Material theory

Plan

1. Closure problem in continuum mechanics and constitutive theory
2. Elastic materials
   - Principle of material frame indifference
   - Material symmetry group
   - A definition of fluids and solids
3. Continuum thermodynamics
   - Energy
   - Entropy
4. Hyperelastic materials
   - State laws
   - Internal constraints
5. Viscosity

Closure problem

- universal laws : conservation laws
  find the fields $(\rho, \underset{\sim}{\boldsymbol{\sigma}}, \underline{\mathbf{v}})$ such that

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \underline{\mathbf{v}}) &= 0 \\
\operatorname{div} \underset{\sim}{\boldsymbol{\sigma}} + \rho \underline{\mathbf{f}} &= \rho \dot{\underline{\mathbf{v}}}
\end{aligned}
$$

  i.e. 1+3=4 equations
  number of unknowns : 1 (ρ) + 3 (v_i) + 6 (σ_{ij}) = 10
  6 equations are missing...

Closure problem

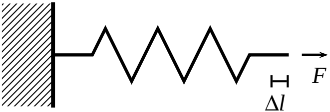
- universal laws : conservation laws
  find the fields $(\rho, \underset{\sim}{\boldsymbol{\sigma}}, \underline{\mathbf{v}})$ such that

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \underline{\mathbf{v}}) &= 0 \\
\operatorname{div} \underset{\sim}{\boldsymbol{\sigma}} + \rho \underline{\mathbf{f}} &= \rho \dot{\underline{\mathbf{v}}}
\end{aligned}
$$

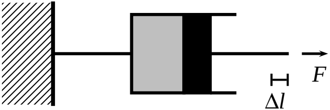
  i.e. 1+3=4 equations
  number of unknowns : $1\ (\rho) + 3\ (v_i) + 6\ (\sigma_{ij}) = 10$
  6 equations are missing...

- material-specific relations: **constitutive equations**
  material behavior laws
  6 relations $\sigma_{ij} \longleftrightarrow L_{ij}, F_{ij}$...

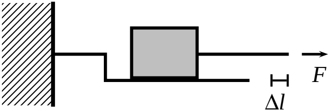
Classification of material behavior



Elasticity $F = k\Delta l$



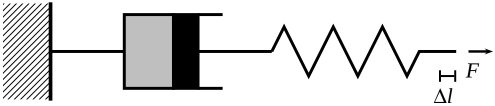
Viscosity $F = \eta\Delta\dot{l}$



Plasticity $F = F_0 \mathrm{sign}(\Delta\dot{l}) \quad \text{if} \quad \Delta\dot{l} \neq 0$

Constitutive functional

- Example : Maxwell **viscoelasticity**



$$\Delta \dot{l}(t) = \Delta \dot{l}_{\text{piston}}(t) + \Delta \dot{l}_{\text{ressort}}(t) = \frac{F(t)}{\eta} + \frac{\dot{F}(t)}{k}$$

$$F(t) = k \int_{-\infty}^{t} \exp(-\frac{k}{\eta}(t-s))\, \Delta \dot{l}\, ds$$

  the current material response depend on the total **history** of deformation

- Extension to a 3D material body : **Constitutive functional**

$$\underset{\sim}{\boldsymbol{\sigma}}(\underline{\mathbf{x}}\,, t) = \underset{0 \leq s \leq t, \underline{\mathbf{Y}} \in \Omega_0}{\mathcal{F}} \left(\Phi(\underline{\mathbf{Y}}\,, s)\right)$$

First simplifications

- principle of **determinism**

$$\underset{\sim}{\boldsymbol{\sigma}}(\underline{\mathbf{x}}\,,t) = \underset{0\leq s\leq t,\underline{\mathbf{Y}}\,\in\Omega_0}{\mathcal{F}}(\Phi(\underline{\mathbf{Y}}\,,s))$$

  non local theory

- principle of **local action**

$$\underset{\sim}{\boldsymbol{\sigma}}(\underline{\mathbf{x}}\,,t) = \underset{0\leq s\leq t,n>0}{\mathcal{F}}\left(\Phi(\underline{\mathbf{X}}\,,s), \frac{\partial^n\Phi}{\partial\underline{\mathbf{X}}^{\,n}}(\underline{\mathbf{X}}\,,s)\right)$$

- **simple material** : first gradient theory

$$\underset{\sim}{\boldsymbol{\sigma}}(\underline{\mathbf{x}}\,,t) = \underset{0\leq s\leq t}{\mathcal{F}}\left(\Phi(\underline{\mathbf{X}}\,,s), \underset{\sim}{\mathbf{F}}(\underline{\mathbf{X}}\,,s)\right)$$

Plan

1. Closure problem in continuum mechanics and constitutive theory
2. Elastic materials
   - Principle of material frame indifference
   - Material symmetry group
   - A definition of fluids and solids
3. Continuum thermodynamics
   - Energy
   - Entropy
4. Hyperelastic materials
   - State laws
   - Internal constraints
5. Viscosity

Form of the constitutive law

- Constitutive functional

$$\underset{\sim}{\boldsymbol{\sigma}}(\underline{\mathbf{x}}\,,t) = \underset{0\leq s\leq t}{\mathcal{F}}\left(\Phi(\underline{\mathbf{X}}\,,s), \underset{\sim}{\mathbf{F}}(\underline{\mathbf{X}}\,,s)\right)$$

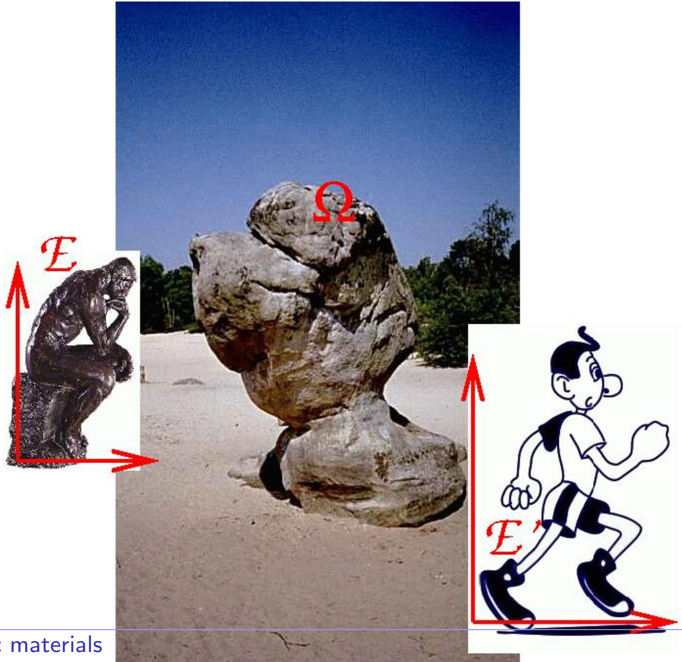
- **Elastic** behavior

$$\underset{\sim}{\boldsymbol{\sigma}} = \mathcal{F}_{\Omega_0}(\underset{\sim}{\mathbf{F}})$$

Plan

1. Closure problem in continuum mechanics and constitutive theory
2. Elastic materials
   - Principle of material frame indifference
   - Material symmetry group
   - A definition of fluids and solids
3. Continuum thermodynamics
   - Energy
   - Entropy
4. Hyperelastic materials
   - State laws
   - Internal constraints
5. Viscosity

Change of observer



Transformation rules under change of observer

observer $(\mathcal{E}, E)$, another observer $(\mathcal{E}', E')$

- Euclidean transformation

$$\underline{\mathbf{x}}' = \underset{\sim}{\mathbf{Q}}(t).\underline{\mathbf{x}} + \underline{\mathbf{c}}(t), \quad t' = t - t_0$$

  convention : $\underset{\sim}{\mathbf{Q}}(t_0) = \underset{\sim}{\mathbf{1}}$ $(\underline{\mathbf{X}}' = \underline{\mathbf{X}})$, $\quad \underline{\mathbf{c}}(t_0) = 0$

- Transformation rules for mechanical quantities?
  - deformation gradient
  - strain tensors
  - velocity gradient
  - stress tensors

Transformation rules under change of observer

observer $(\mathcal{E}, E)$, another observer $(\mathcal{E}', E')$

- Euclidean transformation (trafo)

$$\underline{\mathbf{x}}' = \underset{\sim}{\mathbf{Q}}(t).\underline{\mathbf{x}} + \underline{\mathbf{c}}(t), \quad t' = t - t_0$$

convention : $\underset{\sim}{\mathbf{Q}}(t_0) = \underset{\sim}{\mathbf{1}}$ $(\underline{\mathbf{X}}' = \underline{\mathbf{X}})$, $\quad \underline{\mathbf{c}}(t_0) = 0$

- Objective and invariant tensors
  - ⋆ deformation gradient $\quad \underline{\mathbf{dx}}' = \underset{\sim}{\mathbf{Q}}.\underline{\mathbf{dx}} \quad \underset{\sim}{\mathbf{F}}' = \underset{\sim}{\mathbf{Q}}.\underset{\sim}{\mathbf{F}}$
  - ⋆ Cauchy–Green tensors

$$\underset{\sim}{\mathbf{C}}' = \underset{\sim}{\mathbf{C}}, \quad \underset{\sim}{\mathbf{B}}' = \underset{\sim}{\mathbf{Q}}.\underset{\sim}{\mathbf{B}}.\underset{\sim}{\mathbf{Q}}^T$$

  - ⋆ velocity gradient

$$\underset{\sim}{\mathbf{L}}' = \underset{\sim}{\mathbf{Q}}.\underset{\sim}{\mathbf{L}}.\underset{\sim}{\mathbf{Q}}^T + \underset{\sim}{\dot{\mathbf{Q}}}.\underset{\sim}{\mathbf{Q}}^T, \quad \underset{\sim}{\mathbf{D}}' = \underset{\sim}{\mathbf{Q}}.\underset{\sim}{\mathbf{D}}.\underset{\sim}{\mathbf{Q}}^T, \quad \underset{\sim}{\mathbf{W}}' = \underset{\sim}{\mathbf{Q}}.\underset{\sim}{\mathbf{W}}.\underset{\sim}{\mathbf{Q}}^T + \underset{\sim}{\dot{\mathbf{Q}}}.\underset{\sim}{\mathbf{Q}}^T$$

  - ⋆ stress tensor $\underline{\mathbf{t}}' = \underset{\sim}{\mathbf{Q}}.\underline{\mathbf{t}}$, $\underline{\mathbf{df}}' = \underset{\sim}{\boldsymbol{\sigma}}'.\underline{\mathbf{ds}}'$, $\underset{\sim}{\boldsymbol{\sigma}}' = \underset{\sim}{\mathbf{Q}}.\underset{\sim}{\boldsymbol{\sigma}}.\underset{\sim}{\mathbf{Q}}^T$
    consequence : *the power density of internal forces is invariant with respect to Euclidean trafo*

$$p^{i\prime} = \underset{\sim}{\boldsymbol{\sigma}}' : \underset{\sim}{\mathbf{D}}' = \underset{\sim}{\boldsymbol{\sigma}}' : (\underset{\sim}{\mathbf{Q}}.\underset{\sim}{\mathbf{D}}.\underset{\sim}{\mathbf{Q}}^T) = (\underset{\sim}{\mathbf{Q}}^T.\underset{\sim}{\boldsymbol{\sigma}}'.\underset{\sim}{\mathbf{Q}}) : \underset{\sim}{\mathbf{D}}^\star = p^i$$

Form invariance of the constitutive law

How does the constitutive law transform under Euclidean trafo?

$$\underset{\sim}{\boldsymbol{\sigma}} = \mathcal{F}_{\Omega_0}(\underset{\sim}{\mathbf{F}})$$

$$\underset{\sim}{\boldsymbol{\sigma}}' = \mathcal{F}'_{\Omega_0}(\underset{\sim}{\mathbf{F}}')$$

$$\Longrightarrow \mathcal{F}'_{\Omega_0}(\underset{\sim}{\mathbf{F}}') = \underset{\sim}{\mathbf{Q}}.\mathcal{F}_{\Omega_0}(\underset{\sim}{\mathbf{Q}}^T.\underset{\sim}{\mathbf{F}}').\underset{\sim}{\mathbf{Q}}^T$$

Form invariance principle (also called **principle of material frame indifference** or principle of **material objectivity**) :

$$\mathcal{F}'_{\Omega_0} = \mathcal{F}_{\Omega_0}$$

the physical property (ex: stiffness) does not depend on the observer...

$$\Longrightarrow \mathcal{F}_{\Omega_0}(\underset{\sim}{\mathbf{Q}}.\underset{\sim}{\mathbf{F}}) = \underset{\sim}{\mathbf{Q}}.\mathcal{F}_{\Omega_0}(\underset{\sim}{\mathbf{F}}).\underset{\sim}{\mathbf{Q}}^T$$

Reduced form of the constitutive law

The elasticity law takes the reduced form

$$\underset{\sim}{\boldsymbol{\sigma}}(\underline{\mathbf{X}}\,,t) = \underset{\sim}{\mathbf{R}}(\underline{\mathbf{X}}\,,t).\mathcal{F}_{\Omega_0}(\underset{\sim}{\mathbf{U}}(\underline{\mathbf{X}}\,,t)).\underset{\sim}{\mathbf{R}}^T(\underline{\mathbf{X}}\,,t)$$

or, equivalently,

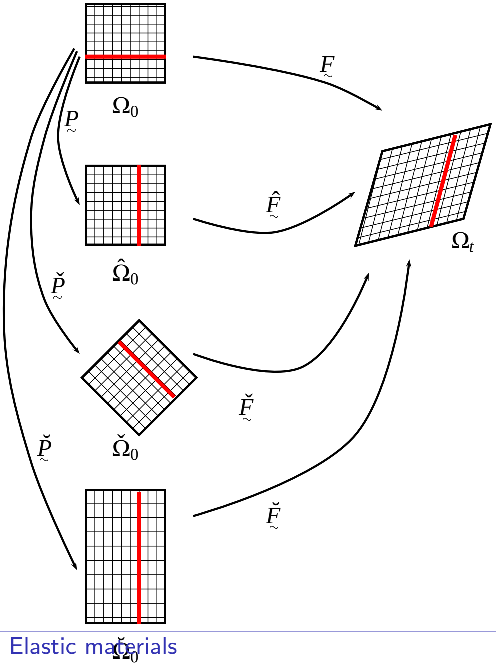
$$\underset{\sim}{\boldsymbol{\Pi}} = \mathcal{F}^{\Pi}_{\Omega_0}(\underset{\sim}{\mathbf{C}})$$

$$\underset{\sim}{\boldsymbol{\Pi}} = \mathcal{F}^{\Pi}_{\Omega_0}(\underset{\sim}{\mathbf{E}})$$

Plan

1. Closure problem in continuum mechanics and constitutive theory
2. Elastic materials
   - Principle of material frame indifference
   - Material symmetry group
   - A definition of fluids and solids
3. Continuum thermodynamics
   - Energy
   - Entropy
4. Hyperelastic materials
   - State laws
   - Internal constraints
5. Viscosity

Change of reference configuration



$\underset{\sim}{\mathbf{P}}$ is an arbitrary linear mapping (invertible but not necessarily a rotation)

$$\hat{\underset{\sim}{\mathbf{F}}} = \underset{\sim}{\mathbf{F}}.\underset{\sim}{\mathbf{P}}^{-1}$$

Change of reference configuration

$\underline{\mathbf{X}} \longrightarrow \underline{\hat{\mathbf{X}}}$

- forms of the constitutive law

$$\underset{\sim}{\boldsymbol{\sigma}}(\underline{\mathbf{x}}\,,t) = \mathcal{F}_{\Omega_0}(\underset{\sim}{\mathbf{F}}(\underline{\mathbf{X}}\,,t))$$

$$\underset{\sim}{\boldsymbol{\sigma}}(\underline{\mathbf{x}}\,,t) = \mathcal{F}_{\hat{\Omega}_0}(\underset{\sim}{\hat{\mathbf{F}}}(\underline{\hat{\mathbf{X}}}\,,t))$$

$$\mathcal{F}_{\hat{\Omega}_0}(\underset{\sim}{\hat{\mathbf{F}}}) = \mathcal{F}_{\Omega_0}(\underset{\sim}{\hat{\mathbf{F}}}.\underset{\sim}{\mathbf{P}})$$

- two reference configurations are **materially indistinguishable** if

$$\mathcal{F}_{\Omega_0} = \mathcal{F}_{\hat{\Omega}_0}$$

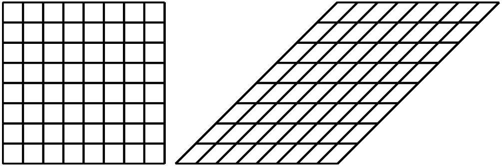
  For such a $\underset{\sim}{\mathbf{P}}$, the constitutive law must satisfy

$$\mathcal{F}_{\Omega_0}(\underset{\sim}{\mathbf{F}}) = \mathcal{F}_{\Omega_0}(\underset{\sim}{\mathbf{F}}.\underset{\sim}{\mathbf{P}})$$

  such a transformation $\underset{\sim}{\mathbf{P}}$ is called **material symmetry** (even it is not necessarily a symmetry in the mathematical sense!)

Material symmetry group

- The set of all material symmetries with respect to a given reference configuration Ω_0 has the mathematical structure of a group
$$\mathfrak{G}_{\Omega_0} \subset \mathcal{U}(E)$$
where $\mathcal{U}(E) \subset GL(E)$ is the **unimodular group** ($|\det \underset{\sim}{\mathbf{P}}| = 1$)
- Let Ω_0 and $\hat{\Omega}_0$ be two configurations of a material body, $\mathfrak{G}_{\hat{\Omega}_0}$ and $\mathfrak{G}_{\Omega_0}$ the corresponding symmetry groups and $\underset{\sim}{\hat{\mathbf{P}}}$ the deformation gradient from Ω_0 to $\hat{\Omega}_0$, then
$$\mathfrak{G}_{\hat{\Omega}_0} = \underset{\sim}{\hat{\mathbf{P}}} . \mathfrak{G}_{\Omega_0} . \underset{\sim}{\hat{\mathbf{P}}}^{-1}$$



Isotropic elastic materials

- a material is **isotropic** with respect to the reference configuration Ω_0 if

$$\mathfrak{G}_{\Omega_0} = GO(E)$$

- function $\mathcal{F}^{\Pi}_{\Omega_0}$ is said to be isotropic

$$\underset{\sim}{\mathbf{Q}}.\mathcal{F}^{\Pi}_{\Omega_0}(\underset{\sim}{\mathbf{C}}).\underset{\sim}{\mathbf{Q}}^T = \mathcal{F}^{\Pi}_{\Omega_0}(\underset{\sim}{\mathbf{Q}}.\underset{\sim}{\mathbf{C}}.\underset{\sim}{\mathbf{Q}}^T)$$

Isotropic elastic materials

- a material is **isotropic** with respect to the reference configuration Ω_0 if

$$\mathfrak{G}_{\Omega_0} = GO(E)$$

- function $\mathcal{F}^{\Pi}_{\Omega_0}$ is said to be isotropic

$$\underset{\sim}{\mathbf{Q}}.\mathcal{F}^{\Pi}_{\Omega_0}(\underset{\sim}{\mathbf{C}}).\underset{\sim}{\mathbf{Q}}^T = \mathcal{F}^{\Pi}_{\Omega_0}(\underset{\sim}{\mathbf{Q}}.\underset{\sim}{\mathbf{C}}.\underset{\sim}{\mathbf{Q}}^T)$$

- **representation theorem** of isotropic functions

$$\underset{\sim}{\mathbf{\Pi}} = \alpha_0 \underset{\sim}{\mathbf{1}} + \alpha_1 \underset{\sim}{\mathbf{C}} + \alpha_2 \underset{\sim}{\mathbf{C}}^2$$

  α_i are arbitrary functions of the principal invariants of $\underset{\sim}{\mathbf{C}}$ which gives, in the Eulerian representation,

$$\underset{\sim}{\boldsymbol{\sigma}} = \beta_0 \underset{\sim}{\mathbf{1}} + \beta_1 \underset{\sim}{\mathbf{B}} + \beta_2 \underset{\sim}{\mathbf{B}}^2$$

- specific case :

$$\mathfrak{G}_{\Omega_0} = \mathcal{U}(E)$$

- the constitutive function is therefore such that

$$\mathcal{F}_{\Omega_0} = \mathcal{F}_{\hat{\Omega}_0}, \quad \forall \underset{\sim}{\mathbf{P}} \in \mathcal{U}(E)$$

- specific case :

$$\mathfrak{G}_{\Omega_0} = \mathcal{U}(E)$$

- the constitutive function is therefore such that

$$\mathcal{F}_{\Omega_0} = \mathcal{F}_{\hat{\Omega}_0}, \quad \forall \underset{\sim}{\mathbf{P}} \in \mathcal{U}(E)$$

- decomposition de $\underset{\sim}{\mathbf{F}}$ into a spherical and unimodular parts

$$\underset{\sim}{\mathbf{F}} = \underset{\sim}{\mathbf{F}}^{sph}.\underset{\sim}{\bar{\mathbf{F}}}, \qquad \underset{\sim}{\bar{\mathbf{F}}} : \Omega_0 \longrightarrow \hat{\Omega}_0$$

  it can be shown that $\mathcal{F}_{\Omega_0}(\underset{\sim}{\mathbf{F}}) = \mathcal{F}_{\Omega_0}(\underset{\sim}{\mathbf{F}}^{sph}) = \mathcal{F}_{\hat{\Omega}_0}(\underset{\sim}{\mathbf{F}}^{sph})$
  the constitutive law takes the form $\underset{\sim}{\boldsymbol{\sigma}} = f(J\underset{\sim}{\mathbf{1}})$

- principle of material frame indifference

$$\underset{\sim}{\boldsymbol{\sigma}}' = f(\underset{\sim}{\mathbf{F}}^{sph'}) = f(J\underset{\sim}{\mathbf{1}}) = \underset{\sim}{\boldsymbol{\sigma}}$$

$$\underset{\sim}{\boldsymbol{\sigma}} = \underset{\sim}{\mathbf{Q}}.\underset{\sim}{\boldsymbol{\sigma}}.\underset{\sim}{\mathbf{Q}}^T, \quad \forall \underset{\sim}{\mathbf{Q}} \in GO(E)$$

  $\underset{\sim}{\boldsymbol{\sigma}}$ is nothing but a homothety... Such a material is a...???

Elastic fluids

- specific case :

$$\mathfrak{G}_{\Omega_0} = \mathcal{U}(E)$$

- the constitutive function is therefore such that

$$\mathcal{F}_{\Omega_0} = \mathcal{F}_{\hat{\Omega}_0}, \quad \forall \underset{\sim}{\mathbf{P}} \in \mathcal{U}(E)$$

- the constitutive law takes the form

$$\underset{\sim}{\boldsymbol{\sigma}} = -p(\rho)\underset{\sim}{\mathbf{1}}$$

or the corresponding Lagrangean representation

$$\underset{\sim}{\mathbf{\Pi}} = J\underset{\sim}{\mathbf{F}}^{-1}.\underset{\sim}{\boldsymbol{\sigma}}.\underset{\sim}{\mathbf{F}}^{-T} = -pJ\underset{\sim}{\mathbf{C}}^{-1}$$

this is an elastic **fluid**!

Plan

1. Closure problem in continuum mechanics and constitutive theory
2. Elastic materials
   - Principle of material frame indifference
   - Material symmetry group
   - A definition of fluids and solids
3. Continuum thermodynamics
   - Energy
   - Entropy
4. Hyperelastic materials
   - State laws
   - Internal constraints
5. Viscosity

A (mechanical) definition of fluids and des solids

- A material is a **fluid** if, for a given reference configuration Ω_0, its symmetry group coincides with the unimodular group :

$$\mathfrak{G}_{\Omega_0} = \mathcal{U}(E)$$

  The symmetry group is then invariant with respect to change of reference configuration.
  A fluid admits no priviledged reference configuration.
  A fluid is isotropic with respect to any of its configurations.

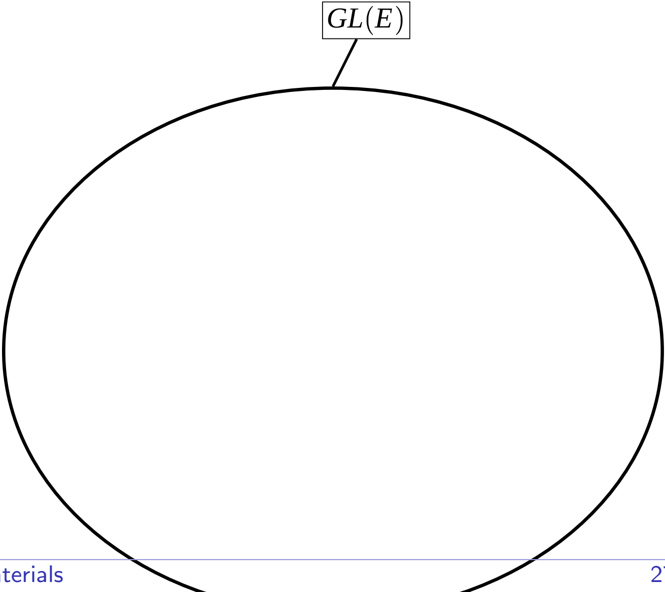
- A material is a **solid** if there exists a configuration Ω_0 for which the symmetry group is a sub–group of the orthogonal group :

$$\mathfrak{G}_{\Omega_0} \subset GO(E)$$

  Such a configuration is called **undistorted**. It is a priviledged configuration to write the consitutive law.

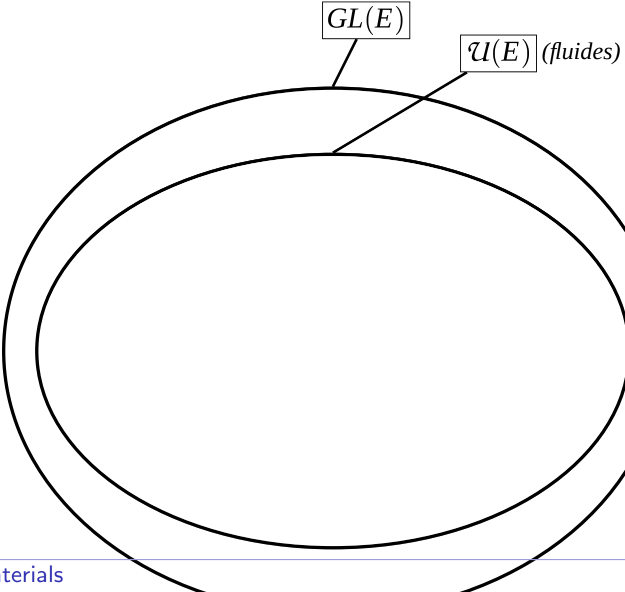
Classification of materials

materials (non necessarily elastic) are classified with respect to their symmetry groups



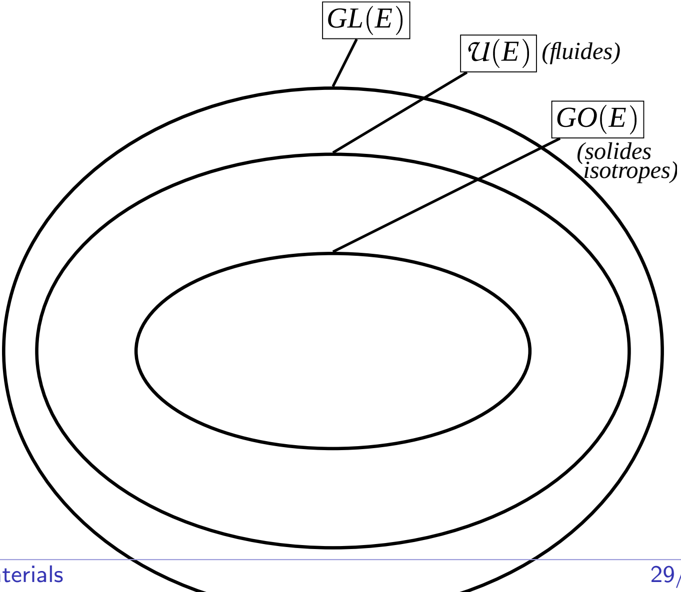
Classification of materials

materials (non necessarily elastic) are classified with respect to their symmetry groups

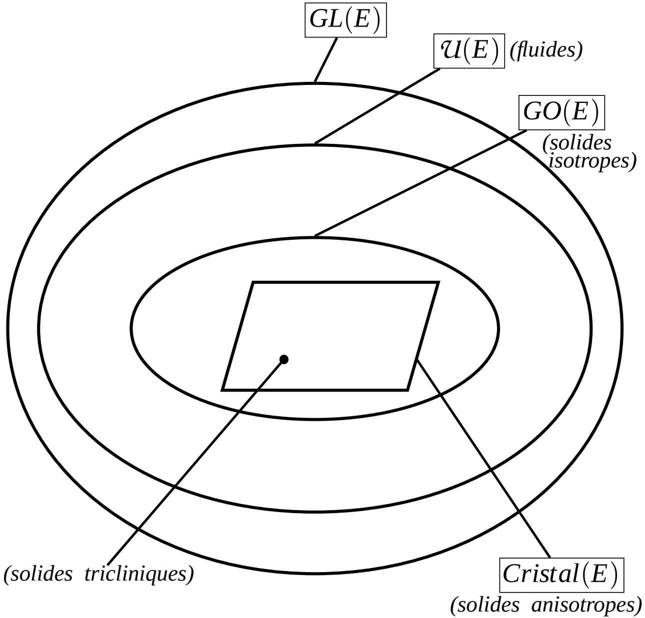


Classification of materials

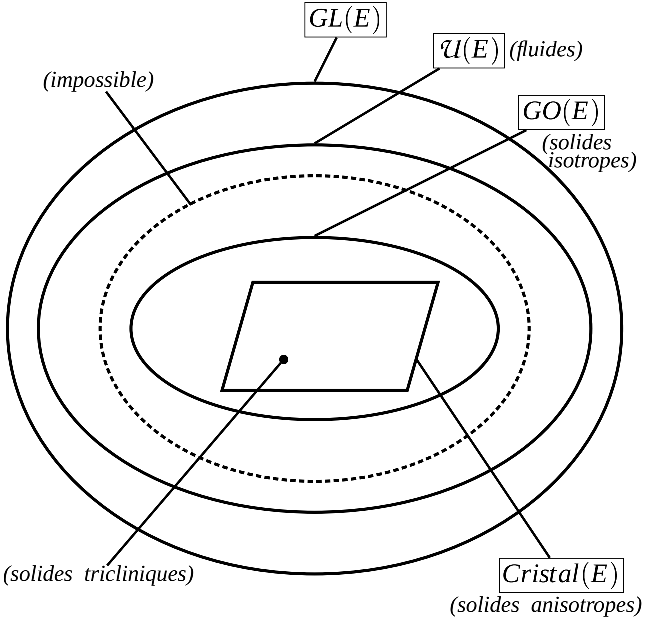
materials (non necessarily elastic) are classified with respect to their symmetry groups



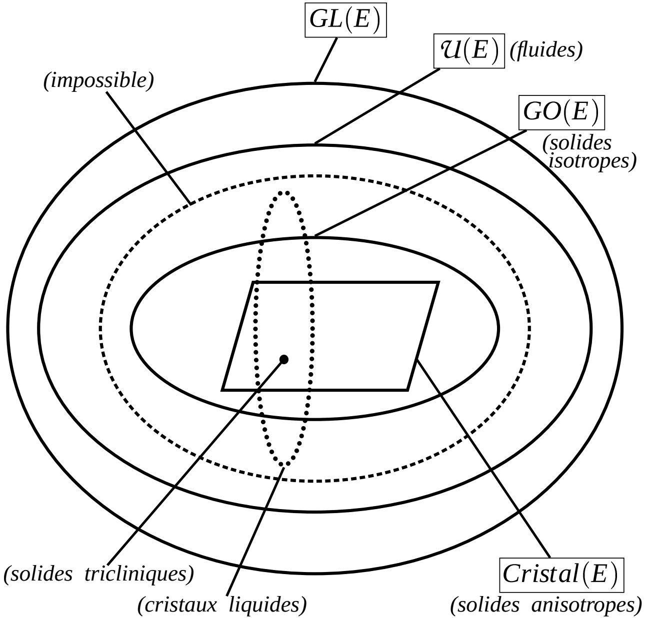
Classification of materials



Classification of materials



Classification of materials



Plan

1. Closure problem in continuum mechanics and constitutive theory
2. Elastic materials
   - Principle of material frame indifference
   - Material symmetry group
   - A definition of fluids and solids
3. Continuum thermodynamics
   - Energy
   - Entropy
4. Hyperelastic materials
   - State laws
   - Internal constraints
5. Viscosity

Plan

1. Closure problem in continuum mechanics and constitutive theory
2. Elastic materials
   - Principle of material frame indifference
   - Material symmetry group
   - A definition of fluids and solids
3. Continuum thermodynamics
   - Energy
   - Entropy
4. Hyperelastic materials
   - State laws
   - Internal constraints
5. Viscosity

Energy balance

- **kinetic energy**

$$\mathcal{K} := \frac{1}{2}\int_{\mathcal{D}} \rho \underline{\mathbf{v}}\,.\underline{\mathbf{v}}\;dv$$

- **power of external forces**

$$\mathcal{P} := \mathcal{P}^c + \mathcal{P}^e = \int_{\partial\mathcal{D}} \underline{\mathbf{t}}\,.\underline{\mathbf{v}}\;ds + \int_{\mathcal{D}} \rho \underline{\mathbf{f}}\,.\underline{\mathbf{v}}\;dv$$

- l'**internal energy** $\mathcal{E}$ of the system, mass density e of internal energy

$$\mathcal{E} := \int_{\mathcal{D}} \rho e(\underline{\mathbf{x}}\,, t)\, dv$$

- **heat** supply $\mathcal{Q}$ to the system in the form of contact heat supply $h(\underline{\mathbf{x}}\,, t, \partial\mathcal{D})$ and volume heat supply $\rho r(\underline{\mathbf{x}}\,, t)$

$$\mathcal{Q} := \int_{\partial\mathcal{D}} h\, ds + \int_{\mathcal{D}} \rho r\, dv$$

heat flux vector $\underline{\mathbf{q}}$ $\qquad h(\underline{\mathbf{x}}\,, \underline{\mathbf{n}}\,, t) = -\underline{\mathbf{q}}\,(\underline{\mathbf{x}}\,, t).\underline{\mathbf{n}}$

Energy principle

$$\dot{\mathcal{E}} + \dot{\mathcal{K}} = \mathcal{P} + \mathcal{Q}$$

Taking the **theorem of kinetic energy** into account,

$$\dot{\mathcal{K}} = \mathcal{P}^i + \mathcal{P}^e + \mathcal{P}^c$$

where, in the absence of discontinuities, $\mathcal{P}^i = -\int_{\mathcal{D}} \underset{\sim}{\boldsymbol{\sigma}} : \underset{\sim}{\mathbf{D}}\, dv$ is the **power of internal forces**, the first principle can be rewritten as

$$\dot{\mathcal{E}} = -\mathcal{P}^i + \mathcal{Q}$$

$$\int_{\mathcal{D}} \rho \dot{e}\, dv = \int_{\mathcal{D}} \underset{\sim}{\boldsymbol{\sigma}} : \underset{\sim}{\mathbf{D}}\, dv - \int_{\partial\mathcal{D}} \underline{\mathbf{q}} . \underline{\mathbf{n}}\, ds + \int_{\mathcal{D}} \rho r\, dv$$

Local formulation of the energy principle

From the global formulation for any sub–domain $\mathcal{D} \subset \Omega_t$...

$$\int_{\mathcal{D}} \rho \dot{e} \, dv = \int_{\mathcal{D}} \underset{\sim}{\boldsymbol{\sigma}} : \underset{\sim}{\mathbf{D}} \, dv - \int_{\partial \mathcal{D}} \underline{\mathbf{q}} \, . \underline{\mathbf{n}} \; ds + \int_{\mathcal{D}} \rho r \, dv$$

... to the local formulation at a regular point of Ω_t

$$\rho \dot{e} = \underset{\sim}{\boldsymbol{\sigma}} : \underset{\sim}{\mathbf{D}} - \operatorname{div} \underline{\mathbf{q}} + \rho r$$

Lagrangean formulation of the energy principle

Lagrangean representation in continuum thermodynamics

$$e(\underline{\mathbf{x}}\,, t) = e_0(\underline{\mathbf{X}}\,, t), \quad \underline{\mathbf{Q}}\,(\underline{\mathbf{X}}\,, t) = J\underset{\sim}{\mathbf{F}}^{-1}.\underline{\mathbf{q}}$$

From the global formulation for any sub–domain $\mathcal{D}_0 \subset \Omega_0$...

$$\int_{\mathcal{D}_0} \rho_0 \dot{e}_0 \, dV = \int_{\mathcal{D}_0} \underset{\sim}{\mathbf{\Pi}} : \underset{\sim}{\dot{\mathbf{E}}} \, dV - \int_{\partial\mathcal{D}_0} \underline{\mathbf{Q}}\,.\underline{\mathbf{N}} \, dS + \int_{\mathcal{D}_0} \rho_0 r_0 \, dV$$

... to the local formulation at a regular point of Ω_0

$$\rho_0 \dot{e}_0 = \underset{\sim}{\mathbf{\Pi}} : \underset{\sim}{\dot{\mathbf{E}}} - \mathrm{Div}\,\underline{\mathbf{Q}} + \rho_0 r_0$$

Plan

1. Closure problem in continuum mechanics and constitutive theory
2. Elastic materials
   - Principle of material frame indifference
   - Material symmetry group
   - A definition of fluids and solids
3. Continuum thermodynamics
   - Energy
   - Entropy
4. Hyperelastic materials
   - State laws
   - Internal constraints
5. Viscosity

Entropy principle

- entropy of the system / mass entropy density

$$\mathcal{S}(\mathcal{D}) = \int_{\mathcal{D}} \rho s \, dv$$

- entropy supply

$$\varphi(\mathcal{D}) = -\int_{\partial\mathcal{D}} \frac{\underline{\mathbf{q}}}{T} . \underline{\mathbf{n}} \, ds + \int_{\mathcal{D}} \frac{\rho r}{T} \, dv$$

- global formulation of the entropy principle for any sub–domain $\mathcal{D} \subset \Omega_t$

$$\dot{\mathcal{S}}(\mathcal{D}) - \varphi(\mathcal{D}) \geq 0$$

$$\frac{d}{dt} \int_{\mathcal{D}} \rho s \, dv + \int_{\partial\mathcal{D}} \frac{\underline{\mathbf{q}}}{T} . \underline{\mathbf{n}} \, ds - \int_{\mathcal{D}} \rho \frac{r}{T} \, dv \geq 0$$

Lagrangean formulation of the entropy principle

Lagrangean description in continuum thermodynamics

$$s(\underline{\mathbf{x}}\,, t) = s_0(\underline{\mathbf{X}}\,, t), \quad \underline{\mathbf{Q}}\,(\underline{\mathbf{X}}\,, t) = J \underset{\sim}{\mathbf{F}}^{-1}.\underline{\mathbf{q}}$$

From the global formulation valid for any sub–domain $\mathcal{D}_0 \subset \Omega_0$...

$$\frac{d}{dt}\int_{\mathcal{D}_0} \rho_0 s_0(\underline{\mathbf{X}}\,, t)\, dV + \int_{\partial\mathcal{D}_0} \frac{\underline{\mathbf{Q}}}{T}.\underline{\mathbf{N}}\, dS + \int_{\mathcal{D}_0} \rho_0 \frac{r_0}{T}\, dV \geq 0$$

... to the local formulation at a regular point Ω_0

$$\rho_0 \dot{s}_0 + \mathrm{Div}\, \frac{\underline{\mathbf{Q}}}{T} - \rho_0 \frac{r_0}{T} \geq 0$$

Plan

1. Closure problem in continuum mechanics and constitutive theory
2. Elastic materials
   - Principle of material frame indifference
   - Material symmetry group
   - A definition of fluids and solids
3. Continuum thermodynamics
   - Energy
   - Entropy
4. Hyperelastic materials
   - State laws
   - Internal constraints
5. Viscosity

Plan

1. Closure problem in continuum mechanics and constitutive theory
2. Elastic materials
   - Principle of material frame indifference
   - Material symmetry group
   - A definition of fluids and solids
3. Continuum thermodynamics
   - Energy
   - Entropy
4. Hyperelastic materials
   - State laws
   - Internal constraints
5. Viscosity

Dissipation

- functions of state : internal energy $e_0(\underset{\sim}{\mathbf{E}}, s_0)$
  Helmholtz free energy $\psi(\underset{\sim}{\mathbf{E}}, T) = e_0 - Ts_0$
- Clausius–Duhem inequality (volume dissipation rate D)

$$D = \underset{\sim}{\mathbf{\Pi}} : \dot{\underset{\sim}{\mathbf{E}}} - \rho_0(\dot{\psi}_0 + \dot{T}s_0) - \underline{\mathbf{Q}} \,. \frac{\mathrm{Grad}\ T}{T} \geq 0$$

Dissipation

- functions of state : internal energy $e_0(\underset{\sim}{\mathbf{E}}, s_0)$
  Helmholtz free energy $\psi(\underset{\sim}{\mathbf{E}}, T) = e_0 - Ts_0$
- Clausius–Duhem inequality (volume dissipation rate D)

$$D = \underset{\sim}{\mathbf{\Pi}} : \dot{\underset{\sim}{\mathbf{E}}} - \rho_0(\dot{\psi}_0 + \dot{T}s_0) - \underline{\mathbf{Q}} \,.\, \frac{\mathrm{Grad}\ T}{T} \geq 0$$

- Elastic materials $\underset{\sim}{\mathbf{\Pi}} = \mathcal{F}^{\Pi}_{\Omega_0}(\underset{\sim}{\mathbf{E}}, T), \quad \psi_0(\underset{\sim}{\mathbf{E}}, T)$

$$\dot{\psi}_0 = \frac{\partial \psi_0}{\partial \underset{\sim}{\mathbf{E}}} : \dot{\underset{\sim}{\mathbf{E}}} + \frac{\partial \psi_0}{\partial T} \dot{T}$$

$$D = (\underset{\sim}{\mathbf{\Pi}} - \rho_0 \frac{\partial \psi_0}{\partial \underset{\sim}{\mathbf{E}}}) : \dot{\underset{\sim}{\mathbf{E}}} - \rho_0(\frac{\partial \psi_0}{\partial T} + s_0)\dot{T} - \underline{\mathbf{Q}} \,.\, \frac{\mathrm{Grad}\ T}{T} \geq 0$$

State laws for hyperelastic materials

- hyperelastic relations

$$\underset{\sim}{\mathbf{\Pi}} = \rho_0 \frac{\partial \psi_0}{\partial \underset{\sim}{\mathbf{E}}}$$

$$s_0 = -\frac{\partial \psi_0}{\partial T}$$

  ψ_0 is also called **elastic potential** (vanishing intrinsic dissipation)

- thermal dissipation

$$D = -\underline{\mathbf{Q}}\,.\frac{\mathrm{Grad}\,T}{T} = -\frac{\rho_0}{\rho}\underline{\mathbf{q}}\,.\mathrm{grad}\,T \geq 0$$

  Fourier law *(thermal constitutive equation)*

$$\underline{\mathbf{Q}} = -\underset{\sim}{\mathbf{K}}(\underset{\sim}{\mathbf{E}}, T).\mathrm{Grad}\,T$$

  there is no thermal potential (total dissipation)

Isotropic hyperelastic materials

- **representation theorem** for isotropic functions of second order tensors

$$\psi_0(\underset{\sim}{\mathbf{E}}, T) \equiv \psi_0(I_1, I_2, I_3, T)$$

the **principal invariants** of $\underset{\sim}{\mathbf{E}}$ are

$$I_1(\underset{\sim}{\mathbf{E}}) := \mathrm{trace}\,\underset{\sim}{\mathbf{E}}, \quad I_2(\underset{\sim}{\mathbf{E}}) := \frac{1}{2}\mathrm{trace}\,\underset{\sim}{\mathbf{E}}^2, \quad I_3(\underset{\sim}{\mathbf{E}}) := \frac{1}{3}\mathrm{trace}\,\underset{\sim}{\mathbf{E}}^3$$

- hyperelastic constitutive equations

$$\underset{\sim}{\mathbf{\Pi}} = \rho_0 \frac{\partial \psi_0}{\partial I_1}\frac{\partial I_1}{\partial \underset{\sim}{\mathbf{E}}} + \rho_0 \frac{\partial \psi_0}{\partial I_2}\frac{\partial I_2}{\partial \underset{\sim}{\mathbf{E}}} + \rho_0 \frac{\partial \psi_0}{\partial I_3}\frac{\partial I_3}{\partial \underset{\sim}{\mathbf{E}}}$$

$$\underset{\sim}{\mathbf{\Pi}} = \rho_0 \frac{\partial \psi_0}{\partial I_1}\underset{\sim}{\mathbf{1}} + \rho_0 \frac{\partial \psi_0}{\partial I_2}\underset{\sim}{\mathbf{E}} + \rho_0 \frac{\partial \psi_0}{\partial I_3}\underset{\sim}{\mathbf{E}}^2$$

compare with the previously established isotropic elastic law

$$\underset{\sim}{\mathbf{\Pi}} = \alpha_0\underset{\sim}{\mathbf{1}} + \alpha_1\underset{\sim}{\mathbf{E}} + \alpha_2\underset{\sim}{\mathbf{E}}^2$$

Plan

1 Closure problem in continuum mechanics and constitutive theory

2 Elastic materials
- Principle of material frame indifference
- Material symmetry group
- A definition of fluids and solids

3 Continuum thermodynamics
- Energy
- Entropy

4 Hyperelastic materials
- State laws
- Internal constraints

5 Viscosity

Incompressible hyperelastic materials

- notion of internal constraint; the licit deformation states are such that

$$g(\underset{\sim}{\mathbf{C}}) = 0, \quad \dot{g}(\underset{\sim}{\mathbf{C}}) = \frac{\partial g}{\partial \underset{\sim}{\mathbf{C}}} : \dot{\underset{\sim}{\mathbf{C}}} = 0$$

- exploitation of the second principle in the presence of internal constraint

$$(\underset{\sim}{\mathbf{\Pi}} - 2\rho_0 \frac{\partial \psi_0}{\partial \underset{\sim}{\mathbf{C}}}) : \dot{\underset{\sim}{\mathbf{C}}} \geq 0$$

$$\Longrightarrow \underset{\sim}{\mathbf{\Pi}} - 2\rho_0 \frac{\partial \psi_0}{\partial \underset{\sim}{\mathbf{C}}} = \lambda \frac{\partial g}{\partial \underset{\sim}{\mathbf{C}}} = \underset{\sim}{\mathbf{\Pi}}^R$$

  the stress $\underset{\sim}{\mathbf{\Pi}}^R$ is the **reaction** to the constraint g

- incompressibility

$$g(\underset{\sim}{\mathbf{C}}) = \det \underset{\sim}{\mathbf{C}} - 1, \quad \frac{\partial g}{\partial \underset{\sim}{\mathbf{C}}} = (\det \underset{\sim}{\mathbf{C}})\underset{\sim}{\mathbf{C}}^{-1}$$

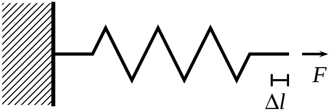
$$\underset{\sim}{\mathbf{\Pi}} = \lambda \underset{\sim}{\mathbf{C}}^{-1} + 2\rho_0 \frac{\partial \psi_0}{\partial \underset{\sim}{\mathbf{C}}}, \quad \underset{\sim}{\boldsymbol{\sigma}} = -p\underset{\sim}{\mathbf{1}} + 2\rho \underset{\sim}{\mathbf{F}}.\frac{\partial \psi_0}{\partial \underset{\sim}{\mathbf{C}}}.\underset{\sim}{\mathbf{F}}^T$$

Plan

1. Closure problem in continuum mechanics and constitutive theory
2. Elastic materials
   - Principle of material frame indifference
   - Material symmetry group
   - A definition of fluids and solids
3. Continuum thermodynamics
   - Energy
   - Entropy
4. Hyperelastic materials
   - State laws
   - Internal constraints
5. Viscosity

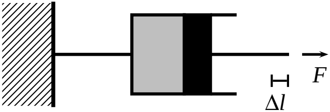
Viscous materials

- we have generalized the “spring” behavior to 3D solids



$$F = k\delta l \qquad \underset{\sim}{\boldsymbol{\sigma}} = \underset{\sim}{\mathbf{R}}.\mathcal{F}_{\Omega_0}(\underset{\sim}{\mathbf{C}}).\underset{\sim}{\mathbf{R}}^T$$

- how to generalize the damping behavior?



$$F = \eta \dot{l} \qquad \underset{\sim}{\boldsymbol{\sigma}} = \mathcal{F}_{\Omega_0}(\underset{\sim}{\mathbf{D}})$$

Viscous materials

- change of reference configuration

$$\underset{\sim}{\mathbf{L}} = \dot{\underset{\sim}{\mathbf{F}}}.\underset{\sim}{\mathbf{F}}^{-1} = \overset{\bullet}{\overbrace{\hat{\underset{\sim}{\mathbf{F}}}.\underset{\sim}{\mathbf{P}}}}.(\hat{\underset{\sim}{\mathbf{F}}}.\underset{\sim}{\mathbf{P}})^{-1} = \hat{\underset{\sim}{\mathbf{L}}}$$

such materials are necessarily **fluids !**

Viscous materials

- change of reference configuration

$$\underset{\sim}{\mathbf{L}} = \dot{\underset{\sim}{\mathbf{F}}}.\underset{\sim}{\mathbf{F}}^{-1} = \overbrace{\hat{\underset{\sim}{\mathbf{F}}}.\underset{\sim}{\mathbf{P}}}^{\bullet}.(\hat{\underset{\sim}{\mathbf{F}}}.\underset{\sim}{\mathbf{P}})^{-1} = \hat{\underset{\sim}{\mathbf{L}}}$$

such materials are necessarily **fluids !**

- change of observers

$$\underset{\sim}{\mathbf{Q}}.\mathcal{F}_{\Omega_0}(\underset{\sim}{\mathbf{D}}).\underset{\sim}{\mathbf{Q}}^T = \mathcal{F}_{\Omega_0}(\underset{\sim}{\mathbf{Q}}.\underset{\sim}{\mathbf{D}}.\underset{\sim}{\mathbf{Q}}^T)$$

$\mathcal{F}_{\Omega_0}$ is an isotropic function of its argument

$$\underset{\sim}{\boldsymbol{\sigma}} = \alpha_0 \underset{\sim}{\mathbf{1}} + \alpha_1 \underset{\sim}{\mathbf{D}} + \alpha_2 \underset{\sim}{\mathbf{D}}^2$$

the α_i are of the invariants of $\underset{\sim}{\mathbf{D}}$ (and possibly of ρ)
these are **Reiner–Rivlin viscous fluids** (1945)

- linearization of the previous law leads to (compressible) **Navier–Stokes or Newtonian fluids**

$$\underset{\sim}{\boldsymbol{\sigma}} = -p(\rho)\underset{\sim}{\mathbf{1}} + \eta_1(\mathrm{trace}\,\underset{\sim}{\mathbf{D}})\underset{\sim}{\mathbf{1}} + 2\eta_2\underset{\sim}{\mathbf{D}}$$

(in the incompressible case, p is a reaction and $\mathrm{trace}\,\underset{\sim}{\mathbf{D}} = 0$)