FINITE DEFORMATIONS

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Notations

scalar	a	a
vector	$\vec{a}$	a_i
2[nd] order tensor	$\underline{a}$	a_{ij}
4[th] order tensor	$\underline{\underline{a}}$	a_{ijkl}
matrices	$\boldsymbol{a}$	
Voigt notation	$\underline{a} \rightarrow \boldsymbol{a}$ $\mathcal{V}(\underline{a}.\underline{b})$	

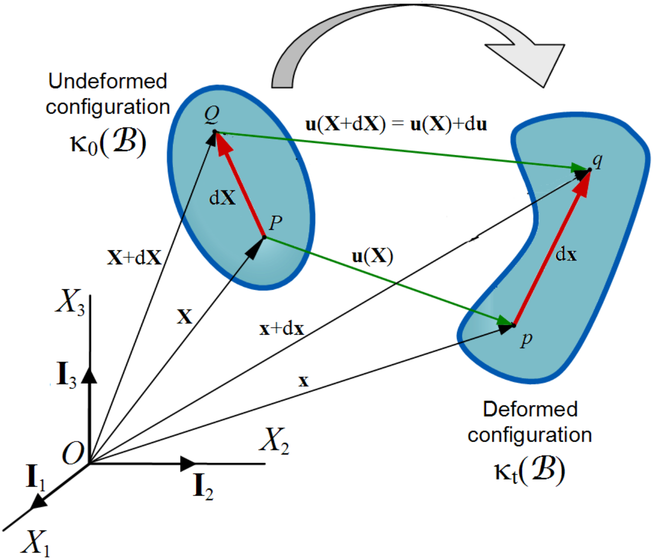
	Products	
.	$c = \vec{a}.\vec{b}$	$c = a_i b_i$
	$\vec{c} = \underline{a}.\vec{b}$	$c_i = a_{ij} b_j$
	$\underline{c} = \underline{a}.\underline{b}$	$c_{ij} = a_{ik} b_{kj}$
:	$c = \underline{a} : \underline{b}$	$c = a_{ij} b_{ij}$
	$\underline{c} = \underline{\underline{a}} : \underline{b}$	$c_{ij} = a_{ijkl} b_{kl}$
$\otimes$	$\underline{c} = \vec{a} \otimes \vec{b}$	$c_{ij} = a_i b_j$
	$\underline{\underline{c}} = \underline{a} \otimes \underline{b}$	$c_{ijkl} = a_{ij} b_{kl}$

[Einstein convention]

Outline

- Displacements and deformation
- Stress measures
- Constitutive equations

Displacement and Deformation



- Displacement field around $\vec{X}$

$$\vec{u}(\vec{X}+d\vec{X}) = \vec{u}(\vec{X}) + d\vec{u} = \vec{u}(\vec{X}) + \frac{\partial \vec{u}}{\partial \vec{X}}.d\vec{X}$$

$$d\vec{x} = d\vec{X} + d\vec{u} = \left(\underline{1} + \frac{\partial \vec{u}}{\partial \vec{X}}\right).d\vec{X}$$

- Transformation gradient

$$\underline{F} = \frac{\partial \vec{x}}{\partial \vec{X}} \qquad F_{il} = \frac{\partial x_i}{\partial X_l}$$

Objectivity

- Rigid body transformation
$$\vec{x}' = \underline{Q}(t).\vec{x} + \vec{c}(t)$$
- Quantities are objective if they are related by the rotation tensor as:
$$\begin{aligned} m' &= m \\ \vec{u}' &= \underline{Q}(t).\vec{u} \\ \underline{T}' &= \underline{Q}(t).\underline{T}.\underline{Q}(t)^T \end{aligned}$$
- Generalization
$$T_{(n)} = \vec{u}_1 \otimes \cdots \otimes \vec{u}_n \quad \text{objective if} \quad T'_{(n)} = \vec{u}'_1 \otimes \cdots \otimes \vec{u}'_n \quad \text{where} \quad \vec{u}'_i = \underline{Q}.\vec{u}_1$$
- $\underline{F}$ is not objective
$$\underline{F}' = \frac{\partial \vec{x}'}{\partial \vec{X}} = \frac{\partial \vec{x}'}{\partial \vec{x}} . \frac{\partial \vec{x}}{\partial \vec{X}} = \underline{Q}.\underline{F}$$
- $\underline{B} = \underline{F}.\underline{F}^T$ is objective
$$\underline{B}' = \underline{F}'.\underline{F}'^T = \underline{Q}.\underline{F}.\underline{F}^T.\underline{Q}^T = \underline{Q}.\underline{B}.\underline{Q}^T$$

. . . and invariance

- Quantities are invariant if they remain unchanged by the transformation

$$m' = m, \qquad \vec{u}' = \vec{u}, \qquad \underline{T}' = \underline{T}$$

- $\underline{C} = \underline{F}^T.\underline{F}$ is invariant

$$\underline{C}' = \underline{F}'^T.\underline{F}' = \underline{F}^T.\underline{Q}^T.\underline{Q}.\underline{F} = \underline{C}$$

Rates

- Let $\delta\vec{x} = \underline{F}.\delta\vec{X}$ be an infinitesimal segment in the current configuration. One gets:

$$\frac{d\delta\vec{x}}{dt} = \frac{d\underline{F}.\delta\vec{X}}{dt} = \frac{d\underline{F}}{dt}.\delta\vec{X} = \dot{\underline{F}}.\delta\vec{X} = \dot{\underline{F}}.\underline{F}^{-1}.\delta\vec{x} = \underline{L}.\delta\vec{x}$$

- $\underline{L}$ can be separated into symmetric ($\underline{D}$) and an skew-symmetric ($\underline{W}$) parts:

$$\underline{L} = \dot{\underline{F}}.\underline{F}^{-1} = \underline{D} + \underline{W}$$

- $\underline{D}$ characterizes strain rate in the following way:

$$\begin{aligned}
\frac{d}{dt}\left(\delta\vec{x}^1.\delta\vec{x}^2\right) &= \frac{d\delta\vec{x}^1}{dt}.\delta\vec{x}^2 + \delta\vec{x}^1.\frac{d\delta\vec{x}^2}{dt} = (D_{ij} + W_{ij})\delta x_j^1 \delta x_i^2 + \delta x_i^1 (D_{ij} + W_{ij})\delta x_j^2 \\
&= \left[D_{ij}\delta x_j^1 \delta x_i^2 + D_{ij}\delta x_i^1 \delta x_j^2\right] + \left[W_{ij}\delta x_j^1 \delta x_i^2 + W_{ij}\delta x_i^1 \delta x_j^2\right] \\
&= 2\delta\vec{x}^1.\underline{D}.\delta\vec{x}^2
\end{aligned}$$

- For a transformation such that: $\underline{F}' = \underline{Q}(t).\underline{F}$

$$\begin{aligned}
\underline{L}' &= \dot{\underline{F}}'.\underline{F}'^{-1} = (\dot{\underline{Q}}.\underline{F} + \underline{Q}.\dot{\underline{F}}).\underline{F}^{-1}.\underline{Q}^{-1} \\
&= \dot{\underline{Q}}.\underline{Q}^T + \underline{Q}.(\underline{D} + \underline{W}).\underline{Q}^T \qquad\qquad \underline{Q}^{-1} = Q^T \\
&= \underline{Q}.\underline{D}.\underline{Q}^T + \dot{\underline{Q}}.\underline{Q}^T + \underline{Q}.\underline{W}.\underline{Q}^T \\
&= \underline{D}' + \underline{W}'
\end{aligned}$$

- With

$$\begin{aligned}
\underline{D}' &= \underline{Q}.\underline{D}.\underline{Q}^T \\
\underline{W}' &= \underline{Q}.\underline{W}.\underline{Q}^T + \dot{\underline{Q}}.\underline{Q}^T
\end{aligned}$$

- Note that $\dot{\underline{Q}}.\underline{Q}^T$ is skew-symmetric as:

$$\underline{Q}.\underline{Q}^T = \underline{1} \qquad \Rightarrow \qquad \dot{\underline{Q}}.\underline{Q}^T + \underline{Q}.\dot{\underline{Q}}^T = \underline{0}$$

- Only $\underline{D}$ is objective

$\underline{R}.\underline{U}$ — $\underline{V}.\underline{U}$ decomposition

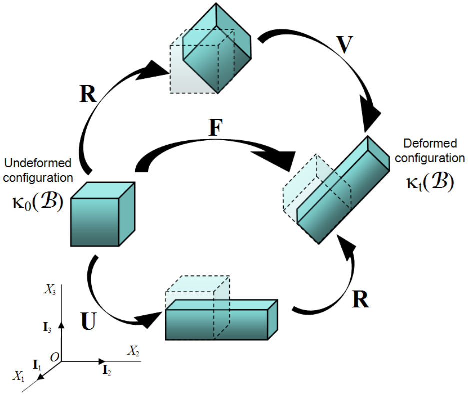
- The deformation gradient $\underline{F}$ can be decomposed, using the polar decomposition theorem, into a product of two second-order tensors

$$\underline{F} = \underline{R}.\underline{U} = \underline{V}.\underline{R}$$

$$F_{iJ} = R_{iK} U_{KJ} = V_{ik} R_{kJ}$$

with ($\underline{R}$ rotation tensor)

$$\underline{R}.\underline{R}^T = \underline{1} \qquad \underline{U} = \underline{U}^T \qquad \underline{V} = \underline{V}^T \qquad \underline{U} = \underline{R}^T.\underline{V}.\underline{R}$$



Calculation of $\underline{R}$ and $\underline{U}$

- $\underline{C} = \underline{F}^T.\underline{F} \equiv \underline{U}.\underline{U}$
- $\det \underline{C} = (\det \underline{F})^2 > 0$ and $\det \underline{C} \neq 0$
- Eigen frame for $\underline{C} = \underline{P}^T.\underline{C}_0.\underline{P}$

$$\underline{C}_0 = \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix} \rightarrow \underline{U}_0 = \begin{pmatrix} \sqrt{c_1} & 0 & 0 \\ 0 & \sqrt{c_2} & 0 \\ 0 & 0 & \sqrt{c_3} \end{pmatrix}$$

$$\underline{U} = \underline{P}^T.\underline{U}_0.\underline{P}$$

- Consequently

$$\underline{R} = \underline{F}.\underline{U}^{-1} \text{ and } \underline{R}^T = \underline{U}^{-1}.\underline{F}^T$$

- so that

$$\underline{R}.\underline{R}^T = \underline{F}.\underline{U}^{-1}.\underline{U}^{-1}.\underline{F}^T = \underline{F}.\underline{C}^{-1}.\underline{F}^T = \underline{F}.(\underline{F}^{-1}.\underline{F}^{-T}).\underline{F}^T = \underline{1}$$

- idem for $\underline{V}$

Volume variation

- Jacobian of the transformation

$$J = \det \underline{F} = \frac{V}{V_0} > 0$$

- so that:

$$\int_{\Omega} \bullet \, d\Omega = \int_{\Omega_0} \bullet \, J d\Omega_0$$

Some strain measures

- Several **rotation-independent** symmetric deformation tensors are used in mechanics.
- Right Cauchy-Green deformation tensor [Lagrangian tensor]

$$\underline{C} = \underline{F}^T.\underline{F} \qquad C_{IJ} = F^T_{Ik}F_{kJ} = F_{kI}F_{kJ}$$

- Left Cauchy-Green deformation tensor [Eulerian tensor]

$$\underline{B} = \underline{F}.\underline{F}^T \qquad B_{ij} = F_{iK}F^T_{Kj} = F_{iK}F_{Kj}$$

- Some finite strain tensors
  - Objective or invariant
  - Must be $\underline{0}$ for $\underline{F} = \underline{1}$
  - Must correspond to the small deformation theory for a first order Taylor expansion with respect to $\underline{F}$

Green-Lagrange	$\underline{E} = \frac{1}{2}(\underline{C} - \underline{1}) = \frac{1}{2}\left(\underline{U}^2 - \underline{1}\right)$
Biot strain tensor	$\underline{E}^{\mathrm{Biot}} = \underline{U} - \underline{1}$
Logarithmic strain tensor	$\underline{E}^{\log} = \log \underline{U}$

Principal stretches: λ_i

- $\underline{U}$ and $\underline{V}$ have the same eigenvalues λ_i and can be expressed as:

$$\underline{U} = \sum_{i=1}^{3} \lambda_i \vec{N}_i \otimes \vec{N}_i \qquad \underline{V} = \sum_{i=1}^{3} \lambda_i \vec{n}_i \otimes \vec{n}_i$$

- so that:

$$\underline{C} = \sum_{i=1}^{3} \lambda_i^2 \vec{N}_i \otimes \vec{N}_i \qquad \underline{B} = \sum_{i=1}^{3} \lambda_i^2 \vec{n}_i \otimes \vec{n}_i$$

- One has:

$$\underline{V} = \underline{R}.\underline{U}.\underline{R}^T = \sum_{i=1}^{3} \lambda_i (\underline{R}.\vec{N}_i) \otimes (\underline{R}.\vec{N}_i)$$

- Note also that:

$$\frac{\partial \lambda_i}{\partial \underline{C}} = \frac{1}{2\lambda_i} \vec{N}_i \otimes \vec{N}_i$$

Some strain measures: examples

- Tension loading

$$u_x = (\Delta L/L)x,\; u_y = (\Delta l/l)y,\; u_z = (\Delta l/l)z$$

$$\underline{F} = \begin{pmatrix} 1+\frac{\Delta L}{L} & 0 & 0 \\ 0 & 1+\frac{\Delta l}{l} & 0 \\ 0 & 0 & 1+\frac{\Delta l}{l} \end{pmatrix}$$

$$C_{11} = \left(1+\frac{\Delta L}{L}\right)^2 \quad C_{22} = C_{33} = \left(1+\frac{\Delta l}{l}\right)^2$$

$$E_{11} = \frac{1}{2}\left(\left(1+\frac{\Delta L}{L}\right)^2 - 1\right) \approx \frac{\Delta L}{L} \quad E_{22} = E_{33} = \frac{1}{2}\left(\left(1+\frac{\Delta l}{l}\right)^2 - 1\right)$$

$$E_{11}^{\mathrm{Biot}} = \frac{\Delta L}{L},\; E_{22}^{\mathrm{Biot}} = E_{33}^{\mathrm{Biot}} = \frac{\Delta L}{L}$$

- Simple shear

$$\underline{F} = \begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \underline{C} = \begin{pmatrix} 1 & \gamma & 0 \\ \gamma & \gamma^2+1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \underline{B} = \begin{pmatrix} 1+\gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \underline{R}.\underline{C}.\underline{R}^T$$

- $\det \underline{F} = \det \underline{C} = 1$
- Eigenvalues of $\underline{C}$

$$1 \qquad 1 + \frac{1}{2}\gamma^2 + \frac{1}{2}\sqrt{4\gamma^2+\gamma^4} \qquad 1 + \frac{1}{2}\gamma^2 - \frac{1}{2}\sqrt{4\gamma^2+\gamma^4}$$

- $\underline{R}$ has the form

$$\underline{R} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Solving $\underline{B} = \underline{R}.\underline{C}.\underline{R}^T$, yields

$$\theta = -\arctan \gamma/2$$

- Finally

$$\underline{U} = \underline{R}^T.\underline{F} = \begin{pmatrix} \dfrac{2}{\sqrt{4+\gamma^2}} & \dfrac{\gamma}{\sqrt{4+\gamma^2}} & 0 \\ \dfrac{\gamma}{\sqrt{4+\gamma^2}} & \dfrac{\gamma^2}{\sqrt{4+\gamma^2}} + \dfrac{2}{\sqrt{4+\gamma^2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- First order Taylor expansion

$$\underline{C} = \begin{pmatrix} 1 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \underline{U}$$

$$\theta = -\frac{\gamma}{2}$$

Stress measures and work equivalence

- **Cauchy** stress $\underline{\sigma}$ (σ_{ij})
- Stress measure in the final (t) configuration
- Velocity gradient

$$\underline{L} = \underline{\dot{F}}.\underline{F}^{-1} = \underline{D} + \underline{W}$$

- Work

$$w = \int_{\Omega} \underline{\sigma} : \underline{D}\, d\Omega$$

- **Kirchhoff** stress

$$w = \int_{\Omega} \underline{\sigma} : \underline{D}\, d\Omega = \int_{\Omega_0} J\underline{\sigma} : \underline{D}\, d\Omega_0 = \int_{\Omega_0} \underline{\tau} : \underline{D}\, d\Omega_0$$

$$\boxed{\underline{\tau} = J\underline{\sigma} \qquad \tau_{ij} = J\sigma_{ij}}$$

- Green-Lagrange strain tensor

$$
\begin{aligned}
\underline{E} &= \frac{1}{2}\left(\underline{F}.\underline{F}^T - \underline{1}\right) \\
\underline{\dot{E}} &= \frac{1}{2}\left(\underline{\dot{F}}.\underline{F}^T + \underline{F}.\underline{\dot{F}}^T\right) \\
\text{or } \underline{F}^{-1}.\underline{\dot{E}}.\underline{F}^{-T} &= \frac{1}{2}\left(\underline{F}^{-1}.\underline{\dot{F}} + \underline{\dot{F}}^T.\underline{F}^{-T}\right)
\end{aligned}
$$

$$
\underline{F}^{-1}.\underline{\dot{E}}.\underline{F}^{-T}\frac{1}{2}\left(\underline{L} + \underline{L}^T\right) = \underline{D} \qquad \underline{\dot{E}} = \underline{F}.\underline{D}.\underline{F}^T
$$

- **Second Piola-Kirchhoff** stress tensor

$$
\begin{aligned}
\sigma_{ij} D_{ij} d\Omega &= \sigma_{ij} F_{iK}^{-T} \dot{E}_{KL} F_{Lj}^{-1} J d\Omega_0 \\
&= J\sigma_{ij} F_{Ki}^{-1} F_{jL}^{-T} \dot{E}_{KL}\Omega_0 \\
&= JF_{Ki}^{-1}\sigma_{ij} F_{jL}^{-T} \dot{E}_{KL}\Omega_0 \\
&= S_{KL}\dot{E}_{KL}\Omega_0
\end{aligned}
$$

- with

$$
\boxed{\underline{S} = J\underline{F}^{-1}.\underline{\sigma}.\underline{F}^{-T} \qquad \text{or} \qquad \underline{\sigma} = \frac{1}{J}\underline{F}.\underline{S}.\underline{F}^T}
$$

- The Second Piola-Kirchhoff stress, $\underline{S}$, is the work conjugate of the Green-Lagrange strain tensor, $\underline{E}$.

- **First Piola-Kirchhoff/Boussinesq** stress tensor

$$
\begin{aligned}
\underline{\sigma} : \underline{D} d\Omega &= \underline{\sigma} : \underline{\dot{F}}.\underline{F}^{-1} J d\Omega_0 \qquad \text{because } \underline{\sigma} \text{ is symmetric} \\
&= J \sigma_{ij} \dot{F}_{iK} F^{-1}_{Kj} d\Omega_0 = J \sigma_{ij} F^{-T}_{jK} \dot{F}_{iK} d\Omega_0 \\
&= \Pi_{iK} \dot{F}_{iK} d\Omega_0 \\
&= \underline{\Pi} : \underline{\dot{F}} d\Omega_0
\end{aligned}
$$

- The First Piola-Kirchhoff stress, $\underline{\Pi}$, is the work conjugate of the transformation gradient $\underline{F}$

$$\boxed{\underline{\Pi} = J \underline{\sigma}.\underline{F}^{-T} \qquad \text{or} \qquad \underline{\sigma} = \frac{1}{J} \underline{\Pi}.\underline{F}^{T}}$$

- $\underline{\Pi}$ is not symmetric

- Finally

$$\boxed{\int_{\Omega} \underline{\sigma} : \underline{D} \, d\Omega = \int_{\Omega_0} \underline{\tau} : \underline{D} \, d\Omega_0 = \int_{\Omega_0} \underline{S} : \underline{\dot{E}} \, d\Omega_0 = \int_{\Omega_0} \underline{\Pi} : \underline{\dot{F}} \, d\Omega_0}$$

Interpretation of the various stress measures: tensile test

- Transformation gradient — Cauchy stress [final configuation]

$$\underline{F} = \begin{pmatrix} F_{\|} & 0 & 0 \\ 0 & F_{\perp} & 0 \\ 0 & 0 & F_{\perp} \end{pmatrix} \qquad J = F_{\|} F_{\perp}^2 \qquad \underline{\sigma} = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- Kirchhoff stress

$$\underline{\tau} = J\underline{\sigma} \qquad \tau = F_{\|} F_{\perp}^2 \sigma \qquad \tau = \sigma \quad \text{for incompressible materials}$$

- Second Piola-Kirchhoff

$$\underline{S} = J\underline{F}^{-1}.\underline{\sigma}.\underline{F}^{-T} \qquad S = \frac{F_{\perp}^2}{F_{\|}}\sigma \quad \text{for incompressible materials} \quad S = \frac{1}{F_{\|}^2}\sigma$$

- First Piola-Kirchhoff stress

$$\underline{\Pi} = J\underline{\sigma}.\underline{F}^{-T} \qquad \Pi = \frac{1}{F_{\|}}\sigma \approx \text{engineering stress}$$

Stress rates - Jaumann stress rate

- Recall of the relation
$$\underline{W}' = \underline{Q}.\underline{W}.\underline{Q}^T + \dot{\underline{Q}}.\underline{Q}^T$$
- so that:
$$\begin{aligned} \dot{\underline{Q}} &= \underline{W}'.\underline{Q} - \underline{Q}.\underline{W} \\ \dot{\underline{Q}}^T &= -\underline{Q}^T.\underline{W}' + \underline{W}.\underline{Q}^T \end{aligned}$$
- For an objective displacement vector $\vec{u}' = \underline{Q}.\vec{u}$, one gets:
$$\dot{\vec{u}}' = \dot{\underline{Q}}.\vec{u} + \underline{Q}.\dot{\vec{u}} = (\underline{W}'.\underline{Q} - \underline{Q}.\underline{W}).\vec{u} + \underline{Q}.\dot{\vec{u}} = \underline{W}'.\vec{u}' - \underline{Q}.\underline{W}.\vec{u} + \underline{Q}.\dot{\vec{u}}$$
- So that:
$$\dot{\vec{u}}' - \underline{W}'.\vec{u}' = \underline{Q}.(\dot{\vec{u}} - \underline{W}.\vec{u})$$
- This allows to define an objective derivative of vectors (Jaumann rate):
$$\vec{u}^J = \dot{\vec{u}} - \underline{W}.\vec{u}$$

- Following the same methodology for second order tensors:

$$\begin{aligned}\dot{\underline{T}}' &= \dot{\underline{Q}}.\underline{T}.\underline{Q}^T + \underline{Q}.\underline{T}.\dot{\underline{Q}}^T + \underline{Q}.\dot{\underline{T}}.\underline{Q}^T \\ &= (\underline{W}'.\underline{Q} - \underline{Q}.\underline{W}).\underline{T}.\underline{Q}^T + \underline{Q}.\underline{T}.(-\underline{Q}^T.\underline{W}' + \underline{W}.\underline{Q}^T) + \underline{Q}.\dot{\underline{T}}.\underline{Q}^T \\ &= \underline{W}'.\underline{T}' - \underline{Q}.\underline{W}.\underline{T}.\underline{Q}^T - \underline{T}'.\underline{W}' + \underline{Q}.\underline{T}.\underline{W}.\underline{Q}^T + \underline{Q}.\dot{\underline{T}}.\underline{Q}^T\end{aligned}$$

- which is rewritten as:

$$\dot{\underline{T}}' - \underline{W}'.\underline{T}' + \underline{T}'.\underline{W}' = \underline{Q}.\left(\dot{\underline{T}} - \underline{W}.\underline{T} + \underline{T}.\underline{W}\right).\underline{Q}^T$$

- An objective derivative (Jaumann derivative) is then obtained for second order tensors:

$$\underline{T}^J = \dot{\underline{T}} - \underline{W}.\underline{T} + \underline{T}.\underline{W}$$

Stress rates - Truesdell stress rate

- Recall the relation between the Cauchy and the second Piola-Kirchhoff stress:

$$\underline{\sigma} = \frac{1}{J}\underline{F}.\underline{S}.\underline{F}^T \qquad \underline{S} = J\underline{F}^{-1}.\underline{\sigma}.\underline{F}^{-T}$$

- As $\underline{S}$ is invariant the following stress rate will be objective (it corresponds to the transport of the rate of the second Piola-Kirchhoff stress):

$$\overset{\circ}{\underline{\sigma}} = \frac{1}{J}\underline{F}.\underline{\dot{S}}.\underline{F}^T \neq \underline{\dot{\sigma}}$$

- Noting that:

$$\underline{\dot{S}} = \dot{J}\underline{F}^{-1}.\underline{\sigma}.\underline{F}^{-T} + J\underline{\dot{F}}^{-1}.\underline{\sigma}.\underline{F}^{-T} + J\underline{F}^{-1}.\underline{\dot{\sigma}}.\underline{F}^{-T} + J\underline{F}^{-1}.\underline{\sigma}.\underline{\dot{F}}^{-T}$$

- So that:

$$\overset{\circ}{\underline{\sigma}} = \frac{\dot{J}}{J}\underline{\sigma} + \underline{F}.\underline{\dot{F}}^{-1}.\underline{\sigma} + \underline{\sigma}.\underline{\dot{F}}^{-T}.\underline{F}^T + \underline{\dot{\sigma}}$$

  - Note that
$$\frac{d}{dt}(\underline{F}.\underline{F}^{-1}) = \underline{0} = \underline{\dot{F}}.\underline{F}^{-1} + \underline{F}.\underline{\dot{F}}^{-1}$$
  - so that
$$\underline{F}.\underline{\dot{F}}^{-1} = -\underline{L} \quad \text{and} \quad \underline{\dot{F}}^{-T}.\underline{F}^T = -\underline{L}^T$$
  - One also has
$$\dot{J}/J = \mathrm{Tr}\underline{L}$$

- Finally, one obtains the Truesdell stress rate

$$\boxed{\overset{\circ}{\underline{\sigma}} = \mathrm{Tr}\underline{L}\,\underline{\sigma} - \underline{L}.\underline{\sigma} - \underline{\sigma}.\underline{L}^T + \underline{\dot{\sigma}}}$$

Stress rates - Green-Naghdi stress rate

- One defines the rotated stress $\underline{\sigma}_R$

$$\underline{\sigma}_R = \underline{R}.\underline{\sigma}.\underline{R}^T \quad \text{or} \quad \underline{\sigma} = \underline{R}^T.\underline{\sigma}_R.\underline{R}$$

- Following the same methodology as for the Truesdell rate, one gets:

$$\overset{\square}{\underline{\sigma}} = \underline{R}^T.\dot{\underline{\sigma}}_R.\underline{R}$$

which defines an objective rate

- Noting that

$$\dot{\underline{\sigma}}_R = \dot{\underline{R}}.\underline{\sigma}.\underline{R}^T + \underline{R}.\underline{\sigma}.\dot{\underline{R}}^T + \underline{R}.\dot{\underline{\sigma}}.\underline{R}^T$$

- one gets:

$$\overset{\square}{\underline{\sigma}} = \dot{\underline{\sigma}} + \underline{R}^T.\dot{\underline{R}}.\underline{\sigma} + \underline{\sigma}.\dot{\underline{R}}^T.\underline{R} = \dot{\underline{\sigma}} - \underline{\Omega}.\underline{\sigma} + \underline{\sigma}.\underline{\Omega}$$

- with

$$\underline{\Omega} = \dot{\underline{R}}.\underline{R}^T$$

- Green-Naghdi stress rate

$$\boxed{\overset{\square}{\underline{\sigma}} = \dot{\underline{\sigma}} - \underline{\Omega}.\underline{\sigma} + \underline{\sigma}.\underline{\Omega}}$$

Constitutive equations : hyperelasticity

- Hyperelasticity is often used for elastomers
- One first defines a strain energy density function W which depends on $\underline{C}$
- For isotropic materials, W only depends on the invariants of $\underline{C}$

$$
\begin{aligned}
I_1 &= \mathrm{Tr}\underline{C} \\
I_2 &= \frac{1}{2}\left((\mathrm{Tr}\underline{C})^2 - \mathrm{Tr}\underline{C}.\underline{C}\right) \\
I_3 &= \det \underline{C} \qquad \text{for incompressible materials: } I_3 = 1 \\
& \quad J = \det \underline{F} \qquad I_3 = J^2
\end{aligned}
$$

- The second Piola-Kirchhoff stress is then given by:

$$\underline{S} = \frac{\partial W}{\partial \underline{E}} = 2\frac{\partial W}{\partial \underline{C}}$$

- Mooney-Rivlin law

$$W = C_1(I_1 - 3) + C_2(I_2 - 3)$$

- Ogden

$$W(\lambda_1, \lambda_2, \lambda_3) = \sum_{p=1}^{N} \frac{\mu_p}{\alpha_p}\left(\lambda_1^{\alpha_p} + \lambda_2^{\alpha_p} + \lambda_3^{\alpha_p} - 3\right)$$

- Use of Penn invariant for nearly incompressible materials:

$$\underline{F} \rightarrow \overline{\underline{F}} = \frac{1}{(\det \underline{F})^{1/3}} \underline{F} \qquad \text{such that } \det \overline{\underline{F}} = 1$$

- then

$$\overline{\underline{C}} = \overline{\underline{F}}^T . \overline{\underline{F}}$$

- and

$$\begin{aligned} \bar{I}_1 &= \mathrm{Tr}\overline{\underline{C}} \\ \bar{I}_2 &= \mathrm{Tr}\overline{\underline{C}}.\overline{\underline{C}} \\ \det \overline{\underline{C}} = 1 & \end{aligned}$$

- Modified strain energy density function

$$W = C_1(\bar{I}_1 - 3) + C_2(\bar{I}_2 - 3)$$

Constitutive equations : hypo-elasticity

- The constitutive equations are written in a rate form relating any objective stress rate to the deformation rate $\underline{D}$:

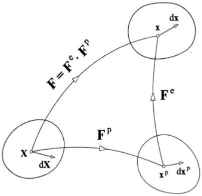
$$\underline{\sigma}^J, \overset{\circ}{\underline{\sigma}}, \overset{\square}{\underline{\sigma}}, \cdots = \underline{\underline{\Lambda}} : \underline{D}$$

- These constitutive equations may be path dependent . . . not physical

- One assume an elastic ($\underline{F}^e$) / plastic ($\underline{F}^p$) transformation decomposition

$$\underline{F} = \underline{F}^e.\underline{F}^p$$

- The decomposition defines an intermediate state :



- The deformation rate is given by:

$$\underline{L} = \underline{\dot{F}}.\underline{F}^{-1} = \underline{\dot{F}}^e.\underline{F}^{e-1} + \underline{F}^e.\underline{\dot{F}}^p.\underline{F}^{p-1}.\underline{F}^{e-1} = \underline{L}^e + \underline{F}^e.\underline{L}^p.\underline{F}^{e-1}$$

- Express $\underline{L}^p = \underline{D}^p + \underline{W}^p$

- Crystal plasticity

$$\underline{D}^p = \sum_s \dot{\gamma}_s \left(\vec{m}_s \otimes \vec{n}_s + \vec{m}_s \otimes \vec{n}_s\right) \qquad \underline{W}^p = \sum_s \dot{\gamma}_s \left(\vec{m}_s \otimes \vec{n}_s - \vec{m}_s \otimes \vec{n}_s\right)$$

$$\dot{\gamma}_s = \dot{\gamma}_s(\underline{T} : (\vec{m}_s \otimes \vec{n}_s))$$

- Isotropic von Mises plasticity

$$\underline{D}^p = \frac{3}{2}\dot{p}\frac{\underline{T}'}{T'_{\mathrm{eq}}} \qquad \underline{W}^p = \underline{0}$$

- $\underline{T}$ rotated stress (various possibilities)

Constitutive equations : corotational formulations

- The constitutive equation is expressed between the rotated stress

$$\underline{\sigma}_R = \underline{R}.\underline{\sigma}.\underline{R}^T$$

  and any stress measure constructed using $\underline{U}$
- The small strain formalism can be used for the constitutive equation
- The corresponding objective stress stress rate is the Green-Naghdi rate.

Constitutive equations : corotational formulations

- The constitutive equations are expressed using:

$$\underline{\sigma}_Q = \underline{Q}.\underline{\sigma}.\underline{Q}^T$$

where $\underline{Q}$ is obtained so that the instantaneous rotation rate of the medium wih respect to the frame is zero:

$$\underline{W}' = \underline{\dot{Q}}.\underline{Q}^T + \underline{Q}.\underline{W}.\underline{Q}^T = \underline{0}$$

so that

$$\underline{\dot{Q}} = -\underline{Q}.\underline{W}$$

- The corresponding strain tensor is:

$$\underline{\varepsilon}_Q = \int_t \underline{Q}.\underline{D}.\underline{Q}^T \, dt$$

- The constitutive equations then relate:

$$\underline{\sigma}_Q = f(\underline{\varepsilon}_Q) \qquad \text{small strain formalism}$$

- The corresponding objective stress stress rate is the Jaumann rate.