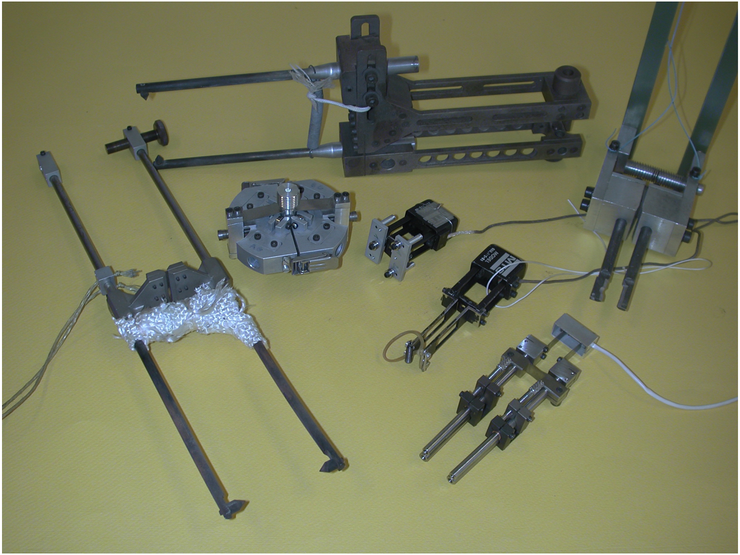
Deformation of continuous media

Plan

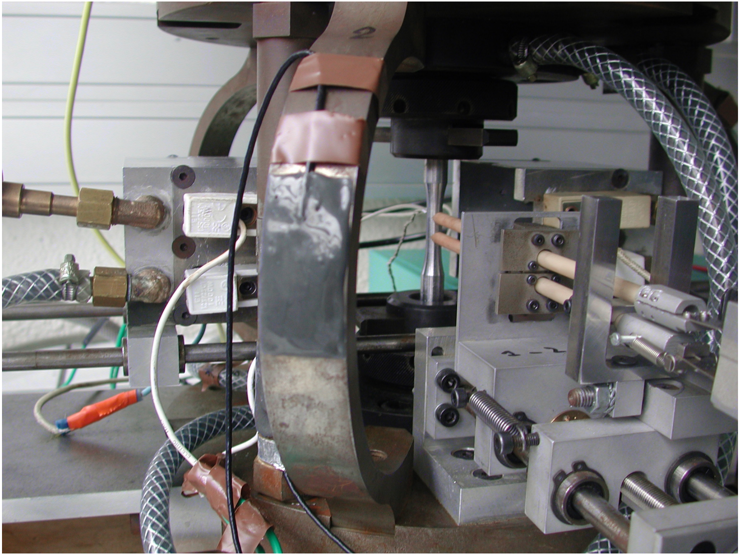
Extensometry

measuring the relative **displacement** of two material points

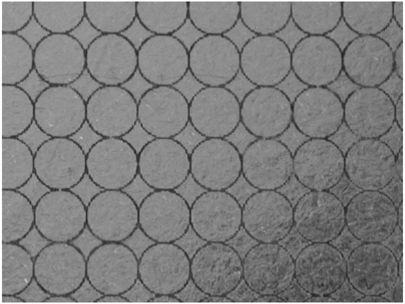


Extensometry

measuring the relative **displacement** of two material points



Strain field measurements



Strain field measurements

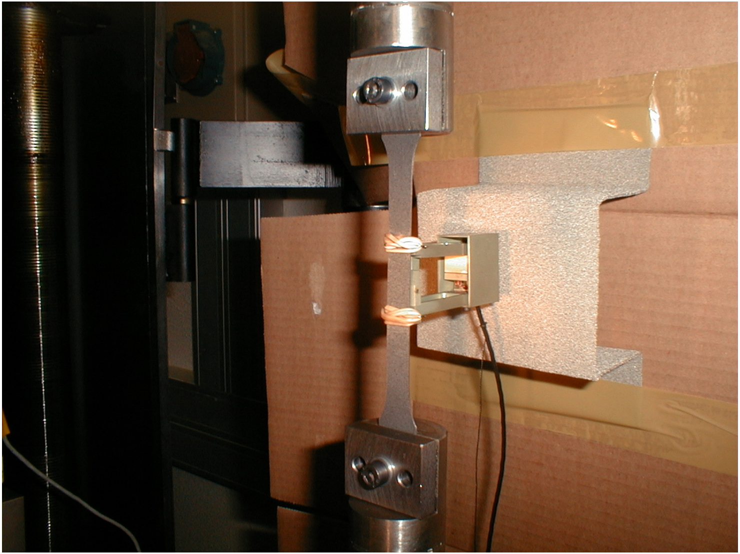
tensile test

Strain field measurements

tensile test

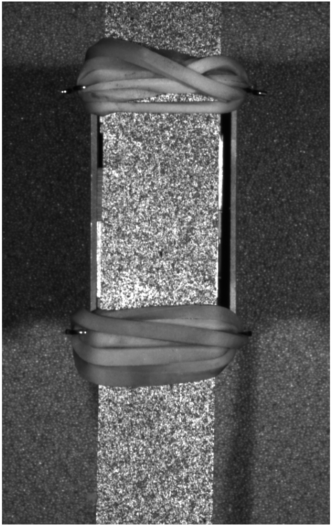
Strain field measurements

tensile test

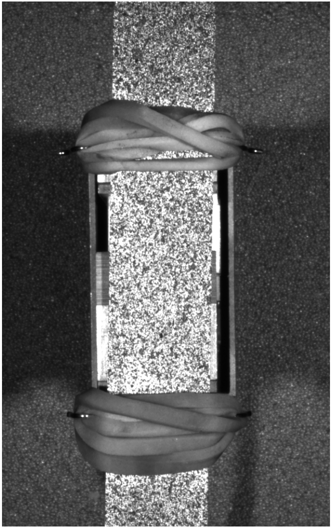


Strain field measurements

elongation of the gauge length



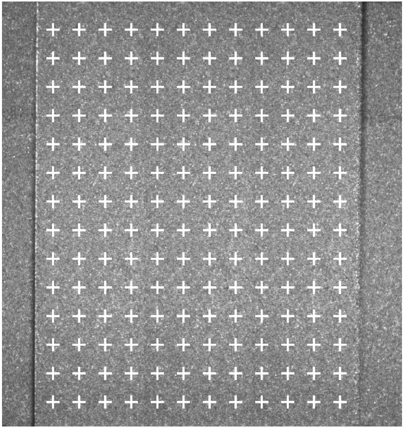
initial state

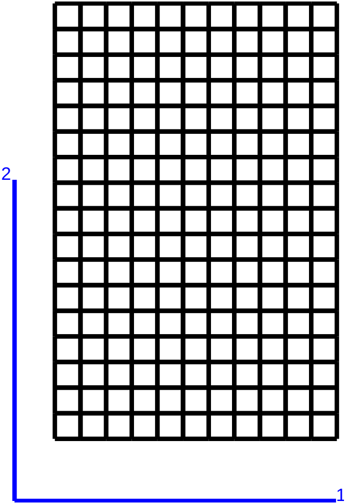


deformed state

Strain field measurements

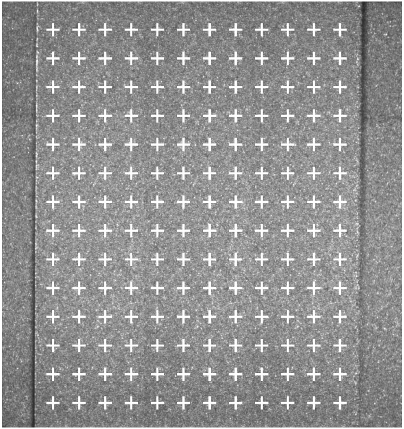
image correlation technique : locating patterns around a grid of material points during deformation

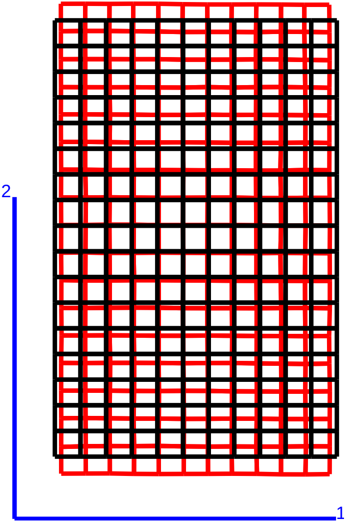




Strain field measurements

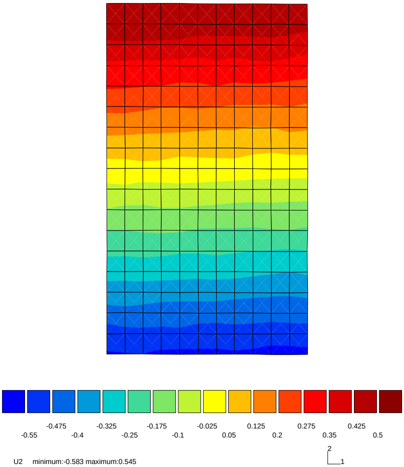
image correlation technique : locating patterns around a grid of material points during deformation



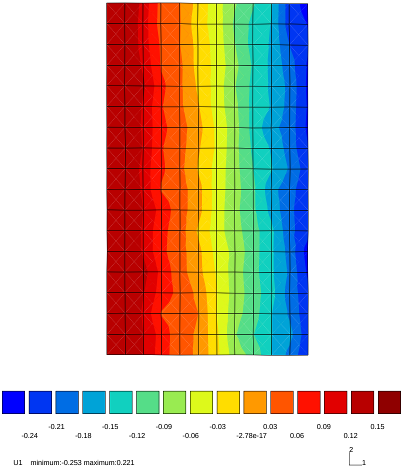


Strain field measurements

field of ...



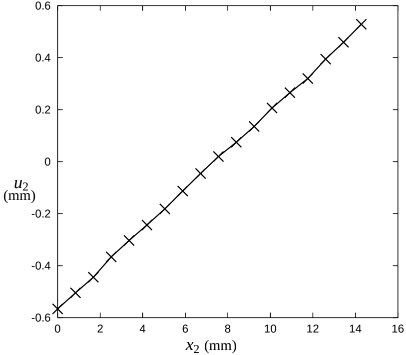
displacement u_2



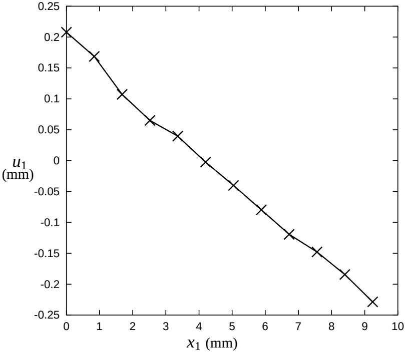
displacement u_1 in mm

Strain field measurements

displacement of points along a line



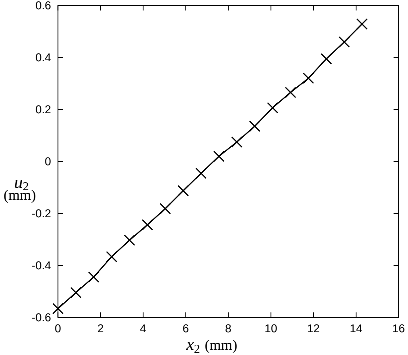
vertical

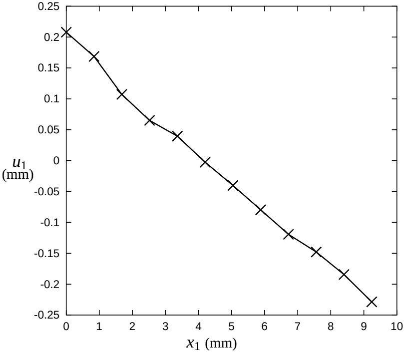


horizontal

Strain field measurements

displacement of points along a line





vertical

horizontal

$$F_{22} - 1 = \frac{\partial u_2}{\partial x_2}$$

$$F_{11} - 1 = \frac{\partial u_1}{\partial x_1}$$

Plan

Notations

Euclidean tensor fields; orthonormal basis $(\underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2, \underline{\mathbf{e}}_3)$

- zeroth order tensors : scalar field $\qquad f(\underline{\mathbf{X}}, t)$
- first order tensors : vectors $\qquad \underline{\mathbf{x}}(\underline{\mathbf{X}}, t)$

$$\underline{\mathbf{x}} = x_i \underline{\mathbf{e}}_i, \quad [\underline{\mathbf{x}}] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

- second order tensors : linear mapping / bilinear forms $\qquad \underset{\sim}{\mathbf{C}}(\underline{\mathbf{X}}, t)$

$$\underset{\sim}{\mathbf{C}} = C_{ij}\, \underline{\mathbf{e}}_i \otimes \underline{\mathbf{e}}_j, \quad [\underset{\sim}{\mathbf{C}}] = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

- operations between tensors with respect to an **orthonormal** basis

$$\underline{\mathbf{a}} . \underline{\mathbf{b}} = a_i b_i = [\underline{\mathbf{a}}]^T [\underline{\mathbf{b}}], \quad \underset{\sim}{\boldsymbol{\sigma}} . \underline{\mathbf{n}} = \sigma_{ij} n_j\, \underline{\mathbf{e}}_i, \quad [\underset{\sim}{\boldsymbol{\sigma}} . \underline{\mathbf{n}}] = [\underset{\sim}{\boldsymbol{\sigma}}]\, [\underline{\mathbf{n}}]$$

$$\underline{\mathbf{m}} . \underset{\sim}{\boldsymbol{\sigma}} . \underline{\mathbf{n}} = m_i \sigma_{ij} n_j = [\underline{\mathbf{m}}]^T [\underset{\sim}{\boldsymbol{\sigma}}][\underline{\mathbf{n}}]$$

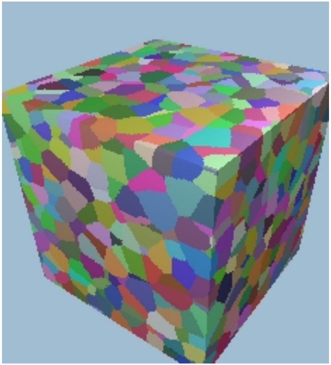
$$\underset{\sim}{\boldsymbol{\sigma}} : \underset{\sim}{\mathbf{L}} = \sigma_{ij} L_{ij} = \mathrm{trace}\,(\underset{\sim}{\boldsymbol{\sigma}} . \underset{\sim}{\mathbf{L}}^T) = \mathrm{trace}\,([\underset{\sim}{\boldsymbol{\sigma}}]\, [\underset{\sim}{\mathbf{L}}]^T)$$

Volume element dV of continuum mechanics

- Notion of **material point** : An infinitsimal volume dV around $\underline{\mathbf{X}}$
- $dV \sim$ Representative Volume Element

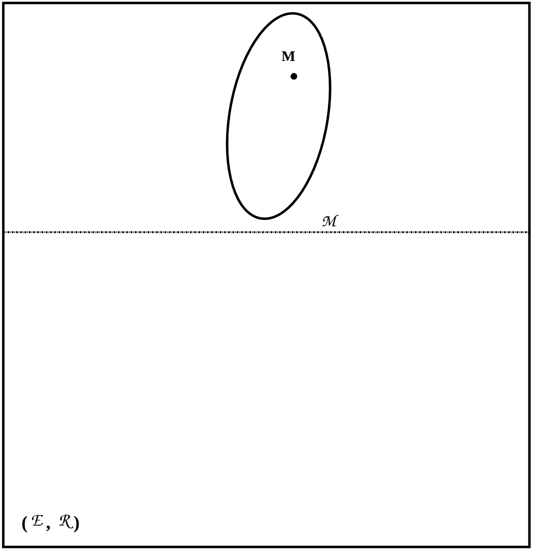
  $$d \ll L_{VER} \ll L$$

  d size of heterogeneities
  L_{RVE} size of RVE
  L structural size
- we follow the material point without considering the particles inside the RVE...



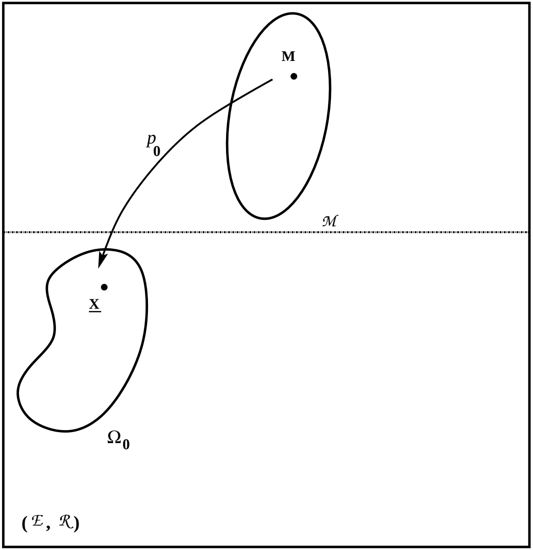
RVE for a metal polycrystal

Material body $\mathcal{M}$



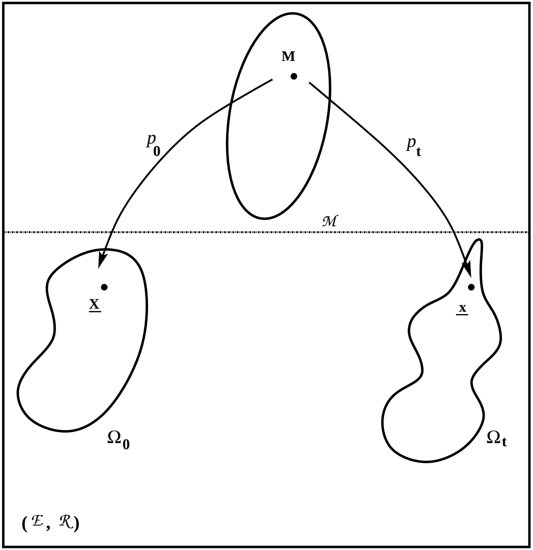
Material body $\mathcal{M}$ is a set of material points M

Reference placement in physical space



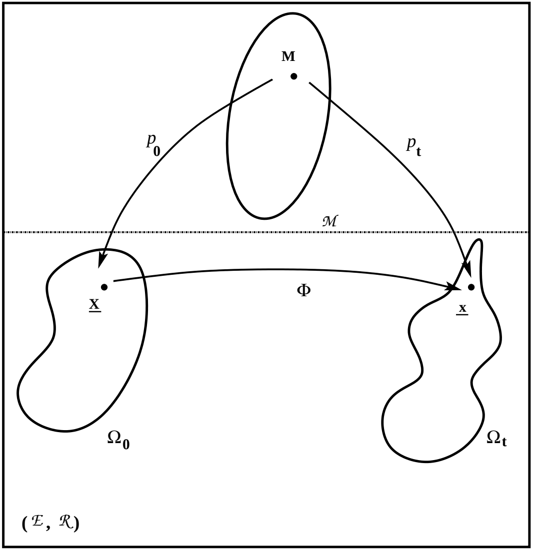
- Ω_0 is the configuration of $\mathcal{M}$ in physical space $\mathcal{E}$ with respect to the observer $\mathcal{R}$ at time t_0
- material point $M \in \mathcal{M}$ occupies place $\underline{\mathbf{X}}$ in this configuration
- we choose it as the **reference configuration**

Current configuration in physical space



- Ω_t is the configuration of material body $\mathcal{M}$ in physical space $\mathcal{E}$ at time t
- material point $M \in \mathcal{M}$ occupies place $\underline{\mathbf{x}}$ in this configuration
- we call it **current configuration**

Current configuration in physical space



- The motion

  $$\underline{\mathbf{x}} = \Phi(\underline{\mathbf{X}}, t)$$

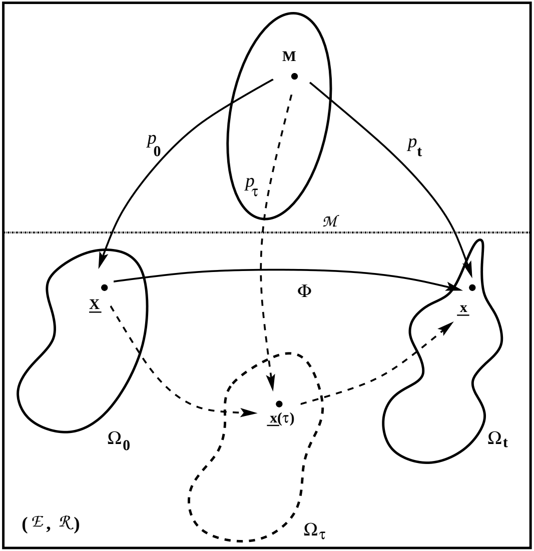
  links Ω_0 to Ω_t

- The motion is **bijective and bi–continuous**

  $$\underline{\mathbf{X}} = \Phi^{-1}(\underline{\mathbf{x}}, t)$$

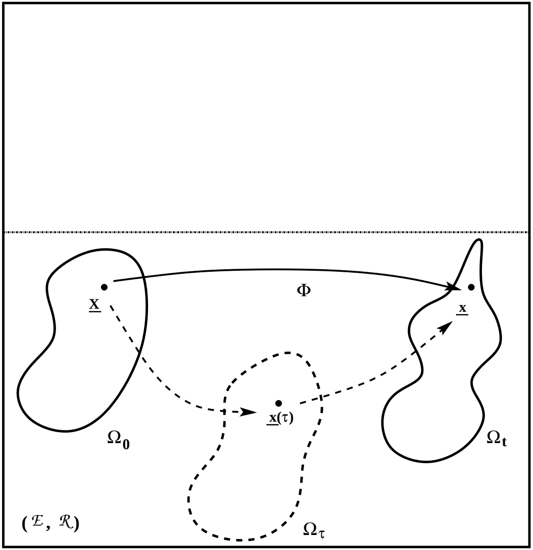
  - ⋆ no fission!
  - ⋆ no fusion!

Intermediate configuration of the material body



- The configuration fo $\mathcal{M}$ at $0 \leq \tau \leq t$ is denoted Ω_τ and called **intermediate configuration** of the material body
- The motion $\Phi(\underline{\mathbf{X}}, \tau)_{0 \leq \tau \leq t}$ records **the deformation history** of the material body

Deformation of continuous media



- The motion

$$\underline{\mathbf{x}} = \Phi(\underline{\mathbf{X}}, t)$$

links Ω_0 to Ω_t

- **Displacement** of material point is

$$\underline{\mathbf{u}}(\underline{\mathbf{X}}, t) = \underline{\mathbf{x}} - \underline{\mathbf{X}}$$

$$\underline{\mathbf{u}}(\underline{\mathbf{X}}, t) = \Phi(\underline{\mathbf{X}}, t) - \underline{\mathbf{X}}$$

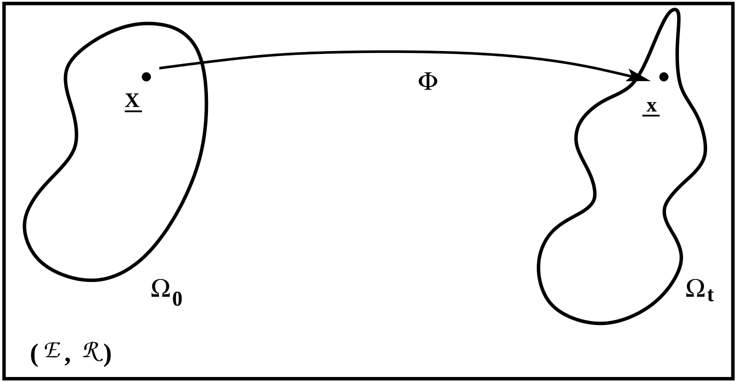
Lagrangean vs. Eulerian approaches

- materials with an underlying **microstructure** : generally "solids"
  The volume element $dV \in \Omega_0$ around $\underline{\mathbf{X}}$ becomes $dv \in \Omega_t$ around $\underline{\mathbf{x}}$. dV and dv contain the same particles.
  field of tensor function $F(\underline{\mathbf{X}}, t)$
  Lagrangean approach
- materials without any underlying **microstructure** : generally "fluids"
  Particles can be interchanged, they are not labelled. One is concerned with the mean velocity of particles going through dv around the geometrical point $\underline{\mathbf{x}} \in \Omega_t$ at time t
  field of tensor function $f(\underline{\mathbf{x}}, t)$
  Eulerian approach
- Les points de vue lagrangien et eulriens ont quivalents :

$$f(\underline{\mathbf{x}}, t) := F(\Phi^{-1}(\underline{\mathbf{x}}, t), t), \quad F(\underline{\mathbf{X}}, t) := f(\Phi(\underline{\mathbf{X}}, t), t)$$

Plan

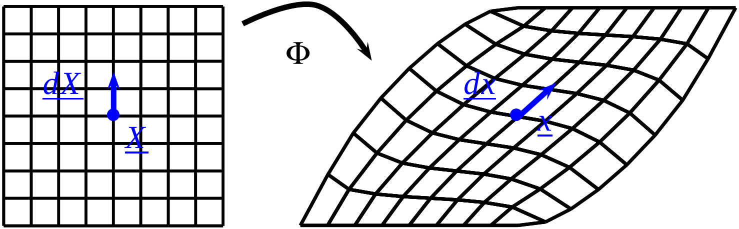
Deformation of a continuum medium



motion Φ $\qquad \underline{\mathbf{x}} = \Phi(\underline{\mathbf{X}}, t)$

displacement field $\qquad \underline{\mathbf{u}}(\underline{\mathbf{X}}, t) = \underline{\mathbf{x}} - \underline{\mathbf{X}} = \Phi(\underline{\mathbf{X}}, t) - \underline{\mathbf{X}}$

Deformation gradient



tangent linear mapping associated with Φ

$$\Phi(\underline{\mathbf{X}} + \underline{\mathbf{dX}}) - \Phi(\underline{\mathbf{X}}) = \frac{\partial \Phi}{\partial \underline{\mathbf{X}}}.\underline{\mathbf{dX}} + o(\underline{\mathbf{X}}, \underline{\mathbf{dX}})$$

material line element $\underline{\mathbf{dX}}$ initial and $\underline{\mathbf{dx}}$ current

$$\underset{\sim}{\mathbf{F}}(\underline{\mathbf{X}}, t) = \frac{\partial \Phi}{\partial \underline{\mathbf{X}}} = \mathrm{Grad}\,\Phi = \underset{\sim}{\mathbf{1}} + \mathrm{Grad}\,\underline{\mathbf{u}}, \quad \underline{\mathbf{dx}} = \underset{\sim}{\mathbf{F}}.\underline{\mathbf{dX}}$$

The deformation gradient $\underset{\sim}{\mathbf{F}}$ is a local characterization of the motion ($\underset{\sim}{\mathbf{F}}(\underline{\mathbf{X}}, t = 0) = \underset{\sim}{\mathbf{1}}$)

Deformation gradient

with respect to an orthonormal basis $(\underline{\mathbf{e}}_i)_{i=1,3}$

$$dx_i = F_{ij} dX_j, \quad \text{avec} \quad F_{ij} = \frac{\partial x_i}{\partial X_j} = \delta_{ij} + \frac{\partial u_i}{\partial X_j} \quad \text{et} \quad \underset{\sim}{\mathbf{F}} = F_{ij}\, \underline{\mathbf{e}}_i \otimes \underline{\mathbf{e}}_j$$

$$\begin{aligned}
dx_1 &= \frac{\partial \Phi_1}{\partial X_1} dX_1 + \frac{\partial \Phi_1}{\partial X_2} dX_2 + \frac{\partial \Phi_1}{\partial X_3} dX_3 \\
dx_2 &= \frac{\partial \Phi_2}{\partial X_1} dX_1 + \frac{\partial \Phi_2}{\partial X_2} dX_2 + \frac{\partial \Phi_2}{\partial X_3} dX_3 \\
dx_3 &= \frac{\partial \Phi_3}{\partial X_3} dX_1 + \frac{\partial \Phi_3}{\partial X_3} dX_2 + \frac{\partial \Phi_3}{\partial X_3} dX_3
\end{aligned}$$

$$\begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} \begin{bmatrix} dX_1 \\ dX_2 \\ dX_3 \end{bmatrix}$$

the components of $\underset{\sim}{\mathbf{F}}$ are dimensionless

Plan

Transformation of a material volume element

- volume element: initial dV and current dv

$$dV = \underline{\mathbf{dX}}_1 . (\underline{\mathbf{dX}}_2 \wedge \underline{\mathbf{dX}}_3) = [\underline{\mathbf{dX}}_1, \underline{\mathbf{dX}}_2, \underline{\mathbf{dX}}_3] = \det(\underline{\mathbf{dX}}_1, \underline{\mathbf{dX}}_2, \underline{\mathbf{dX}}_3)$$

$$dv = [\underline{\mathbf{dx}}_1, \underline{\mathbf{dx}}_2, \underline{\mathbf{dx}}_3] = [\underset{\sim}{\mathbf{F}} . \underline{\mathbf{dX}}_1, \underset{\sim}{\mathbf{F}} . \underline{\mathbf{dX}}_2, \underset{\sim}{\mathbf{F}} . \underline{\mathbf{dX}}_3]$$

$$dv = J\, dV$$

$$J = \det \underset{\sim}{\mathbf{F}} > 0$$

Jacobian of deformation

- the motion is **isochoric** at a point or at all points if $J = 1$
a material is **incompressible** if it can undergo only isochoric motions

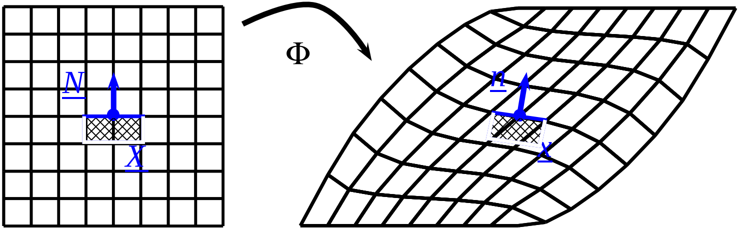
Mass conservation

$$\rho\, dv = \rho_0\, dV = \rho\, J\, dV \Longrightarrow \rho_0 = J\rho$$

$$\int_{\mathcal{D}(t)} \rho(\underline{\mathbf{x}}\,, t)\, dv = \int_{\mathcal{D}_0} \underbrace{\rho(\Phi(\underline{\mathbf{X}}\,, t), t)\, J}_{\rho_0(\underline{\mathbf{X}}\,)}\, dV$$

with $\mathcal{D}_0 = \Phi^{-1}(\mathcal{D}(t))$

Transformation of a material surface element



$$\underline{\mathbf{dS}} = \underline{\mathbf{dX}}_1 \wedge \underline{\mathbf{dX}}_3 = dS\,\underline{\mathbf{N}}, \quad \underline{\mathbf{ds}} = \underline{\mathbf{dx}}_1 \wedge \underline{\mathbf{dx}}_3 = ds\,\underline{\mathbf{n}}$$

the surface element is defined by orthogonal material directions $\underline{\mathbf{dX}}_1$ and $\underline{\mathbf{dX}}_3$. The surface element vector $\underline{\mathbf{dS}}$ does NOT transform like a line element:

$$\underline{\mathbf{ds}} = J\, \underset{\sim}{\mathbf{F}}^{-T}.\underline{\mathbf{dS}}$$

$\underline{\mathbf{dS}}$ et $\underline{\mathbf{ds}}$ (resp. $\underline{\mathbf{N}}$ and $\underline{\mathbf{n}}$) are not made of the same material points in the initial and current configurations

Plan

Polar decomposition of deformation gradient

For all invertible $\underset{\sim}{\mathbf{F}}$*, there exist two unique symmetric positive definite tensors* $\underset{\sim}{\mathbf{U}}$ *et* $\underset{\sim}{\mathbf{V}}$ *and a unique orthogonal tensor* $\underset{\sim}{\mathbf{R}}$ *such that*

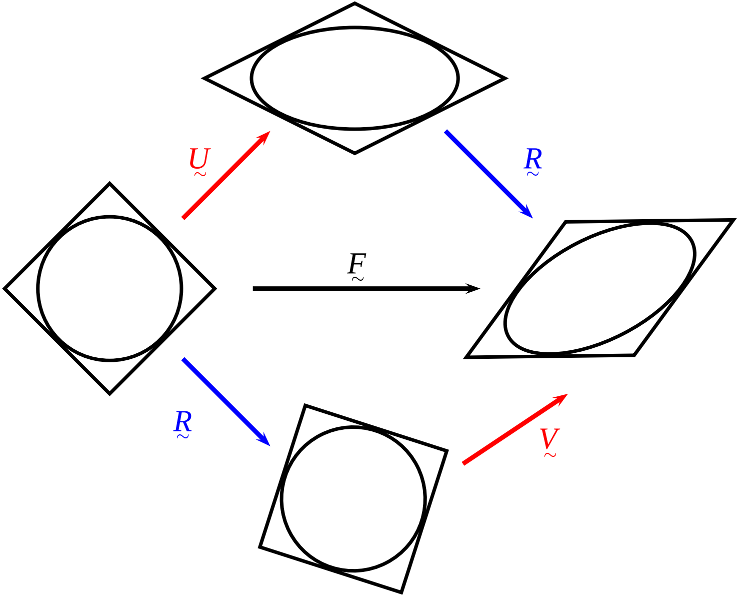
$$\underset{\sim}{\mathbf{F}} = \underset{\sim}{\mathbf{R}}.\underset{\sim}{\mathbf{U}} = \underset{\sim}{\mathbf{V}}.\underset{\sim}{\mathbf{R}}$$

If $\det \underset{\sim}{\mathbf{F}} > 0$, $\underset{\sim}{\mathbf{R}}$ *is a rotation (i.e.* $\det \underset{\sim}{\mathbf{R}} = +1$*).*

$\underset{\sim}{\mathbf{R}}$ polar rotation (3 degrees of freedom) $\qquad \underset{\sim}{\mathbf{R}}^{-1} = \underset{\sim}{\mathbf{R}}^T$

$\underset{\sim}{\mathbf{U}}, \underset{\sim}{\mathbf{V}}$ stretch tensors (6 degrees of freedom) $\qquad \underset{\sim}{\mathbf{U}}^T = \underset{\sim}{\mathbf{U}}, \quad \underset{\sim}{\mathbf{V}}^T = \underset{\sim}{\mathbf{V}}$

Polar decomposition of deformation gradient



Transformation of a principal triad of $\underset{\sim}{\mathbf{U}}$

- **spectral decomposition** of $\underset{\sim}{\mathbf{U}}$ et $\underset{\sim}{\mathbf{V}}$

$$\underset{\sim}{\mathbf{U}}.\underline{\mathbf{V}}_r = \lambda_r \underline{\mathbf{V}}_r, \quad \lambda_r > 0 \quad \text{(no sum)}, \quad \underset{\sim}{\mathbf{U}} = \sum_{r=1}^{3} \lambda_r \underline{\mathbf{V}}_r \otimes \underline{\mathbf{V}}_r$$

the eigen vectors are called **principal directions** or **principal axes** of $\underset{\sim}{\mathbf{U}}$, and eigen values are called **principal stretches**

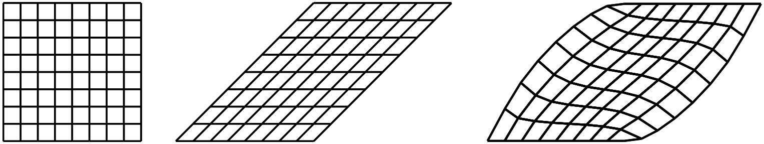
$$\underline{\mathbf{v}}_r = \underset{\sim}{\mathbf{R}}.\underline{\mathbf{V}}_r, \quad \underset{\sim}{\mathbf{V}} = \sum_{r=1}^{3} \lambda_r \underline{\mathbf{v}}_r \otimes \underline{\mathbf{v}}_r$$

- A principal triad $\underset{\sim}{\mathbf{U}}$ transforms into an orthogonal triad. The orientation of the deformed triad with respect to the initial triad is exactly given by polar rotation $\underset{\sim}{\mathbf{R}}$

Plan

Homogeneous deformation

- **homogeneous** deformation : $\underset{\sim}{\mathbf{F}}(\underline{\mathbf{X}}, t) = \underset{\sim}{\mathbf{F}}(t)$



initial homogeneous heterogeneous

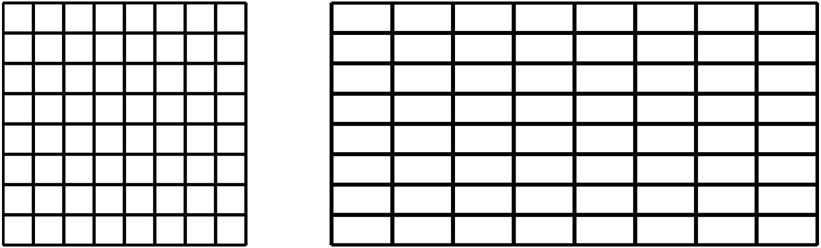
- corresponding motion / displacement field:

$$\underline{\mathbf{x}}(t) = \underset{\sim}{\mathbf{F}}(t).\underline{\mathbf{X}} + \underline{\mathbf{c}}(t)$$

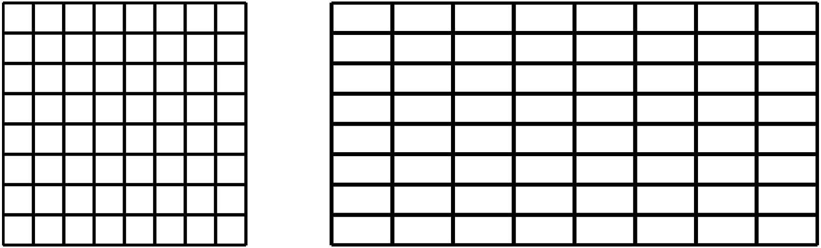
for any pair of material points (extensometry)

$$\underline{\mathbf{x}}_1 - \underline{\mathbf{x}}_2 = \underset{\sim}{\mathbf{F}}.(\underline{\mathbf{X}}_1 - \underline{\mathbf{X}}_2)$$

Pure extension



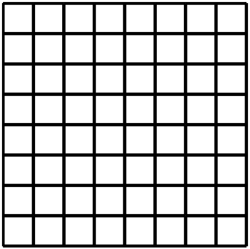
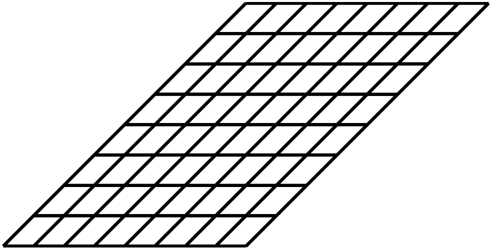
Pure extension



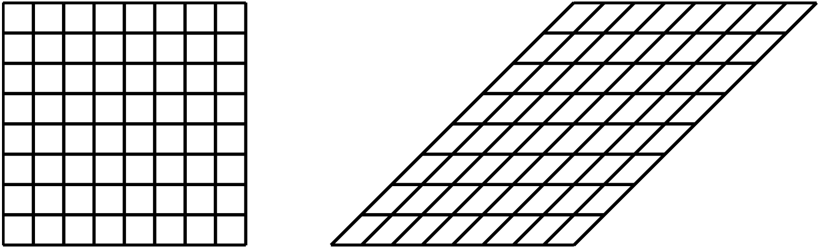
$$\begin{cases} x_1 = X_1(1+\lambda) \\ x_2 = X_2 \\ x_3 = X_3 \end{cases} \qquad \underset{\sim}{\mathbf{F}} = \underset{\sim}{\mathbf{1}} + \lambda \underline{\mathbf{e}}_1 \otimes \underline{\mathbf{e}}_1, \quad [\underset{\sim}{\mathbf{F}}] = \begin{bmatrix} 1+\lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{\mathbf{R}} = \underset{\sim}{\mathbf{1}}, \quad \underset{\sim}{\mathbf{U}} = \underset{\sim}{\mathbf{F}}$$

Simple glide

Simple glide



$$\left\{\begin{array}{l} x_1 = X_1 + \gamma X_2 \\ x_2 = X_2 \\ x_3 = X_3 \end{array}\right. , \quad \underset{\sim}{\mathbf{F}} = \underset{\sim}{\mathbf{1}} + \gamma \underline{\mathbf{e}}_1 \otimes \underline{\mathbf{e}}_2, \quad [\underset{\sim}{\mathbf{F}}] = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Simple glide

$$\underset{\sim}{\mathbf{U}} = \begin{bmatrix} \dfrac{1}{\sqrt{1+(\gamma/2)^2}} & \dfrac{\gamma}{2\sqrt{1+(\gamma/2)^2}} & 0 \\ \dfrac{\gamma}{2\sqrt{1+(\gamma/2)^2}} & \dfrac{1+\gamma^2/2}{\sqrt{1+(\gamma/2)^2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

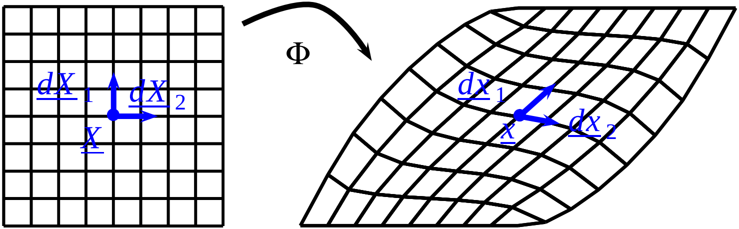
$$\underset{\sim}{\mathbf{V}} = \begin{bmatrix} \dfrac{1+\gamma^2/2}{\sqrt{1+(\gamma/2)^2}} & \dfrac{\gamma}{2\sqrt{1+(\gamma/2)^2}} & 0 \\ \dfrac{\gamma}{2\sqrt{1+(\gamma/2)^2}} & \dfrac{1}{\sqrt{1+(\gamma/2)^2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{\mathbf{R}} = \begin{bmatrix} \dfrac{1}{\sqrt{1+(\gamma/2)^2}} & \dfrac{\gamma}{2\sqrt{1+(\gamma/2)^2}} & 0 \\ \dfrac{-\gamma}{2\sqrt{1+(\gamma/2)^2}} & \dfrac{1}{\sqrt{1+(\gamma/2)^2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

the polar rotation is a rotation with respect to axis $\underline{\mathbf{e}}_3$ and angle $\tan\theta = -\frac{\gamma}{2}$

Plan

Cauchy–Green tensors



$$\underline{\mathbf{dx}}_1.\underline{\mathbf{dx}}_2 = (\underset{\sim}{\mathbf{F}}.\underline{\mathbf{dX}}_1).(\underset{\sim}{\mathbf{F}}.\underline{\mathbf{dX}}_2) = \underline{\mathbf{dX}}_1.\underset{\sim}{\mathbf{F}}^T.\underset{\sim}{\mathbf{F}}.\underline{\mathbf{dX}}_2 = \underline{\mathbf{dX}}_1.\underset{\sim}{\mathbf{C}}.\underline{\mathbf{dX}}_2$$

right Cauchy–Green tensor $\underset{\sim}{\mathbf{C}} = \underset{\sim}{\mathbf{F}}^T.\underset{\sim}{\mathbf{F}}$ induces a metric on Ω_0

$$\underline{\mathbf{dX}}_1.\underline{\mathbf{dX}}_2 = \underline{\mathbf{dx}}_1.\underset{\sim}{\mathbf{B}}^{-1}.\underline{\mathbf{dx}}_2$$

left Cauchy–Green tensor $\underset{\sim}{\mathbf{B}} = \underset{\sim}{\mathbf{F}}.\underset{\sim}{\mathbf{F}}^T$ induces a metric on Ω_t

($\underset{\sim}{\mathbf{C}}$ et $\underset{\sim}{\mathbf{B}}$ are symmetric positive definite, $\underset{\sim}{\mathbf{B}} \neq \underset{\sim}{\mathbf{C}}^T$!!)

Changes in length

- length changes

$$\|\underline{\mathbf{dx}}\|^2 - \|\underline{\mathbf{dX}}\|^2 = \underline{\mathbf{dX}}.(\underset{\sim}{\mathbf{C}} - \underset{\sim}{\mathbf{1}}).\underline{\mathbf{dX}} = \underline{\mathbf{dx}}.(\underset{\sim}{\mathbf{1}} - \underset{\sim}{\mathbf{B}}^{-1}).\underline{\mathbf{dx}}$$

- relative elongation

$$\underline{\mathbf{dX}} = \|\underline{\mathbf{dX}}\|\,\underline{\mathbf{M}}$$

$$\lambda(\underline{\mathbf{M}}) = \frac{\|\underline{\mathbf{dx}}\|}{\|\underline{\mathbf{dX}}\|} = \sqrt{\underline{\mathbf{M}}.\underset{\sim}{\mathbf{C}}.\underline{\mathbf{M}}} = \|\underset{\sim}{\mathbf{F}}.\underline{\mathbf{M}}\| = \|\underset{\sim}{\mathbf{U}}.\underline{\mathbf{M}}\|$$

- interpretation of the components of $\underset{\sim}{\mathbf{C}}$

$$\lambda(\underline{\mathbf{e}}_1) = \sqrt{C_{11}} = \sqrt{F_{11}^2 + F_{21}^2 + F_{31}^2}$$

C_{11} is the square of the relative elongation of the first basis vector

Changes in angles

- variation of the angle between two material line elements

$$\underline{\mathbf{dX}}_1 = |\underline{\mathbf{dX}}_1|\, \underline{\mathbf{M}}_1, \quad \underline{\mathbf{dX}}_2 = |\underline{\mathbf{dX}}_2|\, \underline{\mathbf{M}}_2$$

$$\underline{\mathbf{dx}}_1 = |\underline{\mathbf{dx}}_1|\, \underline{\mathbf{m}}_1, \quad \underline{\mathbf{dx}}_2 = |\underline{\mathbf{dx}}_2|\, \underline{\mathbf{m}}_2$$

$$\cos\Theta = \underline{\mathbf{M}}_1.\underline{\mathbf{M}}_2$$

$$\cos\theta = \underline{\mathbf{m}}_1.\underline{\mathbf{m}}_2 = \frac{\underline{\mathbf{M}}_1.\underset{\sim}{\mathbf{C}}.\underline{\mathbf{M}}_2}{\lambda(\underline{\mathbf{M}}_1)\lambda(\underline{\mathbf{M}}_2)}$$

- glide angle γ

$$\gamma := \Theta - \theta$$

If $\Theta = \pi/2$ (initially orthogonal directions)

$$\sin\gamma = \frac{\underline{\mathbf{M}}_1.\underset{\sim}{\mathbf{C}}.\underline{\mathbf{M}}_2}{\lambda(\underline{\mathbf{M}}_1)\lambda(\underline{\mathbf{M}}_2)}$$

- interpretation of the components of $\underset{\sim}{\mathbf{C}}$: $\underline{\mathbf{M}}_1 = \underline{\mathbf{E}}_1$ et $\underline{\mathbf{M}}_2 = \underline{\mathbf{E}}_2$

$$\sin\gamma = \frac{C_{12}}{\sqrt{C_{11}C_{22}}}$$

Rigid body motion

when the distance between any two material points does not change in the motion :

$$\forall \underline{\mathbf{X}}_1, \underline{\mathbf{X}}_2 \neq 0, \quad \|\underline{\mathbf{x}}_1 - \underline{\mathbf{x}}_2\| = \|\underline{\mathbf{X}}_1 - \underline{\mathbf{X}}_2\|$$

$$(\underset{\sim}{\mathbf{F}}.(\underline{\mathbf{X}}_1 - \underline{\mathbf{X}}_2))^2 = (\underline{\mathbf{X}}_1 - \underline{\mathbf{X}}_2).\underset{\sim}{\mathbf{F}}^T.\underset{\sim}{\mathbf{F}}.(\underline{\mathbf{X}}_1 - \underline{\mathbf{X}}_2) = (\underline{\mathbf{X}}_1 - \underline{\mathbf{X}}_2).(\underline{\mathbf{X}}_1 - \underline{\mathbf{X}}_2)$$

$$\Longrightarrow \quad \underset{\sim}{\mathbf{F}}^T.\underset{\sim}{\mathbf{F}} = \underset{\sim}{\mathbf{C}} = \underset{\sim}{\mathbf{1}}$$

The deformation gradient is a rotation $\underset{\sim}{\mathbf{Q}}(t)$. The corresponding motion is

$$\underline{\mathbf{x}} = \underset{\sim}{\mathbf{Q}}(t).\underline{\mathbf{X}} + \underline{\mathbf{c}}(t)$$

Strain measures

- candidates

$$\underset{\sim}{\mathbf{C}}, \underset{\sim}{\mathbf{B}}, \underset{\sim}{\mathbf{U}}, \underset{\sim}{\mathbf{V}}$$

- additional rules for defining a strain measure:
  - ⋆ symmetric and dimensionless;
  - ⋆ vanish for a rigid body motion and when $\underset{\sim}{\mathbf{F}} = \underset{\sim}{\mathbf{1}}$;
  - ⋆ the Taylor expansion around $\underset{\sim}{\mathbf{F}} = \underset{\sim}{\mathbf{1}}$ is $\frac{1}{2}(\underset{\sim}{\mathbf{H}} + \underset{\sim}{\mathbf{H}}^T) + o(\underset{\sim}{\mathbf{H}})$

$$\underset{\sim}{\mathbf{H}} = \underset{\sim}{\mathbf{F}} - \underset{\sim}{\mathbf{1}} = \mathrm{Grad}\,\underline{\mathbf{u}}$$

- Green–Lagrange and Almansi tensors

$$\underset{\sim}{\mathbf{E}} := \frac{1}{2}(\underset{\sim}{\mathbf{C}} - \underset{\sim}{\mathbf{1}}), \quad \underset{\sim}{\mathbf{A}} := \frac{1}{2}(\underset{\sim}{\mathbf{1}} - \underset{\sim}{\mathbf{B}}^{-1})$$

$$\underset{\sim}{\mathbf{E}} = \frac{1}{2}(\underset{\sim}{\mathbf{H}} + \underset{\sim}{\mathbf{H}}^T + \underset{\sim}{\mathbf{H}}^T.\underset{\sim}{\mathbf{H}})$$

$$E_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j})$$

- **Lagrangean/Eulerian** strain measures

Hill's strain measures

$$\underset{\sim}{\mathbf{E}}_n := \frac{1}{n}(\underset{\sim}{\mathbf{U}}^n - \underset{\sim}{\mathbf{1}}), \quad \underset{\sim}{\mathbf{A}}_n := \frac{1}{n}(\underset{\sim}{\mathbf{V}}^n - \underset{\sim}{\mathbf{1}})$$

In particular,

$$\underset{\sim}{\mathbf{E}}_2 = \underset{\sim}{\mathbf{E}}, \quad \underset{\sim}{\mathbf{A}}_{-2} = \underset{\sim}{\mathbf{A}}$$

Logarithmic strain tensor ($n = 0$):

$$\underset{\sim}{\mathbf{E}}_0 := \log \underset{\sim}{\mathbf{U}}, \quad \underset{\sim}{\mathbf{A}}_0 := \log \underset{\sim}{\mathbf{V}}$$

Logarithm of a symmetric positive definite tensor:

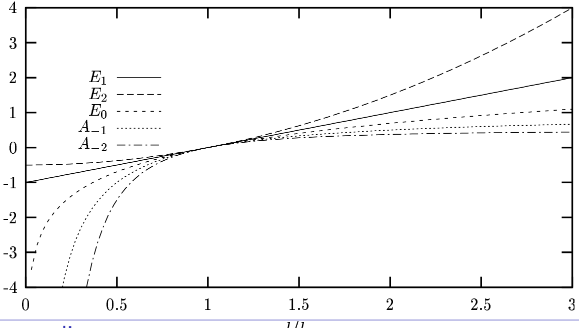
$$\underset{\sim}{\mathbf{U}} = \sum_{r=1}^{3} \lambda_r \underline{\mathbf{V}}_r \otimes \underline{\mathbf{V}}_r \Longrightarrow \log \underset{\sim}{\mathbf{U}} := \sum_{r=1}^{3} (\log \lambda_r) \underline{\mathbf{V}}_r \otimes \underline{\mathbf{V}}_r$$

Extensometry

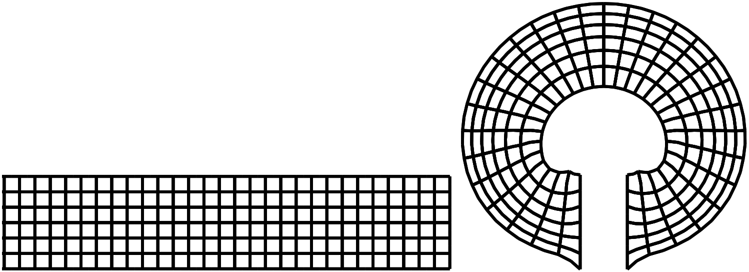
$$C = \left(\frac{l}{l_0}\right)^2, \quad B = \left(\frac{l}{l_0}\right)^2, \quad E_1 = \frac{l - l_0}{l_0}, \quad A_{-1} = \frac{l - l_0}{l}$$

$$E_2 = \frac{1}{2}\left(\left(\frac{l}{l_0}\right)^2 - 1\right), \quad A_{-2} = \frac{1}{2}\left(1 - \left(\frac{l_0}{l}\right)^2\right)$$

$$E_0 = \log\frac{l}{l_0}, \quad A_0 = \log\frac{l}{l_0}$$

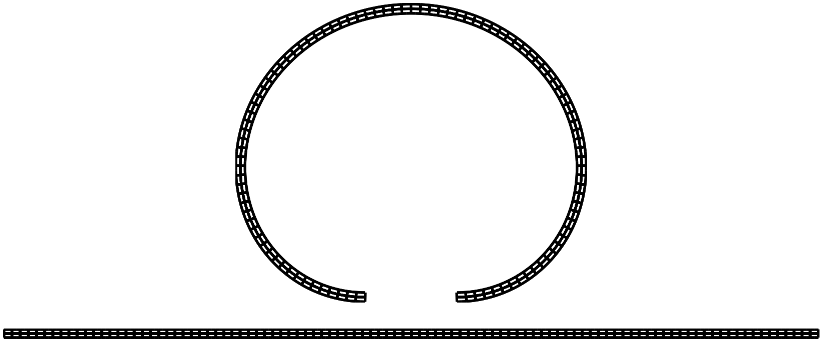


Large strains / Large rotations



$$\underset{\sim}{\mathbf{F}} = \underset{\sim}{\mathbf{R}}.\underset{\sim}{\mathbf{U}}$$

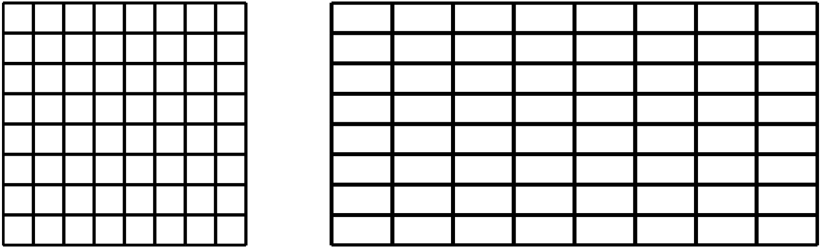
Small strains / Large rotations



$$\underset{\sim}{\mathbf{U}} \simeq \underset{\sim}{\mathbf{1}} + \underset{\sim}{\mathbf{E}}, \quad \|\underset{\sim}{\mathbf{E}}\| \ll 1$$

for slender bodies in one or two directions (beams, plates and shells...), "large deformation" does not necessarily imply "large strain"...

Large strains / Small rotations



representation of small rotations

$$[\underset{\sim}{\mathbf{R}}] = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \simeq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -\phi & 0 \\ \phi & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underset{\sim}{\mathbf{R}} \simeq \underset{\sim}{\mathbf{1}} + \underset{\sim}{\boldsymbol{\omega}}$$

$\underset{\sim}{\boldsymbol{\omega}}$ skew–symmetric tensor : $\underset{\sim}{\boldsymbol{\omega}}^T = -\underset{\sim}{\boldsymbol{\omega}}$

Small strains / small rotations

$$\underset{\sim}{\mathbf{H}} = \mathrm{Grad}\,\underline{\mathbf{u}}\,, \quad \underset{\sim}{\mathbf{F}} = \underset{\sim}{\mathbf{1}} + \underset{\sim}{\mathbf{H}} = \underset{\sim}{\mathbf{1}} + \underset{\sim}{\varepsilon} + \underset{\sim}{\omega}$$

$$\underset{\sim}{\varepsilon} = \frac{1}{2}(\underset{\sim}{\mathbf{H}} + \underset{\sim}{\mathbf{H}}^T), \quad \underset{\sim}{\omega} = \frac{1}{2}(\underset{\sim}{\mathbf{H}} - \underset{\sim}{\mathbf{H}}^T)$$

- context of infinitesimal deformation (with respect to a given observer) :

$$\|\underset{\sim}{\mathbf{H}} = \mathrm{Grad}\,\underline{\mathbf{u}}\,\| \ll 1 \Longleftrightarrow \underset{\sim}{\mathbf{F}} = \mathcal{O}(\underset{\sim}{\mathbf{1}})$$

small deformation = small strain + small rotation

$$\underset{\sim}{\mathbf{F}} = \underset{\sim}{\mathbf{1}} + \underset{\sim}{\varepsilon} + \underset{\sim}{\omega} \simeq (\underset{\sim}{\mathbf{1}} + \underset{\sim}{\omega}).(\underset{\sim}{\mathbf{1}} + \underset{\sim}{\varepsilon})$$

small strain strain tensor $\qquad \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

small rotation tensor $\qquad \omega_{ij} := \frac{1}{2}(u_{i,j} - u_{j,i})$

- Caution! one can always compute $\underset{\sim}{\varepsilon}$ but it has a physical meaning only within the infinitesimal context...

$$\underset{\sim}{\mathbf{F}} = \underset{\sim}{\mathbf{Q}} \Longrightarrow \underset{\sim}{\mathbf{C}} = \underset{\sim}{\mathbf{1}}, \quad \underset{\sim}{\mathbf{E}} = 0 \quad \text{mais} \quad \underset{\sim}{\varepsilon} = \frac{1}{2}(\underset{\sim}{\mathbf{Q}} + \underset{\sim}{\mathbf{Q}}^T) - \underset{\sim}{\mathbf{1}} \neq 0\,!!!$$

- $\underset{\sim}{\mathbf{C}} \simeq \underset{\sim}{\mathbf{1}} + 2\underset{\sim}{\varepsilon}, \quad \underset{\sim}{\mathbf{E}} \simeq \underset{\sim}{\varepsilon}$

Plan

Summary: Deformation of material line, surface and volume elements

material line element: $\underline{\mathbf{dx}} = \underset{\sim}{\mathbf{F}}.\underline{\mathbf{dX}}$

material surface element: $\underline{\mathbf{ds}} = J\,\underset{\sim}{\mathbf{F}}^{-T}\,\underline{\mathbf{dS}}$

material volume element: $dv = J\,dV$

Summary: Finite deformation

$\underset{\sim}{\mathbf{F}} = \underset{\sim}{\mathbf{R}}.\underset{\sim}{\mathbf{U}} = \underset{\sim}{\mathbf{V}}.\underset{\sim}{\mathbf{R}}$	deformation gradient ($\det \underset{\sim}{\mathbf{F}} > 0$)
$\underset{\sim}{\mathbf{R}}$	polar rotation ($\det \underset{\sim}{\mathbf{R}} = 1$)
$\underset{\sim}{\mathbf{U}}$	right stretch tensor
$\underset{\sim}{\mathbf{V}}$	left stretch tensor
$\underset{\sim}{\mathbf{C}} := \underset{\sim}{\mathbf{F}}^T.\underset{\sim}{\mathbf{F}} = \underset{\sim}{\mathbf{U}}^2$	right Cauchy–Green tensor
$\underset{\sim}{\mathbf{B}} := \underset{\sim}{\mathbf{F}}.\underset{\sim}{\mathbf{F}}^T = \underset{\sim}{\mathbf{V}}^2$	left Cauchy–Green tensor
$\underset{\sim}{\mathbf{E}} := \frac{1}{2}(\underset{\sim}{\mathbf{C}} - \underset{\sim}{\mathbf{1}})$	Green–Lagrange strain measure
$\underset{\sim}{\mathbf{A}} := \frac{1}{2}(\underset{\sim}{\mathbf{1}} - \underset{\sim}{\mathbf{B}}^{-1})$	Almansi strain measure

Summary: Infinitesimal deformation

$\underset{\sim}{\mathbf{H}} = \underset{\sim}{\varepsilon} + \underset{\sim}{\omega} = \mathrm{Grad}\,\underline{\mathbf{u}}$	displacement gradient
$\underset{\sim}{\varepsilon} = \dfrac{1}{2}(\mathrm{Grad}\,\underline{\mathbf{u}} + (\mathrm{Grad}\,\underline{\mathbf{u}})^T)$	small strain tensor
$\underset{\sim}{\omega} = \dfrac{1}{2}(\mathrm{Grad}\,\underline{\mathbf{u}} - (\mathrm{Grad}\,\underline{\mathbf{u}})^T)$	small rotation tensor
$\underset{\sim}{\mathbf{F}} = \underset{\sim}{\mathbf{1}} + \underset{\sim}{\varepsilon} + \underset{\sim}{\omega} \simeq (\underset{\sim}{\mathbf{1}} + \underset{\sim}{\varepsilon}).(\underset{\sim}{\mathbf{1}} + \underset{\sim}{\omega})$,	$\underset{\sim}{\mathbf{C}} \simeq \underset{\sim}{\mathbf{1}} + 2\underset{\sim}{\varepsilon} \simeq \underset{\sim}{\mathbf{B}}, \quad \underset{\sim}{\mathbf{E}} \simeq \underset{\sim}{\varepsilon}$
$\dfrac{\lvert\underline{\mathbf{dx}}\rvert - \lvert\underline{\mathbf{dX}}\rvert}{\lvert\underline{\mathbf{dX}}\rvert} \simeq \underline{\mathbf{M}}.\underset{\sim}{\varepsilon}.\underline{\mathbf{M}}$	infinitesimal elongation
$\dfrac{dv - dV}{dV} \simeq \mathrm{div}\,\underline{\mathbf{u}} = \mathrm{trace}\,\underset{\sim}{\varepsilon}$	infinitesimal volume change

Plan

Plan

Velocity field

- Notations

$$\underline{\dot{\mathbf{x}}} = \frac{d\underline{\mathbf{x}}}{dt} = \frac{\partial \Phi}{\partial t}(\underline{\mathbf{X}}\,,t) = \underline{\mathbf{V}}\,(\underline{\mathbf{X}}\,,t)$$

- Lagranfean/Eulerian representations

$$\underline{\mathbf{v}}\,(\underline{\mathbf{x}}\,,t) := \underline{\mathbf{V}}\,(\Phi^{-1}(\underline{\mathbf{x}}\,,t),t)$$

more generally

$$f(\underline{\mathbf{x}}\,,t) := F(\underline{\mathbf{X}}\,,t), \quad \text{avec} \quad \underline{\mathbf{x}} = \Phi(\underline{\mathbf{X}}\,,t)$$

- convective time derivative

$$\begin{aligned}\dot{F}(\underline{\mathbf{X}}\,,t) &:= \frac{d}{dt}F(\underline{\mathbf{X}}\,,t) = \frac{\partial F}{\partial t}(\underline{\mathbf{X}}\,,t)\\ &= \frac{d}{dt}f(\underline{\mathbf{x}}\,,t) = \frac{\partial f}{\partial t}(\underline{\mathbf{x}}\,,t) + \frac{\partial f}{\partial \underline{\mathbf{x}}}.\underline{\mathbf{v}}\,(\underline{\mathbf{x}}\,,t) = \dot{f}(\underline{\mathbf{x}}\,,t)\end{aligned}$$

Velocity gradient field

- instantaneous evolution of a material line element

$$\underline{\mathbf{dx}} = \underset{\sim}{\mathbf{F}}.\underline{\mathbf{dX}}$$

$$\overbrace{\underline{\mathbf{dx}}}^{\bullet} = \underset{\sim}{\mathbf{L}}.\underline{\mathbf{dx}}\,, \quad \text{with} \quad \underset{\sim}{\mathbf{L}} = \underset{\sim}{\dot{\mathbf{F}}}.\underset{\sim}{\mathbf{F}}^{-1}$$

- velocity gradient tensor

$$\begin{aligned} \underset{\sim}{\dot{\mathbf{F}}} &= \frac{\partial^2 \Phi}{\partial t \partial \underline{\mathbf{X}}}(\underline{\mathbf{X}}\,, t) = \frac{\partial^2 \Phi}{\partial \underline{\mathbf{X}}\, \partial t}(\underline{\mathbf{X}}\,, t) \\ &= \mathrm{Grad}\, \underline{\mathbf{V}}\,(\underline{\mathbf{X}}\,, t) = (\mathrm{grad}\, \underline{\mathbf{v}}\,(\underline{\mathbf{x}}\,, t)).\underset{\sim}{\mathbf{F}} \end{aligned}$$

$$\underset{\sim}{\mathbf{L}}(\underline{\mathbf{x}}\,, t) = \mathrm{grad}\, \underline{\mathbf{v}}\,(\underline{\mathbf{x}}\,, t) = \underset{\sim}{\dot{\mathbf{F}}}.\underset{\sim}{\mathbf{F}}^{-1}$$

Strain rate tensor

- instantaneous evolution of the scalar product of two material line elements

  on the one hand...

$$\overbrace{\underline{\mathbf{dx}}_1.\underline{\mathbf{dx}}_2}^{\bullet} = \underline{\mathbf{dx}}_1.\underset{\sim}{\mathbf{L}}^T.\underline{\mathbf{dx}}_2 + \underline{\mathbf{dx}}_1.\underset{\sim}{\mathbf{L}}.\underline{\mathbf{dx}}_2 = 2\underline{\mathbf{dx}}_1.\underset{\sim}{\mathbf{D}}.\underline{\mathbf{dx}}_2$$

  ... and on the other hand

$$\overbrace{\underline{\mathbf{dx}}_1.\underline{\mathbf{dx}}_2}^{\bullet} = \overbrace{\underline{\mathbf{dX}}_1.\underset{\sim}{\mathbf{C}}.\underline{\mathbf{dX}}_2}^{\bullet} = \underline{\mathbf{dX}}_1.\underset{\sim}{\dot{\mathbf{C}}}.\underline{\mathbf{dX}}_2 = 2\underline{\mathbf{dX}}_1.\underset{\sim}{\dot{\mathbf{E}}}.\underline{\mathbf{dX}}_2$$

  hence ...

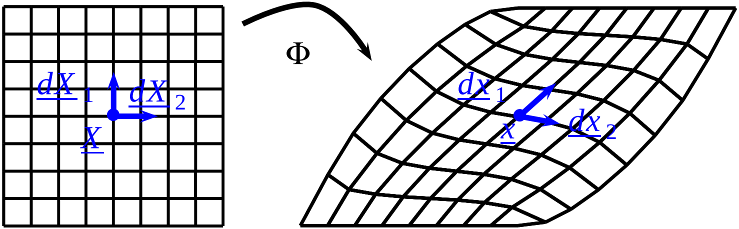
$$\underset{\sim}{\dot{\mathbf{E}}} = \frac{1}{2}\underset{\sim}{\dot{\mathbf{C}}} = \underset{\sim}{\mathbf{F}}^T.\underset{\sim}{\mathbf{D}}.\underset{\sim}{\mathbf{F}}, \quad \underset{\sim}{\mathbf{D}} := \frac{1}{2}(\underset{\sim}{\mathbf{L}} + \underset{\sim}{\mathbf{L}}^T)$$

  strain rate tensor

- elongation rate :

$$\underline{\mathbf{dx}} = \|\underline{\mathbf{dx}}\|\ \underline{\mathbf{m}}, \quad \underline{\mathbf{m}}\ \text{unit vector} \qquad \frac{\dot{\lambda}}{\lambda} = \frac{\overbrace{\|\underline{\mathbf{dx}}\|}^{\bullet}}{\|\underline{\mathbf{dx}}\|} = \underline{\mathbf{m}}.\underset{\sim}{\mathbf{D}}.\underline{\mathbf{m}}$$

Slip rate



- glide angle : $\gamma = \Theta - \theta$

$$\dot{\gamma} = -\dot{\theta}$$

$$\overbrace{\underline{\mathbf{dx}}_1.\underline{\mathbf{dx}}_2}^{\bullet} = \overbrace{\|\underline{\mathbf{dx}}_1\|\,\|\underline{\mathbf{dx}}_2\|\,\cos\theta}^{\bullet} = 2\underline{\mathbf{dx}}_1.\underset{\sim}{\mathbf{D}}.\underline{\mathbf{dx}}_2$$

If $\theta = \frac{\pi}{2}$ at time t,

$$\dot{\gamma} = 2\underline{\mathbf{m}}_1.\underset{\sim}{\mathbf{D}}.\underline{\mathbf{m}}_2$$

o $\underline{\mathbf{m}}_1 = \underline{\mathbf{dx}}_1/\|\underline{\mathbf{dx}}_1\|, \quad \underline{\mathbf{m}}_2 = \underline{\mathbf{dx}}_2/\|\underline{\mathbf{dx}}_2\|$

- particular case, $\underline{\mathbf{m}}_1 = \underline{\mathbf{e}}_1, \quad \underline{\mathbf{m}}_2 = \underline{\mathbf{e}}_2 \Longrightarrow \dot{\gamma} = 2D_{12}$

Plan

Spin tensor

- instantaneous evolution of the orientation of a material line element $\underline{\mathbf{m}} = \underline{\mathbf{dx}} / \|\underline{\mathbf{dx}}\|$

$$\dot{\underline{\mathbf{m}}} = \underset{\sim}{\mathbf{L}}.\underline{\mathbf{m}} - (\underline{\mathbf{m}}.\underset{\sim}{\mathbf{D}}.\underline{\mathbf{m}})\underline{\mathbf{m}}$$

- If $\underline{\mathbf{m}}$ is parallel to a principal vector of $\underset{\sim}{\mathbf{D}}$

$$\underset{\sim}{\mathbf{W}} := \underset{\sim}{\mathbf{L}} - \underset{\sim}{\mathbf{D}} = \frac{1}{2}(\underset{\sim}{\mathbf{L}} - \underset{\sim}{\mathbf{L}}^T)$$

$$\dot{\underline{\mathbf{m}}} = \underset{\sim}{\mathbf{W}}.\underline{\mathbf{m}}$$

  spin tensor

- Consequence : *The orthonormal triad of material vectors coinciding at time t with the triad of unit eigenvectors of $\underset{\sim}{\mathbf{D}}$ transforms like a rigid body with the rotation rate $\underset{\sim}{\mathbf{W}}$ at time t*
- Caution! The triad of eigenvectors of $\underset{\sim}{\mathbf{D}}$ generally does not rotate at the rate $\underset{\sim}{\mathbf{W}}$... (counterexample: simple glide)

Decomposition of the velocity gradient tensor

- strain rate + spin

$$\underset{\sim}{\mathbf{L}} = \underset{\sim}{\mathbf{D}} + \underset{\sim}{\mathbf{W}}$$

  symmetric and skew–symmetric parts

- polar decomposition

$$\underset{\sim}{\mathbf{F}} = \underset{\sim}{\mathbf{R}}.\underset{\sim}{\mathbf{U}}$$

$$\underset{\sim}{\mathbf{L}} = \underset{\sim}{\dot{\mathbf{F}}}.\underset{\sim}{\mathbf{F}}^{-1} = \underset{\sim}{\dot{\mathbf{R}}}.\underset{\sim}{\mathbf{R}}^{T} + \underset{\sim}{\mathbf{R}}.\underset{\sim}{\dot{\mathbf{U}}}.\underset{\sim}{\mathbf{U}}^{-1}.\underset{\sim}{\mathbf{R}}^{T}$$

  Caution! the last term is generally not symmetric... In general,

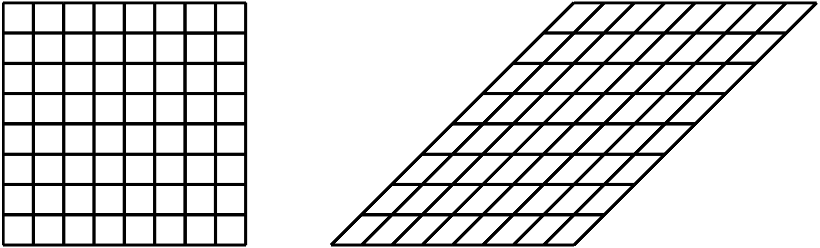
$$\underset{\sim}{\mathbf{W}} \neq \underset{\sim}{\dot{\mathbf{R}}}.\underset{\sim}{\mathbf{R}}^{T}$$

- spin vector $\quad \forall \underline{\mathbf{y}}\,,\quad \underset{\sim}{\mathbf{W}}.\underline{\mathbf{y}} = \overset{\times}{\underline{\mathbf{W}}} \wedge \underline{\mathbf{y}}$

$$\left\{\begin{array}{l} \overset{\times}{W}_1 = -W_{23} = \frac{1}{2}(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}) \\ \overset{\times}{W}_2 = -W_{31} = \frac{1}{2}(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}) \\ \overset{\times}{W}_3 = -W_{12} = \frac{1}{2}(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}) \end{array}\right. \quad , \quad \overset{\times}{\underline{\mathbf{W}}} = \frac{1}{2}\mathrm{rot}\,\underline{\mathbf{v}}$$

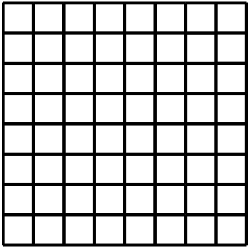
Plan

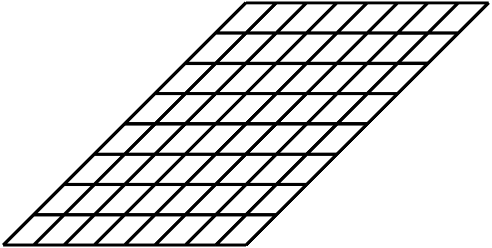
Simple glide



$$[\underset{\sim}{\mathbf{L}}] = \begin{bmatrix} 0 & \dot{\gamma} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad [\underset{\sim}{\mathbf{D}}] = \begin{bmatrix} 0 & \dfrac{\dot{\gamma}}{2} & 0 \\ \dfrac{\dot{\gamma}}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad [\underset{\sim}{\mathbf{W}}] = \begin{bmatrix} 0 & \dfrac{\dot{\gamma}}{2} & 0 \\ -\dfrac{\dot{\gamma}}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Simple glide





$$[\underset{\sim}{\mathbf{W}}] = \begin{bmatrix} 0 & \dfrac{\dot\gamma}{2} & 0 \\ -\dfrac{\dot\gamma}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [\underset{\sim}{\mathbf{R}}] = \begin{bmatrix} \frac{1}{\sqrt{1+(\gamma/2)^2}} & \frac{\gamma}{2\sqrt{1+(\gamma/2)^2}} & 0 \\ \frac{-\gamma}{2\sqrt{1+(\gamma/2)^2}} & \frac{1}{\sqrt{1+(\gamma/2)^2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\dot\theta_W = -\frac{\dot\gamma}{2}, \quad \tan\theta_R = -\frac{\gamma}{2}, \quad \dot\theta_R = -\frac{\dot\gamma}{2}\frac{1}{1+\gamma^2/4}$$

Single vortex

Single vortex

- kinematicse

$$\underline{\mathbf{v}}(r, \theta, z, t) = \frac{\Gamma}{2\pi r}\underline{\mathbf{e}}_{\theta}$$

Current lines are circles arount the vortex center

- velocity gradient

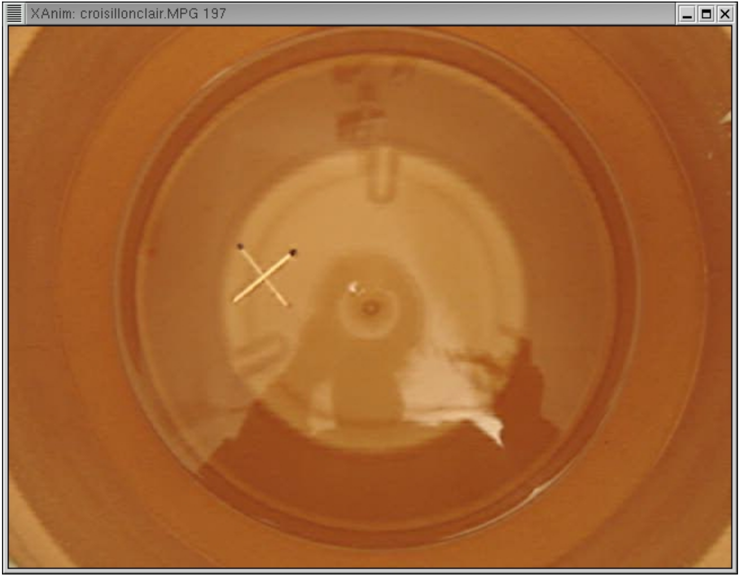
$$\underset{\sim}{\mathbf{L}} = -\frac{\Gamma}{2\pi r^2}(\underline{\mathbf{e}}_r \otimes \underline{\mathbf{e}}_{\theta} + \underline{\mathbf{e}}_{\theta} \otimes \underline{\mathbf{e}}_r)$$

- irrotational flow

$$\underset{\sim}{\mathbf{W}} = 0$$

- circulation of $\underline{\mathbf{v}}$ around O $\qquad \oint \underline{\mathbf{v}} . \underline{\mathbf{e}}_{\theta}\, r d\theta = \Gamma$

Vorticimeter



Vorticimeter (1)

- unit directions characterising the cross $\underline{\mathbf{m}}_1$ et $\underline{\mathbf{m}}_2$

$$\underline{\dot{\mathbf{m}}}_1 = \underset{\sim}{\mathbf{L}}.\underline{\mathbf{m}}_1 - (\underline{\mathbf{m}}_1.\underset{\sim}{\mathbf{D}}.\underline{\mathbf{m}}_1)\underline{\mathbf{m}}_1$$

$$\underline{\dot{\mathbf{m}}}_2 = \underset{\sim}{\mathbf{L}}.\underline{\mathbf{m}}_2 - (\underline{\mathbf{m}}_2.\underset{\sim}{\mathbf{D}}.\underline{\mathbf{m}}_2)\underline{\mathbf{m}}_2$$

- Evolution of the angle between one match and a fixed direction in space $\underline{\mathbf{a}}$

$$-\sin\varphi_1\,\dot{\varphi}_1 = \underline{\dot{\mathbf{m}}}_1.\underline{\mathbf{a}} = \underline{\mathbf{a}}.\underset{\sim}{\mathbf{L}}.\underline{\mathbf{m}}_1 - (\underline{\mathbf{m}}_1.\underset{\sim}{\mathbf{D}}.\underline{\mathbf{m}}_1)\,\underline{\mathbf{m}}_1.\underline{\mathbf{a}}$$

The choice of $\underline{\mathbf{a}}$ does not matter if we are interested in $\dot{\varphi}$. Take

$$\varphi_1 = (\underline{\mathbf{a}} = \underline{\mathbf{m}}_2, \underline{\mathbf{m}}_1) = -\frac{\pi}{2} \Longrightarrow \dot{\varphi}_1 = \underline{\mathbf{m}}_2.\underset{\sim}{\mathbf{L}}.\underline{\mathbf{m}}_1$$

$$\varphi_2 = (\underline{\mathbf{a}} = \underline{\mathbf{m}}_1, \underline{\mathbf{m}}_2) = \frac{\pi}{2} \Longrightarrow \dot{\varphi}_2 = \underline{\mathbf{m}}_1.\underset{\sim}{\mathbf{L}}.\underline{\mathbf{m}}_2$$

Vorticimeter (2)

- For a rigid cross ($\underline{\mathbf{m}}_1.\underline{\mathbf{m}}_2 = 0$ at each time), the spin of the cross is the mean value of the spin of the matches :

$$
\begin{aligned}
\dot{\varphi} &= \frac{\dot{\varphi}_1 + \dot{\varphi}_2}{2} = \underline{\mathbf{m}}_2.\underset{\sim}{\mathbf{W}}.\underline{\mathbf{m}}_1 \\
&= \underline{\mathbf{m}}_2.(\overset{\times}{\underline{\mathbf{W}}} \wedge \underline{\mathbf{m}}_1) = \overset{\times}{\underline{\mathbf{W}}} \wedge (\underline{\mathbf{m}}_1 \wedge \underline{\mathbf{m}}_2) = \overset{\times}{\underline{\mathbf{W}}} .\underline{\mathbf{e}}_z
\end{aligned}
$$

- The spin of the rigid cross is exactly that of the spin tensor of the fluid. The vorticimeter can used to measure $\underset{\sim}{\mathbf{W}}$.
- For a simple vortex, $\underset{\sim}{\mathbf{W}} = 0$. The cross does not rotate...

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Analogy $\underset{\sim}{\mathbf{D}} \longleftrightarrow \underset{\sim}{\boldsymbol{\varepsilon}}$

	strain rate $\underset{\sim}{\mathbf{D}}$ (general case)	small strain $\underset{\sim}{\boldsymbol{\varepsilon}}$ (infinitesimal context)
symmetric gradient operator	$\underset{\sim}{\mathbf{D}} = \frac{1}{2}(\mathrm{grad}\,\underline{\mathbf{v}} + \mathrm{grad}\,\underline{\mathbf{v}})$ $D_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i})$	$\underset{\sim}{\boldsymbol{\varepsilon}} = \frac{1}{2}(\mathrm{Grad}\,\underline{\mathbf{u}} + \mathrm{Grad}\,\underline{\mathbf{u}})$ $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$
volume change	$\dfrac{\overset{\bullet}{\overbrace{dv}}}{dv} = \mathrm{div}\,\underline{\mathbf{v}} = \mathrm{trace}\,\underset{\sim}{\mathbf{D}}$	$\dfrac{dv - dV}{dV} \simeq \mathrm{Div}\,\underline{\mathbf{u}} = \mathrm{trace}\,\underset{\sim}{\boldsymbol{\varepsilon}}$
relative elongation	$\dfrac{\dot{\lambda}}{\lambda} = \underline{\mathbf{m}}\,.\underset{\sim}{\mathbf{D}}.\underline{\mathbf{m}}$	$\lambda - 1 \simeq \underline{\mathbf{M}}\,.\underset{\sim}{\boldsymbol{\varepsilon}}.\underline{\mathbf{M}} \simeq \frac{\lambda-1}{\lambda}$

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Principle of virtual power

- Power of external forces acting on material domain $\mathcal{D} \subset \Omega_t$

$$\mathcal{P}^c(\underline{\mathbf{v}}^\star) + \mathcal{P}^e(\underline{\mathbf{v}}^\star) = \int_{\partial\mathcal{D}} \underline{\mathbf{t}} \,.\, \underline{\mathbf{v}}^\star \, ds + \int_{\mathcal{D}} \rho \underline{\mathbf{f}} \,.\, \underline{\mathbf{v}}^\star \, dv$$

- Power of acceleration forces

$$\mathcal{P}^a(\underline{\mathbf{v}}^\star) := \int_{\mathcal{D}} \rho \underline{\mathbf{a}} \,.\, \underline{\mathbf{v}}^\star \, dv$$

- Power of internal forces, stress tensor

$$\mathcal{P}^i(\underline{\mathbf{v}}^\star) := -\int_{\mathcal{D}} p^{(i)} \, dv, \qquad p^{(i)} = \underset{\sim}{\boldsymbol{\sigma}} : \underset{\sim}{\mathbf{D}}^\star \sim \mathrm{MPa.s}^{-1} = \mathrm{Wm}^{-3}$$

- Principle of virtual power, $\forall \underline{\mathbf{v}}^\star$ (regular), $\forall \mathcal{D} \subset \Omega_t$

$$\mathcal{P}^c(\underline{\mathbf{v}}^\star) + \mathcal{P}^e(\underline{\mathbf{v}}^\star) + \mathcal{P}^i(\underline{\mathbf{v}}^\star) = \mathcal{P}^a(\underline{\mathbf{v}}^\star)$$

$$-\int_{\mathcal{D}} \underset{\sim}{\boldsymbol{\sigma}} : \underset{\sim}{\mathbf{D}}^\star \, dv + \int_{\partial\mathcal{D}} \underline{\mathbf{t}} \,.\, \underline{\mathbf{v}}^\star \, ds + \int_{\mathcal{D}} \rho \underline{\mathbf{f}} \,.\, \underline{\mathbf{v}}^\star \, dv = \int_{\mathcal{D}} \rho \underline{\mathbf{a}} \,.\, \underline{\mathbf{v}}^\star \, dv$$

Principle of virtual power

- Principle of virtual power (regular case, no shock wave)

$$\mathcal{P}^i(\underline{\mathbf{v}}^\star) + \mathcal{P}^c(\underline{\mathbf{v}}^\star) + \mathcal{P}^e(\underline{\mathbf{v}}^\star) = \mathcal{P}^a(\underline{\mathbf{v}}^\star)$$

$$-\int_{\mathcal{D}} \underset{\sim}{\boldsymbol{\sigma}} : \underset{\sim}{\mathbf{D}}^\star \, dv + \int_{\partial\mathcal{D}} \underline{\mathbf{t}} \,.\underline{\mathbf{v}}^\star \, ds + \int_{\mathcal{D}} \rho \underline{\mathbf{f}} \,.\underline{\mathbf{v}}^\star \, dv = \int_{\mathcal{D}} \rho \underline{\mathbf{a}} \,.\underline{\mathbf{v}}^\star \, dv$$

- equivalent to the field equations (balance of momentum and moment of momentum)

$$\mathrm{div}\, \underset{\sim}{\boldsymbol{\sigma}} + \rho \underline{\mathbf{f}} = \rho \vec{a}, \quad \forall \underline{\mathbf{x}} \in \Omega_t$$

$$\underset{\sim}{\boldsymbol{\sigma}}^T = \underset{\sim}{\boldsymbol{\sigma}}$$

$$\underline{\mathbf{t}} = \underset{\sim}{\boldsymbol{\sigma}}.\underline{\mathbf{n}}\,, \quad \forall \underline{\mathbf{x}} \in \partial\Omega_t$$

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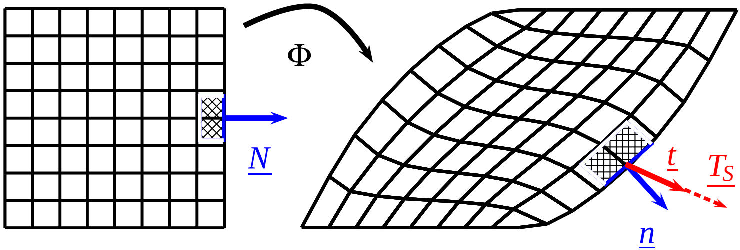
Nominal stress tensor



- Lagrangean version of the field equations

$$\int_{\mathcal{D}} \rho \underline{\mathbf{a}}\; dv = \int_{\partial\mathcal{D}} \underset{\sim}{\boldsymbol{\sigma}}.\underline{\mathbf{n}}\; ds + \int_{\mathcal{D}} \rho \underline{\mathbf{f}}\; dv$$

$$\int_{\mathcal{D}_0} \rho_0 \underline{\mathbf{A}}\; dV = \int_{\partial\mathcal{D}_0} \underset{\sim}{\mathbf{S}}.\underline{\mathbf{N}}\; dS + \int_{\mathcal{D}_0} \rho_0 \underline{\mathbf{F}}\; dV$$

- nominal stress tensor or Boussinesq stress tensor $\underset{\sim}{\mathbf{S}}$
  $\underline{\mathbf{n}}\, ds = J\; \underset{\sim}{\mathbf{F}}^{-T}.\underline{\mathbf{N}}\, dS$

$$\underline{\mathbf{t}}\, ds = \underline{\mathbf{T}}_S dS = \underset{\sim}{\mathbf{S}}.\underline{\mathbf{N}}\, dS, \quad \text{avec} \quad \underset{\sim}{\mathbf{S}} := J\underset{\sim}{\boldsymbol{\sigma}}.\underset{\sim}{\mathbf{F}}^{-T}$$

Piola–Kirchhoff stress tensor

- power density of internal forces

$$\int_{\mathcal{D}} \underset{\sim}{\boldsymbol{\sigma}} : \underset{\sim}{\mathbf{D}}\, dv = \int_{\mathcal{D}} \underset{\sim}{\mathbf{\Pi}} : \underset{\sim}{\dot{\mathbf{E}}}\, dV$$

$$\underset{\sim}{\mathbf{\Pi}} = J\, \underset{\sim}{\mathbf{F}}^{-1} . \underset{\sim}{\boldsymbol{\sigma}} . \underset{\sim}{\mathbf{F}}^{-T} = \underset{\sim}{\mathbf{F}}^{-1} . \underset{\sim}{\mathbf{S}}$$

  Piola–Kirchhoff stress tensor $\underset{\sim}{\mathbf{\Pi}}$

- mass density of power of internal forces

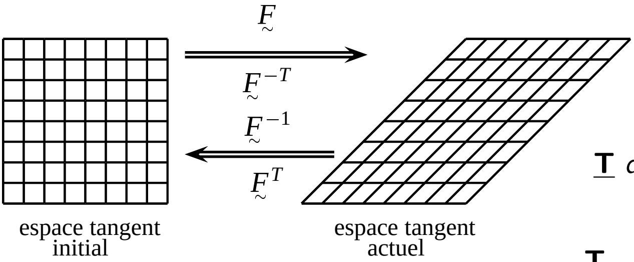
$$\frac{\underset{\sim}{\boldsymbol{\sigma}} : \underset{\sim}{\mathbf{D}}}{\rho} = \frac{\underset{\sim}{\mathbf{\Pi}} : \underset{\sim}{\dot{\mathbf{E}}}}{\rho_0}$$

  conjugate stress–strain measures

- transport of the traction vector

$$\underline{\mathbf{T}}\ dS := \underset{\sim}{\mathbf{F}}^{-1} . \underline{\mathbf{t}}\ ds = \underset{\sim}{\mathbf{F}}^{-1} . \underline{\mathbf{T}}_{S}\, dS = \underset{\sim}{\mathbf{\Pi}} . \underline{\mathbf{N}}\ dS$$

Remark on transport rules



$$\underline{\mathbf{dx}} = \underset{\sim}{\mathbf{F}}.\underline{\mathbf{dX}}$$

$$\underline{\mathbf{ds}} = J\underset{\sim}{\mathbf{F}}^{-T}.\underline{\mathbf{dS}}$$

$$\underline{\mathbf{T}}\ dS = \underset{\sim}{\mathbf{F}}^{-1}.\underline{\mathbf{t}}\ ds$$

$$\begin{aligned}\underline{\mathbf{T}}_M\, dS &= \underset{\sim}{\mathbf{F}}^{T}.\underline{\mathbf{t}}\ ds \\ &= \underset{\sim}{\mathbf{C}}.\underset{\sim}{\mathbf{\Pi}}.\underline{\mathbf{N}}\ dS\end{aligned}$$

Mandel stress tensor