \underline{\mathbf{u}} \otimes \nabla\end{aligned}$$

- Mesures de déformation

$$STRAIN := \{\underline{\varepsilon}, \quad \tilde{\boldsymbol{\epsilon}} := \underline{\mathbf{u}} \otimes \nabla - \tilde{\boldsymbol{\chi}}, \quad \tilde{\mathbf{K}} := \tilde{\boldsymbol{\chi}} \otimes \nabla\}$$

Le milieu micromorphe: statique

- Méthode des puissances virtuelles [Germain, 1973]

$$\mathcal{P}^{(i)} = \int_{\mathcal{D}} p^{(i)} \, dV, \quad \mathcal{P}^{(c)} = \int_{\partial\mathcal{D}} p^{(c)} \, dS$$

$$p^{(i)} = \underline{\sigma} : \dot{\underline{\xi}} + \underline{s} : \dot{\underline{e}} + \underline{\underline{S}} : \dot{\underline{\underline{K}}}$$

$$p^{(c)} = \underline{\underline{t}} \cdot \dot{\underline{\underline{u}}} + \underline{\underline{M}} : \dot{\underline{\underline{\chi}}}$$

- Equations de champ

★ Quantité de mouvement

$$(\underline{\sigma} + \underline{s}) \cdot \nabla = 0$$

★ Moment cinétique généralisé

$$\underline{\underline{S}} \cdot \nabla + \underline{s} = 0$$

- Conditions aux limites

$$\underline{\underline{t}} = (\underline{\sigma} + \underline{s}) \cdot \underline{\underline{n}}, \quad \underline{\underline{M}} = \underline{\underline{S}} \cdot \underline{\underline{n}}$$

Le milieu micromorphe: thermodynamique

- Equation locale de l'énergie

$$\rho \dot{\epsilon} = p^{(i)} - \underline{\mathbf{q}} \cdot \nabla + r$$

- Second principe et inégalité de Clausius–Duhem

$$\rho \dot{\eta} + \left(\frac{\mathbf{q}}{T} \right) \cdot \nabla - \frac{r}{T} \geq 0$$

$$\rho (T \dot{\eta} - \dot{\epsilon}) + p^{(i)} - \frac{\mathbf{q}}{T} \cdot (\nabla T) \geq 0$$

- Variables d'état et énergie libre de Helmholtz

$$\underline{\varepsilon} = \underline{\varepsilon}^e + \underline{\varepsilon}^p, \quad \underline{\mathbf{e}} = \underline{\mathbf{e}}^e + \underline{\mathbf{e}}^p, \quad \underline{\mathbf{K}} = \underline{\mathbf{K}}^e + \underline{\mathbf{K}}^p$$

$$\mathcal{Z} := \{T, \quad \underline{\varepsilon}^e, \quad \underline{\mathbf{e}}^e, \quad \underline{\mathbf{K}}^e, \quad \alpha\}$$

$$\Psi = \epsilon - T\eta$$

Le milieu micromorphe: potentiel de dissipation

- Exploitation du second principe à la Coleman–Noll
 - ★ Lois d'état

$$\eta = -\frac{\partial \Psi}{\partial T}, \quad \tilde{\boldsymbol{\sigma}} = \rho \frac{\partial \Psi}{\partial \tilde{\boldsymbol{\varepsilon}}^e}, \quad \tilde{\mathbf{s}} = \rho \frac{\partial \Psi}{\partial \tilde{\mathbf{e}}^e}, \quad \tilde{\mathbf{S}} = \rho \frac{\partial \Psi}{\partial \tilde{\mathbf{K}}^e}, \quad R = \rho \frac{\partial \Psi}{\partial \alpha}$$

- ★ Lois d'évolution
dissipation résiduelle

$$D = \tilde{\boldsymbol{\sigma}} : \dot{\tilde{\boldsymbol{\varepsilon}}}^p + \tilde{\mathbf{s}} : \dot{\tilde{\mathbf{e}}}^p + \tilde{\mathbf{S}} : \dot{\tilde{\mathbf{K}}}^p - R \dot{\alpha}$$

potentiel de dissipation

$$\Omega(\tilde{\boldsymbol{\sigma}}, \tilde{\mathbf{s}}, \tilde{\mathbf{S}}, R)$$

$$\dot{\tilde{\boldsymbol{\varepsilon}}}^p = \frac{\partial \Omega}{\partial \tilde{\boldsymbol{\sigma}}}, \quad \dot{\tilde{\mathbf{e}}}^p = \frac{\partial \Omega}{\partial \tilde{\mathbf{s}}}, \quad \dot{\tilde{\mathbf{K}}}^p = \frac{\partial \Omega}{\partial \tilde{\mathbf{S}}}, \quad \dot{\alpha} = -\frac{\partial \Omega}{\partial R}$$

Le modèle *microstrain*

Degrés de liberté ($\underline{\mathbf{u}}, \tilde{\boldsymbol{\chi}}^s$)

Mesures de déformation :

$$(\tilde{\boldsymbol{\varepsilon}}, \tilde{\boldsymbol{\varepsilon}} - \tilde{\boldsymbol{\chi}}^s, \tilde{\boldsymbol{\chi}}^s \otimes \nabla)$$

$$S_{ijk,k} + s_{ij} = 0$$

$$A\chi_{ij,kk}^s + b(\varepsilon_{ij} - \chi_{ij}^s) = 0$$

$$\underline{\sigma} = \tilde{\mathbf{C}} : (\tilde{\boldsymbol{\varepsilon}} - \tilde{\boldsymbol{\varepsilon}}^p)$$

$$\varepsilon_{ij} = \chi_{ij}^s - l^2 \Delta \chi_{ij}^s$$

$$\underline{\mathbf{s}} = b(\tilde{\boldsymbol{\varepsilon}} - \tilde{\boldsymbol{\chi}}^s)$$

Lien avec “implicit

$$\underline{\mathbf{S}} = A\tilde{\boldsymbol{\chi}}^s \otimes \nabla$$

gradient-enhanced
elastoplasticity models”

[Engelen *et al.*, 2003]

$$\operatorname{div}(\underline{\sigma} + \underline{\mathbf{s}}) = 0$$

Conditions aux limites :

$$\operatorname{div} \underline{\mathbf{S}} + \underline{\mathbf{s}} = 0$$

$$(u_i, \chi_{ij}^s) \text{ or } (\sigma_{ij} + s_{ij}) n_j, S_{ijk} n_k$$

Programmation en éléments finis (1)

Exemple: milieu micromorphe 2D

$$[\text{DOF}] = [U_1 \ U_2 \ X_{11} \ X_{22} \ X_{12} \ X_{21}]^T$$

$$\begin{aligned} [\text{STRAIN}] = & [\varepsilon_{11} \ \varepsilon_{22} \ \varepsilon_{33} \ \varepsilon_{12} \ e_{11} \ e_{22} \ e_{33} \ e_{12} \ e_{21} \\ & K_{111} \ K_{112} \ K_{121} \ K_{122} \ K_{211} \ K_{212} \ K_{221} \ K_{222}]^T \end{aligned}$$

$$[\text{STRAIN}] = [B] \ [\text{DOF}]$$

$$\begin{aligned} [\text{FLUX}] = & [\sigma_{11} \ \sigma_{22} \ \sigma_{33} \ \sigma_{12} \ s_{11} \ s_{22} \ s_{33} \ s_{12} \ s_{21} \\ & S_{111} \ S_{112} \ S_{121} \ S_{122} \ S_{211} \ S_{212} \ S_{221} \ S_{222}]^T \end{aligned}$$

Programmation en éléments finis (2)

$$[B] = \begin{bmatrix} \partial_{x_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \partial_{x_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2}\partial_{x_2} & \frac{1}{2}\partial_{x_1} & 0 & 0 & 0 & 0 \\ \partial_{x_1} & 0 & -1 & 0 & 0 & 0 \\ 0 & \partial_{x_2} & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \partial_{x_2} & 0 & 0 & 0 & -1 & 0 \\ 0 & \partial_{x_1} & 0 & 0 & 0 & -1 \\ 0 & 0 & \partial_{x_1} & 0 & 0 & 0 \\ 0 & 0 & \partial_{x_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \partial_{x_1} & 0 \\ 0 & 0 & 0 & 0 & \partial_{x_2} & 0 \\ 0 & 0 & 0 & \partial_{x_1} & 0 & 0 \\ 0 & 0 & 0 & \partial_{x_2} & 0 & 0 \end{bmatrix} = [A] \begin{bmatrix} S_{111} \\ S_{112} \\ S_{121} \\ S_{122} \\ S_{211} \\ S_{212} \\ S_{221} \\ S_{222} \end{bmatrix} = [A] \begin{bmatrix} K_{111} \\ K_{112} \\ K_{121} \\ K_{122} \\ K_{211} \\ K_{212} \\ K_{221} \\ K_{222} \end{bmatrix}$$

Programmation en éléments finis (3)

matrice d'élasticité généralisée [Mindlin, 1964]

$$\begin{bmatrix} AA & 0 & 0 & A_{1,4,5} & 0 & A_{2,5,8} & A_{1,2,3} & 0 \\ 0 & A_{3,10,14} & A_{2,11,13} & 0 & A_{1,11,15} & 0 & 0 & A_{1,2,3} \\ 0 & A_{2,11,13} & A_{8,10,15} & 0 & A_{5,11,14} & 0 & 0 & A_{2,5,8} \\ A_{1,4,5} & 0 & 0 & A_{4,10,13} & 0 & A_{5,11,14} & A_{1,11,15} & 0 \\ 0 & A_{1,11,15} & A_{5,11,14} & 0 & A_{4,10,13} & 0 & 0 & A_{1,4,5} \\ A_{2,5,8} & 0 & 0 & A_{5,11,14} & 0 & A_{8,10,15} & A_{2,11,13} & 0 \\ A_{1,2,3} & 0 & 0 & A_{1,11,15} & 0 & A_{2,11,13} & A_{3,10,14} & 0 \\ 0 & A_{1,2,3} & A_{2,5,8} & 0 & A_{1,4,5} & 0 & 0 & AA \end{bmatrix}$$

avec

$$AA = 2A_1 + 2A_2 + A_3 + A_4 + 2A_5 + A_8 + A_{10} + 2A_{11} + A_{13} + A_{14} + A_{15}$$

$$\text{et } A_{i,j,k} = A_i + A_j + A_k$$

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Bandes de localisation de largeur finie

Cas des bandes horizontales obtenues à l'aide d'un critère elliptique

$$\sigma_{eq} = \sqrt{C + F} |\sigma_{22}|, \quad \dot{\varepsilon}_{22}^p = \dot{p} \sqrt{C + F}, \quad \dot{p} = \frac{2\mu\sqrt{C + F}\dot{\varepsilon}_{22}}{2\mu(C + F) + H}$$

$$\begin{cases} (\sigma_{22} + s_{22})_{,2} = 0, \quad S_{222,2} + s_{22} = 0 \\ \sigma_{22} = 2\mu(\varepsilon_{22} - \varepsilon_{22}^p) = \frac{2\mu}{2\mu(C+F)+H}(H\varepsilon_{22} + R_0\sqrt{C+F}) \\ s_{22} = 2\mu(\varepsilon_{22} - \chi_{22}), \quad S_{222} = A\chi_{22,2} \end{cases}$$

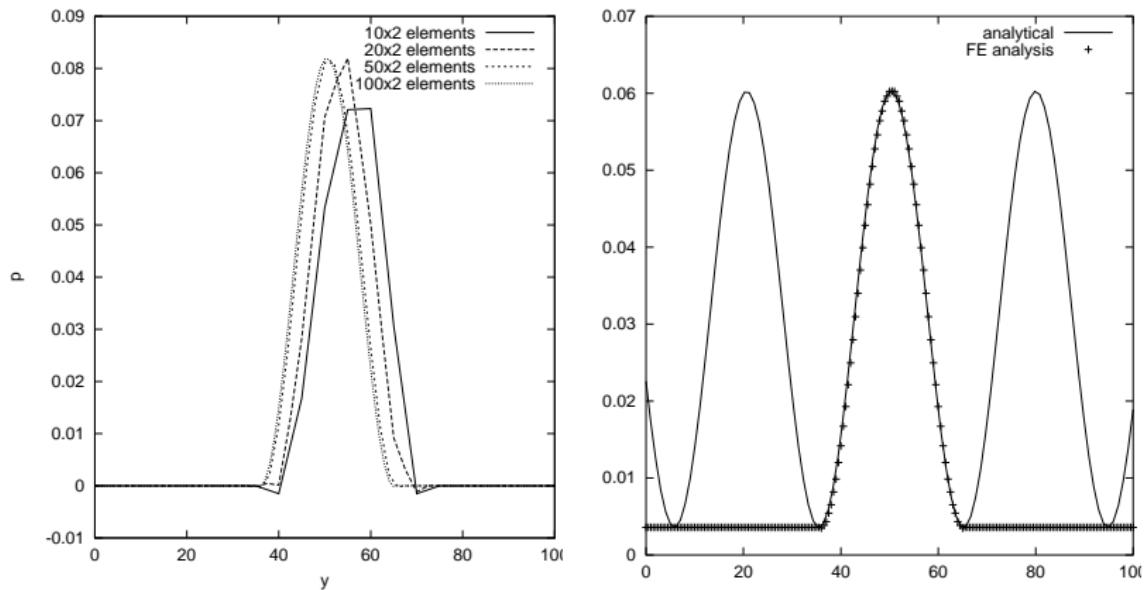
Finalement

$$\chi_{22,222} - \frac{2\mu\bar{H}}{A(\bar{H} + 2\mu)} \chi_{22,2} = 0, \quad \bar{H} = \frac{2\mu H}{2\mu(C + F) + H}$$

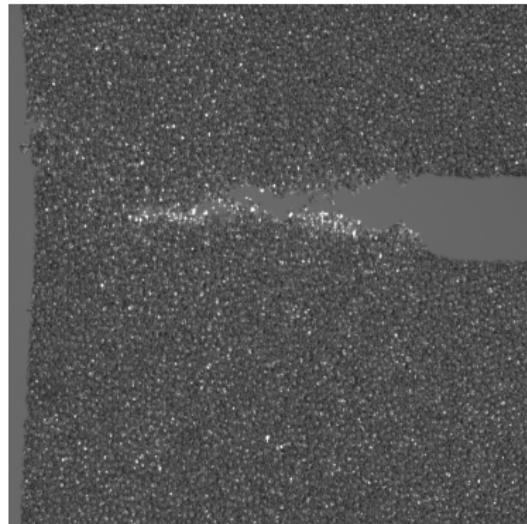
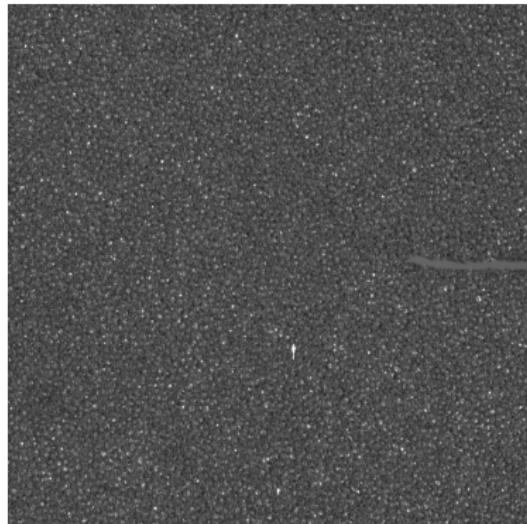
Longueur d'onde

$$\frac{1}{\omega} = \sqrt{\frac{A(\bar{H} + 2\mu)}{2\mu|\bar{H}|}}$$

Simulations par éléments finis à l'aide du milieu micromorphe

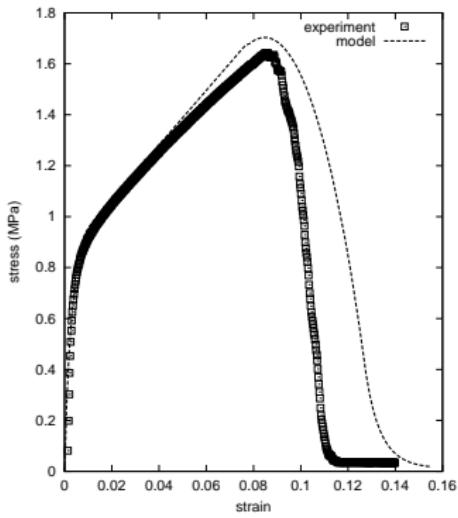


Comportement et rupture des mousses de nickel



longueur de fissure : 10 mm, taille de cellule : 0.5mm

Comportement et rupture des mousses de nickel



Critère simple(iste) de rupture

$$p = p_{crit} = 0.08$$

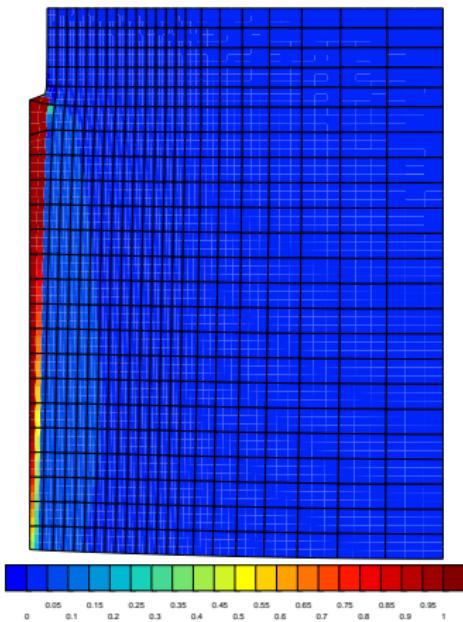
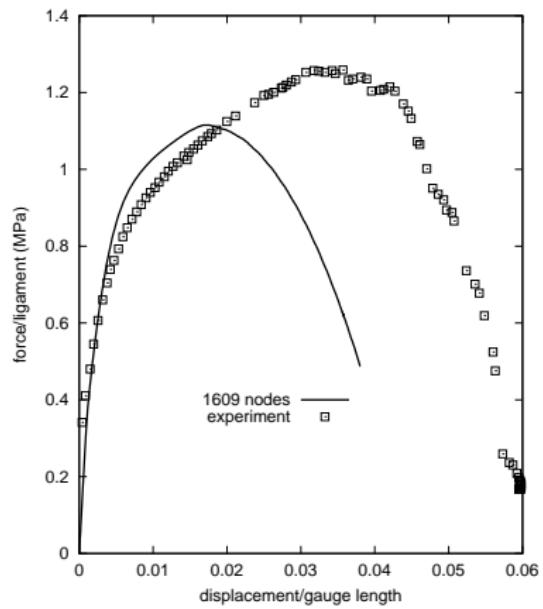
dispersion limitée en p_{crit}

adoucissement explicite pour
 $p > p_{crit}$

$$R = R(p > p_{crit})$$

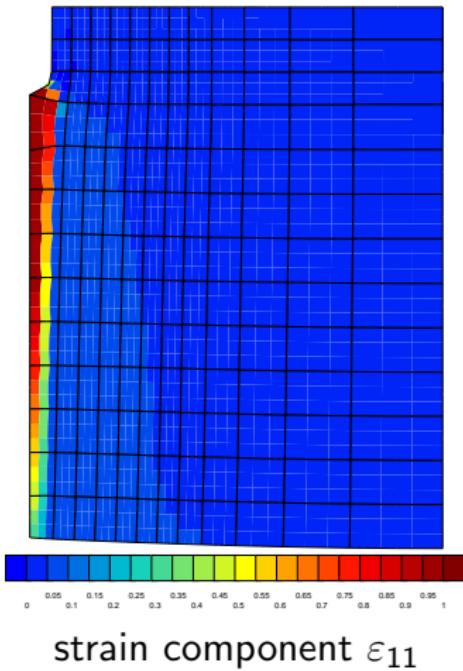
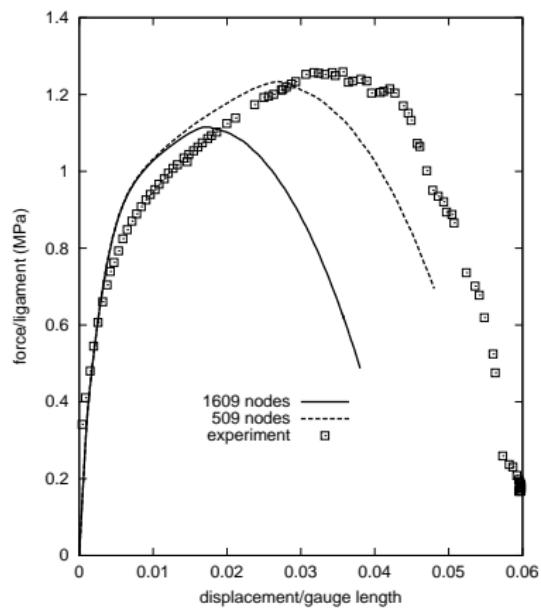
Direction de traction RD

Traction d'une plaque fissurée : cas classique

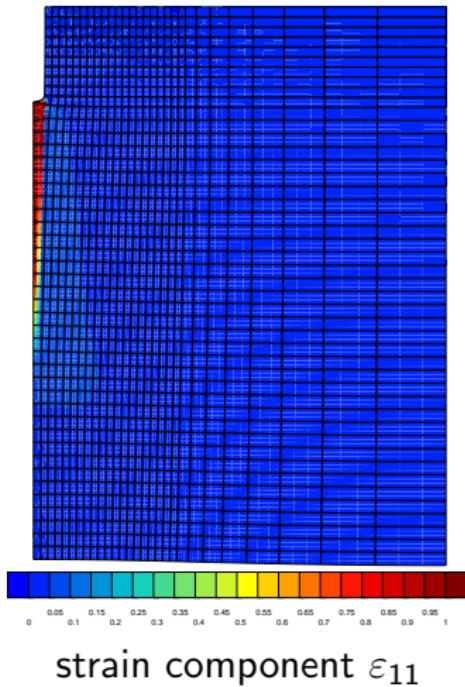
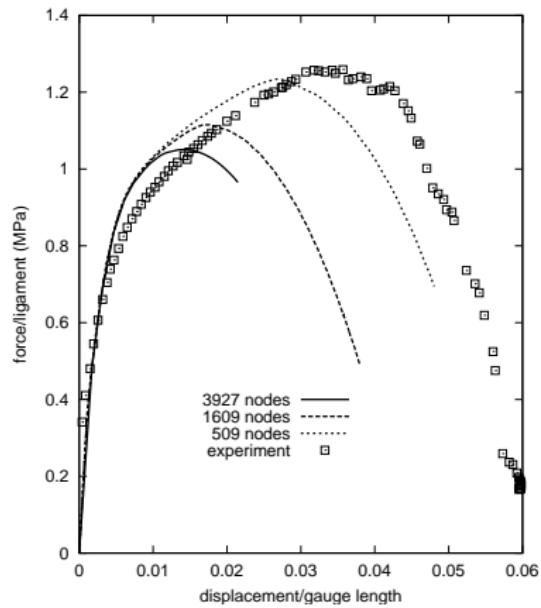


strain component ε_{11}

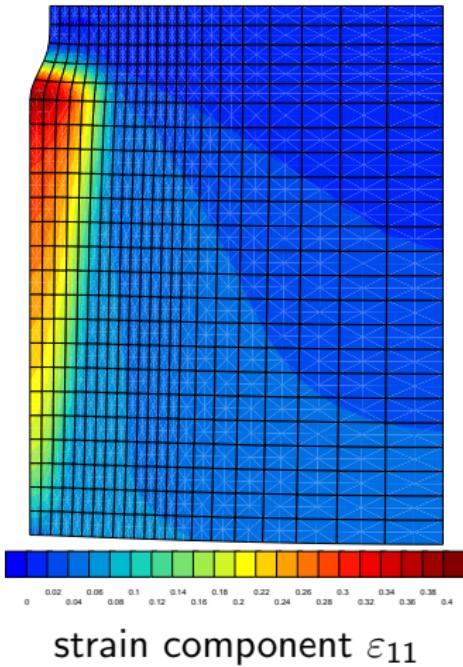
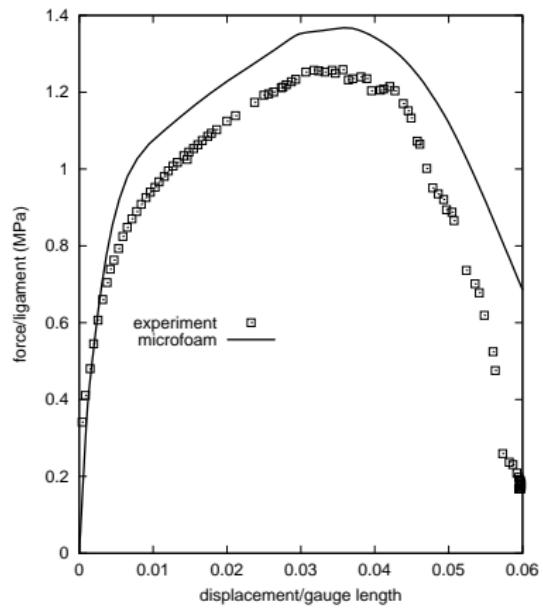
Traction d'une plaque fissurée : cas classique



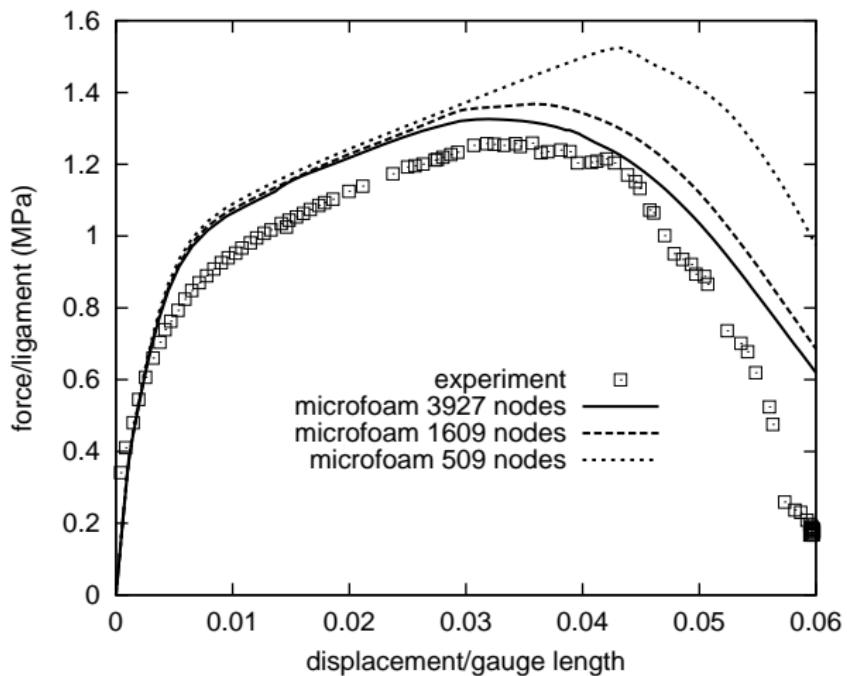
Traction d'une plaque fissurée : cas classique



Traction d'une plaque fissurée : cas micromorphe

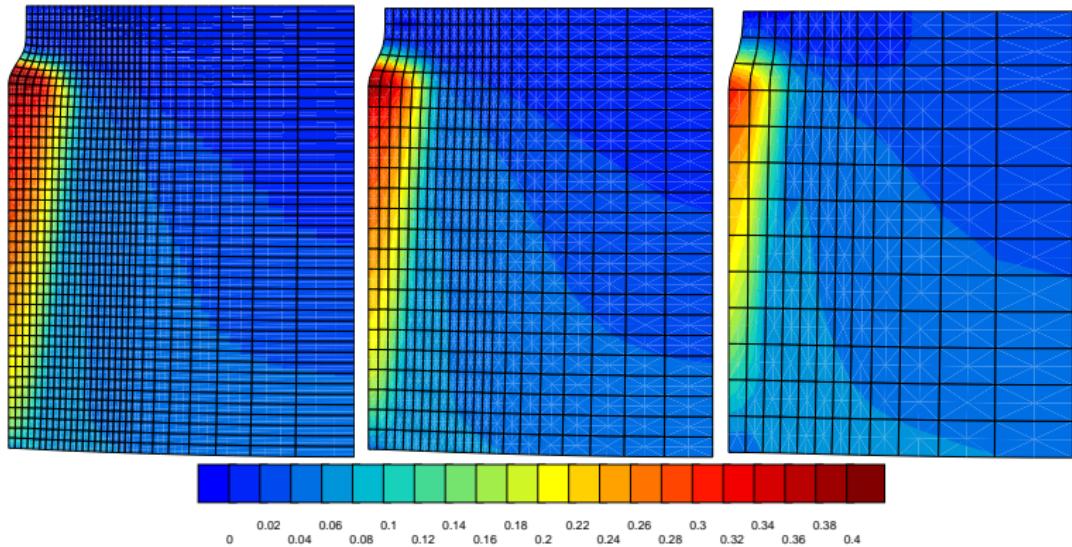


Traction d'une plaque fissurée : cas micromorphe

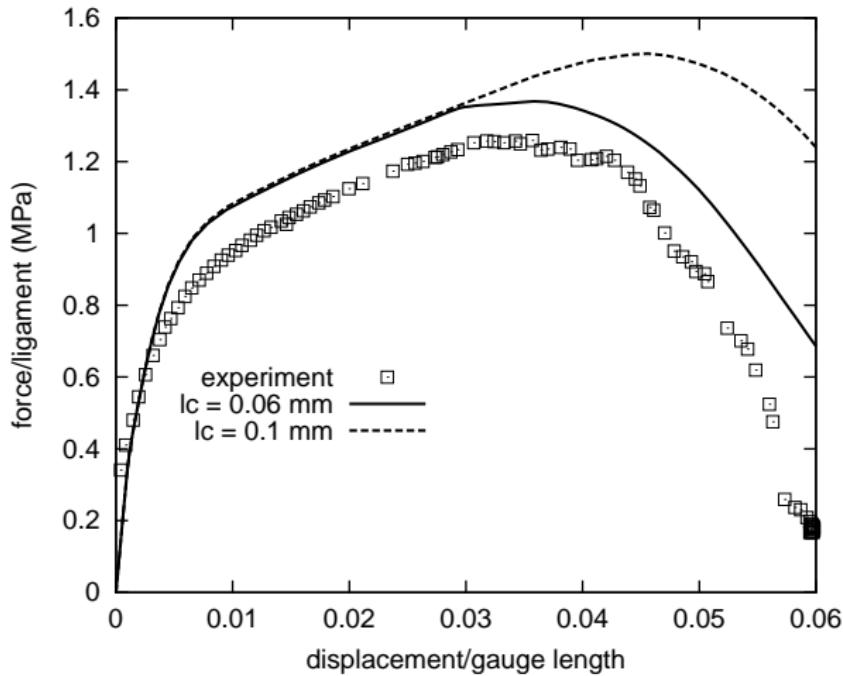


influence of mesh size on the overall curves

Traction d'une plaque fissurée : cas micromorphe



Traction d'une plaque fissurée : cas micromorphe



Influence of the characteristic length on fracture : $l_c^2 = a_8 / \mu$ with
 $\mu = 167 \text{ MPa}$