IMPLEMENTATION OF CONSTITUTIVE EQUATIONS

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Introduction

Why is it important to know how to implement constitutive equations in finite element codes ?

- Few constitutive equations are available in commercial FE codes
- Implement new constitutive models in commercial codes (ABAQUS, ANSYS, MARC, . . . , Code_Aster, CAST3M, . . . , Zset, WARP3D)
- Implement specific treatment of implemented models

Outline

- " Numerical" definition of constitutive equations
- Methods (explicit/implicit) for numerical integration
- Consistant tangent matrox
- Example : von Mises plasticiy model
- Convergence

“ Numerical” definition of constitutive equations

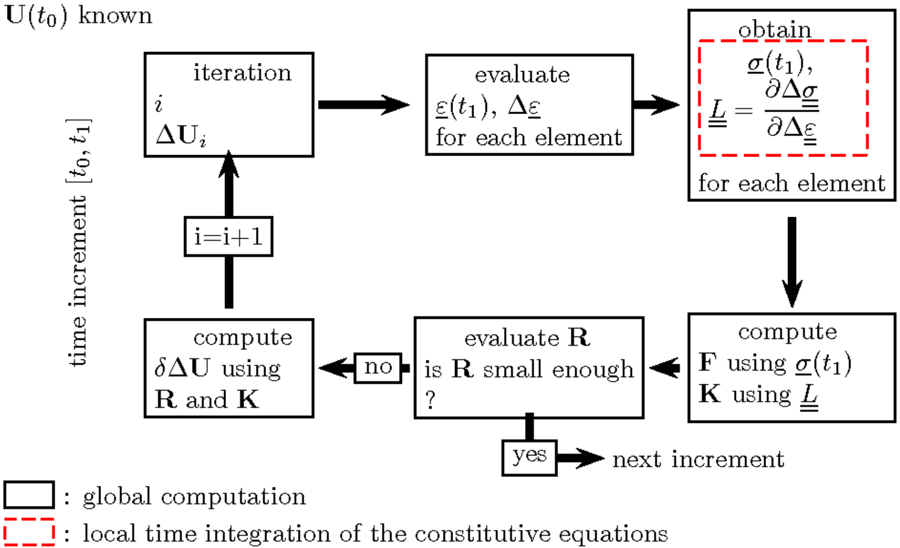
“ Numerical” definition of constitutive equations

- In the case of a displacement based FE formulation, nodal displacements are known and deformations can be computed
- The constitutive equations are used to evaluate:
  1. the stress tensor $\underline{\sigma}$
  2. the “consistent” tangent matrix

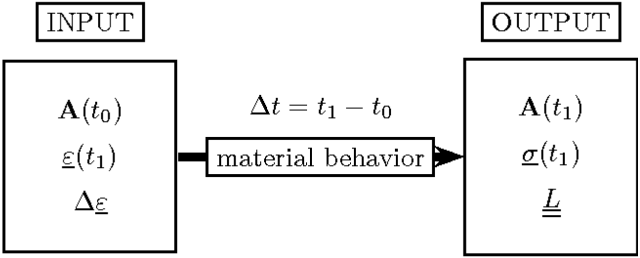
$$\underline{\underline{\mathbf{L}}} = \frac{\partial \Delta \underline{\sigma}}{\partial \Delta \underline{\varepsilon}}$$

  for a given deformation increment $\Delta \underline{\varepsilon}$.
- Material behaviours are also characterized be internal variables **A**.
- The constitutive equations are used to update these variables.

Constitutive equations within the finite element method



Generic interface for constitutive equations



Constitutive equations as a set of differential equations

$$\frac{d\mathbf{A}}{dt} = \dot{\mathbf{A}} = \mathbf{G}(\mathbf{A}, t)$$

$$\frac{dA_i}{dt} = \dot{A}_i = G_i(A_1, \ldots, A_n, t)$$

- Time is used here to represent an imposed deformation as well as imposed **external parameters** depending on both time and position such as an temperature ($T(\vec{x}, t)$).
- Evaluation of the constitutive equations = integrate the previous system from t_0 to t_1.
- Usually: $\mathbf{A} = (\underline{\varepsilon}_e, \ldots)$ so that

$$\underline{\sigma}(t_1) = \underline{E}(t_1) : \underline{\varepsilon}_e(t_1)$$

- $\underline{\underline{L}}$ must still be computed. . .

Integration methods for constitutive equations

Integration methods for constitutive equations : Explicit Euler

$$\mathbf{A}(t_1) = \mathbf{A}(t_0) + \dot{\mathbf{A}}\left(\mathbf{A}(t_0), t_0\right)\Delta t = \mathbf{A}(t_0) + \mathbf{G}\left(\mathbf{A}(t_0), t_0\right)\Delta t$$

$$\Delta\mathbf{A} = \mathbf{G}\left(\mathbf{A}(t_0), t_0\right)\Delta t$$

- The method is not stable ; it must not be employed
- Explicit ? : because $\dot{\mathbf{A}}$ is evaluated at t_0 using $(\mathbf{A}(t_0), t_0)$

Integration methods for constitutive equations : Runge–Kutta

- Numerical estimation of derivatives of $\dot{\mathbf{A}}$: $d\mathbf{G}/d\mathbf{A}$, $d^2\mathbf{G}/d\mathbf{A}^2$, $d^3\mathbf{G}/d\mathbf{A}^3$ ...
- Error estimator to control the solution

The Runge–Kutta integration method is easy to set up as it only uses the differential equation:

$$\dot{\mathbf{A}} = \mathbf{G}(\mathbf{A}, t)$$

It however has some drawbacks:

- Integration may require long CPU time
- In the case of plastic material, the plastic multiplier must be computed. This can be tricky in cases where material coefficients depend on external parameters (e.g. temperature).
- The stability of the integration scheme is not certain for softening materials.

Implicit method (or θ-methods)

- Estimation of $\dot{\mathbf{A}}$ at time t_θ between t_0 and t_1
- $t_\theta = t_0 + \theta\Delta t$ avec $0 \leq \theta \leq 1$
- Two solutions :

$$
\begin{aligned}
\Delta\mathbf{A} &= \mathbf{G}(\mathbf{A}(t_0) + \theta\Delta\mathbf{A}\ ,\ t_0 + \theta\Delta t)\Delta t \\
&= \mathbf{G}(\mathbf{A}_\theta, t_\theta)\Delta t \\
\Delta\mathbf{A} &= \left[(1-\theta)\mathbf{G}(\mathbf{A}(t_0), t_0) + \theta\mathbf{G}(\mathbf{A}(t_1), t_1)\right]\Delta t \\
&= \left[(1-\theta)\mathbf{G}(\mathbf{A}_0, t_0) + \theta\mathbf{G}(\mathbf{A}_0 + \Delta\mathbf{A}, t_0 + \Delta t)\right]\Delta t
\end{aligned}
$$

- Implicit i? : $\Delta\mathbf{A}$ appears on both sides of the previous equations
- Integrrate the constitutive equations = solve the previous equations
- $\theta = 0 \rightarrow$ Euler
- ... in the following we will focus on the first method

. . . implicit method

- Resolution using the Newton-Raphson method
- Residual expressed as:

$$\mathbf{R}(\Delta\mathbf{A}) = \Delta\mathbf{A} - \mathbf{G}(\mathbf{A}(t_0) + \theta\Delta\mathbf{A}\ ,\ t_0 + \theta\Delta t)\Delta t$$
$$R_i(\Delta A_1, \ldots, \Delta A_n) = \Delta A_i - G_i(A_1(t_0) + \theta\Delta A_1, \ldots, A_n(t_0) + \theta\Delta A_n)\Delta t$$

- First order Taylor expansion at s^{th} estimate $\Delta\mathbf{A}_s$

$$\mathbf{R} = \mathbf{R}(\Delta\mathbf{A}_s) + \frac{\partial\mathbf{R}}{\partial\Delta\mathbf{A}}.(\Delta\mathbf{A} - \Delta\mathbf{A}_s) = \mathbf{0}$$

- Calculation of the next estimate $(s+1)$:

$$\Delta\mathbf{A}_{s+1} = \Delta\mathbf{A}_s - \left(\frac{\partial\mathbf{R}}{\partial\Delta\mathbf{A}}\right)^{-1}_{\Delta\mathbf{A}=\Delta\mathbf{A}_s}.\mathbf{R}(\Delta\mathbf{A}_s)$$

- $\mathbf{J} = \partial\mathbf{R}/\partial\Delta\mathbf{A}$ ($J_{ij} = \partial R_j/\partial A_j$): Jacobian matrix, $\mathbf{J}^\star = \mathbf{J}^{-1}$

Remark

- The **A** vector often holds second order tensors
- The Voigt notation is then used
- Notation

Standard notation	Preferred notation
$\underline{\varepsilon} = \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \end{pmatrix}, \quad \underline{\sigma} = \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{pmatrix}$	$\underline{x} = \begin{pmatrix} x_{11} \\ x_{22} \\ x_{33} \\ \sqrt{2}x_{12} \\ \sqrt{2}x_{23} \\ \sqrt{2}x_{31} \end{pmatrix}$

Examples

Integration — Simple example : von Mises plasticity

- Additive decomposition of deformations:

$$\underline{\varepsilon} = \underline{\varepsilon}_e + \underline{\varepsilon}_p$$

- Yield surface

$$\phi = \sigma_{\mathrm{eq}} - R(p)$$

- Normality

$$\underline{\dot{\varepsilon}}_p = \dot{p}\frac{\partial \phi}{\partial \underline{\sigma}} = \frac{3}{2}\dot{p}\frac{\underline{s}}{\sigma_{\mathrm{eq}}} = \dot{p}\underline{n}$$

- Internal variables : $(\underline{\varepsilon}_e, p)$

- Calculation of $\dot{p}$ using the consistency condition: $\dot{\phi} = 0$

$$\dot{\phi} = \frac{\partial \phi}{\partial \underline{\sigma}} : \underline{\dot{\sigma}} + \frac{\partial \phi}{\partial p} \dot{p} = \underline{n} : \underline{\dot{\sigma}} - H\dot{p}$$

- avec $\underline{\sigma} = \underline{\underline{E}} : \underline{\varepsilon}_e = \underline{\underline{E}} : (\underline{\varepsilon} - \underline{\varepsilon}_p) \rightarrow \underline{\dot{\sigma}} = \underline{\underline{E}} : \underline{\dot{\varepsilon}}_e = \underline{\underline{E}} : (\underline{\dot{\varepsilon}} - \underline{\dot{\varepsilon}}_p)$
- Finally one gets

$$\dot{p} = \frac{\underline{n} : \underline{\underline{E}} : \underline{\dot{\varepsilon}}}{\underline{n} : \underline{\underline{E}} : \underline{n} + H}$$

- System of differential equations to be solved

$$\text{sur } \underline{\varepsilon}_e \qquad \underline{\dot{\varepsilon}}_e = \underline{\dot{\varepsilon}} - \dot{p}\underline{n}$$

$$\text{sur } p \qquad \dot{p} = \frac{\underline{n} : \underline{\underline{E}} : \underline{\dot{\varepsilon}}}{\underline{n} : \underline{\underline{E}} : \underline{n} + H}$$

- Attention must be paid on dependences on external parameters (temperature. . .)
- Ready for explicit Runge–Kutta integration !

von Mises plasticity : implicit integration

$$\dot{\underline{\varepsilon}}_e = \dot{\underline{\varepsilon}} - \dot{p}\underline{n} \qquad \rightarrow \qquad \Delta\underline{\varepsilon}_e = \Delta\underline{\varepsilon} - \Delta p\underline{n}$$

$$\dot{p} = \frac{\underline{n} : \underline{\underline{E}} : \dot{\underline{\varepsilon}}}{\underline{n} : \underline{\underline{E}} : \underline{n} + H} \qquad \rightarrow \qquad \Delta p = \frac{\underline{n} : \underline{\underline{E}} : \Delta\underline{\varepsilon}}{\underline{n} : \underline{\underline{E}} : \underline{n} + H}$$

- Evaluation of $\underline{n}$, $\underline{\underline{E}}$, H ? …at time $t_\theta = t_0 + \theta\Delta t$.
- Application :

$$\underline{n} = \frac{3}{2}\frac{\underline{s}^\theta}{\sigma^\theta_{\mathrm{eq}}} \quad \text{avec} \quad \underline{\sigma}^\theta = \underline{\underline{E}}^\theta : \underline{\varepsilon}^\theta_e \quad \underline{\varepsilon}^\theta_e = \underline{\varepsilon}^0_e + \theta\Delta\underline{\varepsilon}_e$$

$$\underline{\underline{E}}^\theta = \underline{\underline{E}}(T^\theta) = \underline{\underline{E}}(T^0 + \theta\Delta T)$$

$$H^\theta = H(p^\theta) = H(p^0 + \theta\Delta p)$$

- The equation

$$\Delta p = \frac{\underline{n} : \underline{\underline{E}} : \Delta\varepsilon}{\underline{n} : \underline{\underline{E}} : \underline{n} + H}$$

is correct but may be replaced by:

$$\phi = \sigma_{\mathrm{eq}} - R(p) = 0$$

This equation is wrong if R depends on an external parameter as in that case:

$$\dot{p} = \frac{\underline{n} : \underline{\underline{E}} : \underline{\dot{\varepsilon}} - R_{,T}\dot{T}}{\underline{n} : \underline{\underline{E}} : \underline{n} + H} \rightarrow \Delta p = \frac{\underline{n} : \underline{\underline{E}} : \Delta\underline{\varepsilon} - R_{,T}\delta T}{\underline{n} : \underline{\underline{E}} : \underline{n} + H}$$

The incremental equation is then

$$\Delta p = \frac{\underline{n}^{\theta} : \underline{\underline{E}}^{\theta} : \Delta\underline{\varepsilon} - R^{\theta}_{,T}\Delta T}{\underline{n}^{\theta} : \underline{\underline{E}}^{\theta} : \underline{n}^{\theta} + H^{\theta}}$$

The method must be avoided and it is better to use the yield condition $\phi = \sigma_{\mathrm{eq}} - R(p) = 0$

- N.B.: in the following the supscript $^{\theta}$ will be omitted for the sake of simplicity

Residuals

$$
\begin{aligned}
\underline{R}_e &= \Delta\underline{\varepsilon}_e + \Delta p\underline{n} - \Delta\varepsilon \\
R_p &= \phi = \sigma_{\mathrm{eq}} - R(p)
\end{aligned}
$$

$$\mathbf{R} = (R_e, R_p)$$

Writing the Jacobian matrix…a matter of some delicacy

- Writing using sub-blocks

$$\mathbf{J} = \left(\begin{array}{c|c} \dfrac{\partial \underline{R}_e}{\partial \Delta \underline{\varepsilon}_e} & \dfrac{\partial \underline{R}_e}{\partial \Delta p} \\ \hline \dfrac{\partial R_p}{\partial \Delta \underline{\varepsilon}_e} & \dfrac{\partial R_p}{\partial \Delta p} \end{array}\right) = \left(\begin{array}{c|c} \mathrm{T4} & \mathrm{T2} \\ \hline \mathrm{T2} & \mathrm{Sc} \end{array}\right)$$

Calculation of the derivatives of $\underline{R}_e = \Delta\underline{\varepsilon}_e + \Delta p \underline{n}^\theta - \Delta\varepsilon$

$$\frac{\partial \underline{R}_e}{\partial \Delta \underline{\varepsilon}_e} = \underline{\underline{1}} + \Delta p \frac{\partial \underline{n}}{\partial \underline{\sigma}} : \frac{\partial \underline{\sigma}}{\partial \underline{\varepsilon}_e} : \frac{\partial \underline{\varepsilon}_e}{\partial \Delta \underline{\varepsilon}_e}$$

$$\frac{\partial \underline{n}}{\partial \underline{\sigma}} = \underline{\underline{N}} = \frac{1}{\sigma_{\mathrm{eq}}} \left(\frac{3}{2} \underline{\underline{J}} - \underline{n} \otimes \underline{n} \right)$$

where $\underline{\underline{J}}$ is such that $\underline{\underline{J}} : \underline{a} = \mathrm{deviator}(\underline{a})$

$$\frac{\partial \underline{\sigma}}{\partial \underline{\varepsilon}_e} = \underline{\underline{E}}$$

$$\frac{\partial \underline{\varepsilon}_e}{\partial \Delta \underline{\varepsilon}_e} = \theta \underline{\underline{1}}$$

so that:

$$\boxed{\frac{\partial \underline{R}_e}{\partial \Delta \underline{\varepsilon}_e} = \underline{\underline{1}} + \theta \underline{\underline{N}} : \underline{\underline{E}}}$$

$$\boxed{\frac{\partial \underline{R}_e}{\partial \Delta p} = \underline{n}}$$

Calculation of the derivatives of $R_p = \sigma_{\text{eq}} - R(p)$

$$\boxed{\frac{\partial R_p}{\partial \Delta \underline{\varepsilon}_e}} = \frac{\partial \sigma_{\text{eq}}}{\partial \underline{\sigma}} : \frac{\partial \underline{\sigma}}{\partial \underline{\varepsilon}_e} : \frac{\partial \underline{\varepsilon}_e}{\partial \Delta \underline{\varepsilon}_e} = \boxed{\theta \underline{n} : \underline{\underline{E}}}$$

$$\boxed{\frac{\partial R_p}{\partial \Delta p}} = -\frac{\partial R}{\partial p} \frac{\partial p}{\partial \Delta p} = \boxed{-\theta H}$$

Tangent matrix / "Consistent" tangent matrix

- **Tangent matrix**

$$\dot{\underline{\sigma}} = \underline{\underline{L}}_p : \dot{\underline{\varepsilon}}$$

- Calculation :

$$\dot{\underline{\sigma}} = \underline{\underline{E}} : (\dot{\underline{\varepsilon}} - \dot{p}\underline{n})$$

and

$$\dot{p} = \frac{\underline{n} : \underline{\underline{E}} : \dot{\underline{\varepsilon}}}{\underline{n} : \underline{\underline{E}} : \underline{n} + H}$$

imply that

$$\underline{\underline{L}}_p = \underline{\underline{E}} - \frac{(\underline{\underline{E}} : \underline{n}) \otimes (\underline{n} : \underline{\underline{E}})}{\underline{n} : \underline{\underline{E}} : \underline{n} + H}$$

- The tangent matrix cannot be defined for viscoplastic materials

- **“Consistent” tangent matrix**

$$\underline{\underline{L}}_c = \frac{\partial \Delta \underline{\sigma}}{\partial \Delta \underline{\varepsilon}}$$

$$\Delta \underline{\sigma} = \underline{\underline{E}} : (\Delta : \underline{\varepsilon} - \Delta p \underline{n})$$

- calculation

$$\begin{aligned}
\delta \Delta \underline{\sigma} \quad &= \quad \underline{\underline{E}} : (\delta \Delta \underline{\varepsilon} - \delta \Delta p \underline{n} - \Delta p \delta \underline{n}) \\
& \quad\ \ \underline{\underline{E}} : (\delta \Delta \underline{\varepsilon} - \delta \Delta p \underline{n}) = \underline{\underline{L}}_p : \delta \Delta \underline{\varepsilon} \\
& \quad\ \ \Delta p \delta \underline{n} = \Delta p \frac{\partial \underline{n}}{\partial \underline{\sigma}} : \frac{\partial \underline{\sigma}}{\partial \Delta \underline{\varepsilon}} : \delta \Delta \underline{\varepsilon} = \Delta p \underline{\underline{N}} : \underline{\underline{E}} : \delta \Delta \underline{\varepsilon} + \dots \\
& \quad\ \ \text{so that} \\
\underline{\underline{L}}_c \quad &\approx \quad \underline{\underline{L}}_p - \Delta p \underline{\underline{E}} : \underline{\underline{N}} : \underline{\underline{E}} + \underline{\underline{O}}(\Delta p^2)
\end{aligned}$$

- The consistent tangent matrix can be defined for viscoplastic materials

Automatic calculation of the consistent tangent matrix

- It is generally possible to write:

$$
\begin{aligned}
\mathbf{A} &= (\underline{\varepsilon}_e, \mathbf{a}) \\
\mathbf{R} &= (\mathbf{R}_e, \mathbf{R}_a) \\
\mathbf{R}_e &= \Delta\underline{\varepsilon}_e + \Delta\underline{\varepsilon}_{\mathrm{irr}} - \Delta\underline{\varepsilon}
\end{aligned}
$$

- Influence of a variation of $\Delta\underline{\varepsilon}$ on the variations of integrated variables $(\underline{\varepsilon}_e, \mathbf{a})$?
- At the solution point **R** must remain equal to **0**

$$
\delta\mathbf{R} = \mathbf{0} = \delta \begin{pmatrix} \Delta\underline{\varepsilon}_e + \Delta\underline{\varepsilon}_{\mathrm{irr}} \\ \mathbf{R}_a \end{pmatrix} - \delta \begin{pmatrix} \Delta\underline{\varepsilon} \\ \mathbf{0} \end{pmatrix}
$$

$$
\delta\mathbf{R} = \frac{\partial\mathbf{R}}{\partial\mathbf{A}} - \begin{pmatrix} \delta\Delta\underline{\varepsilon} \\ \mathbf{0} \end{pmatrix} = \mathbf{J}.\delta\Delta A - \begin{pmatrix} \delta\Delta\underline{\varepsilon} \\ \mathbf{0} \end{pmatrix}
$$

- Therefore

$$
\delta\Delta A = \mathbf{J}^{-1}.\begin{pmatrix} \delta\Delta\underline{\varepsilon} \\ \mathbf{0} \end{pmatrix}
$$

- $\mathbf{J}^{-1}$ can be expressed using sub-blocks

$$\mathbf{J}^{-1} = \mathbf{J}^{\star} = \begin{pmatrix} \mathbf{J}^{\star}_{ee} & \mathbf{J}^{\star}_{ea} \\ \mathbf{J}^{\star}_{ae} & \mathbf{J}^{\star}_{aa} \end{pmatrix},$$

- so that

$$\delta\Delta\underline{\varepsilon}_e = \mathbf{J}^{\star}_{ee}.\delta\Delta\underline{\varepsilon}$$

- Using Hooke's law:

$$\underline{\sigma}(t_1) = \underline{\sigma}(t_0) + \Delta\underline{\sigma} = \underline{\underline{E}}(t_1) : \underline{\varepsilon}_e(t_1) = \underline{\underline{E}}(t_1) : (\underline{\varepsilon}_e(t_0) + \Delta\underline{\varepsilon}_e)$$

therefore

$$\delta\Delta\underline{\sigma} = \underline{\underline{E}}(t_1) : \delta\Delta\underline{\varepsilon}_e = \underline{\underline{E}}(t_1) : \underline{\underline{J}}^{\star}_{ee} : \delta\Delta\varepsilon$$

- Finally

$$\boxed{\underline{\underline{L}}_c = \underline{\underline{E}}(t_1) : \underline{\underline{J}}^{\star}_{ee}}$$

- In cases where $\underline{\underline{E}}$ depends on an integrated variable (e.g. d =damage) the calculation is more difficult

$$\Delta\underline{\sigma} = \frac{\partial \underline{\underline{E}}}{\partial d}\delta\Delta d : \underline{\varepsilon}_e + \underline{\underline{E}}(t_1) : \Delta\underline{\varepsilon}_e$$

$$\boxed{\underline{\underline{L}}_c = \frac{\partial \underline{\underline{E}}}{\partial d} : (\underline{\varepsilon}_e \otimes \mathbf{J}^\star_{de}) + \underline{\underline{E}} : \underline{\underline{J}}^\star_{ee}}$$

Explicit versus Implicit integration

Explicit	Implicit
easy to implement	difficult to implement
time consuming	efficient
$\underline{\underline{L}}_c$?	calculation of $\underline{\underline{L}}_c$

- $\underline{\underline{L}}_c$ can be evaluated using a perturbation method

$$L^c_{ijkl} = \frac{\sigma_{ij}(\Delta\underline{\varepsilon} + \delta\varepsilon\underline{\mu}^{kl}) - \sigma_{ij}(\Delta\underline{\varepsilon})}{\delta\varepsilon}$$

- The tensor $\underline{\mu}^{kl}$ is such that:

$$\mu^{kl}_{ij} = \delta_{ik}\delta_{lj}$$

Viscoplastity

- To introduce rate dependency (viscoplastity) the above equations need to be slightly modified
- Additive decomposition of deformations:

$$\underline{\varepsilon} = \underline{\varepsilon}_e + \underline{\varepsilon}_p$$

- Yield surface

$$\phi = \sigma_{\text{eq}} - R(p)$$

- Normality

$$\underline{\dot{\varepsilon}}_p = \dot{p}\frac{\partial \phi}{\partial \underline{\sigma}}$$

- ϕ can now be positive
- $\dot{p}$ is given by a specific material dependent law

$$\dot{p} = \mathcal{F}(\phi)$$

$\mathcal{F}$ is such that $\mathcal{F}(x) = 0$ if $x \leq 0$

- For instance (Norton low)

$$\dot{p} = \left\langle \frac{\phi}{K} \right\rangle^n$$

- Differential equations to be integrated are therefore

$$\begin{aligned} \underline{\dot{\varepsilon}}_e &= \underline{\dot{\varepsilon}} - \dot{p}\underline{n} \\ \dot{p} &= \left\langle \frac{\phi}{K} \right\rangle^n \end{aligned}$$

Viscoplastity : integration

- Runge-Kutta integration is straightforward
- For the implicit integration

$$
\begin{aligned}
\underline{R}_e &= \Delta\underline{\varepsilon}_e + \Delta p \underline{n} - \Delta\underline{\varepsilon} \\
R_p &= \Delta p - \mathcal{F}(\phi)\Delta t
\end{aligned}
$$

- Evaluation of the Jacobian matrix: only the derivatives of R_p have to be re-computed with $\mathcal{F}' = \partial\mathcal{F}/\partial\phi$

$$
\begin{aligned}
\frac{\partial R_p}{\partial \Delta\underline{\varepsilon}_e} &= -\Delta t \frac{\partial \mathcal{F}}{\partial \phi} \frac{\partial \phi}{\partial \underline{\sigma}} : \frac{\partial \underline{\sigma}}{\partial \underline{\varepsilon}_e} : \frac{\partial \underline{\varepsilon}_e}{\partial \Delta\underline{\varepsilon}_e} = -\theta \Delta t \mathcal{F}' \underline{n} : \underline{\underline{E}} \\
\frac{\partial R_p}{\partial \Delta p} &= -\Delta t \frac{\partial \mathcal{F}}{\partial \phi} \frac{\partial \phi}{\partial R} \frac{\partial R}{\partial p} \frac{\partial p}{\partial \Delta p} = -\theta \Delta t \mathcal{F}' H
\end{aligned}
$$