Reduced order model and simultaneous computation for thermomechanical multidimensional models

Fatima Daïm, D. Ryckelynck, D. M. Benziane

Centre des Matériaux
Mines-ParisTech

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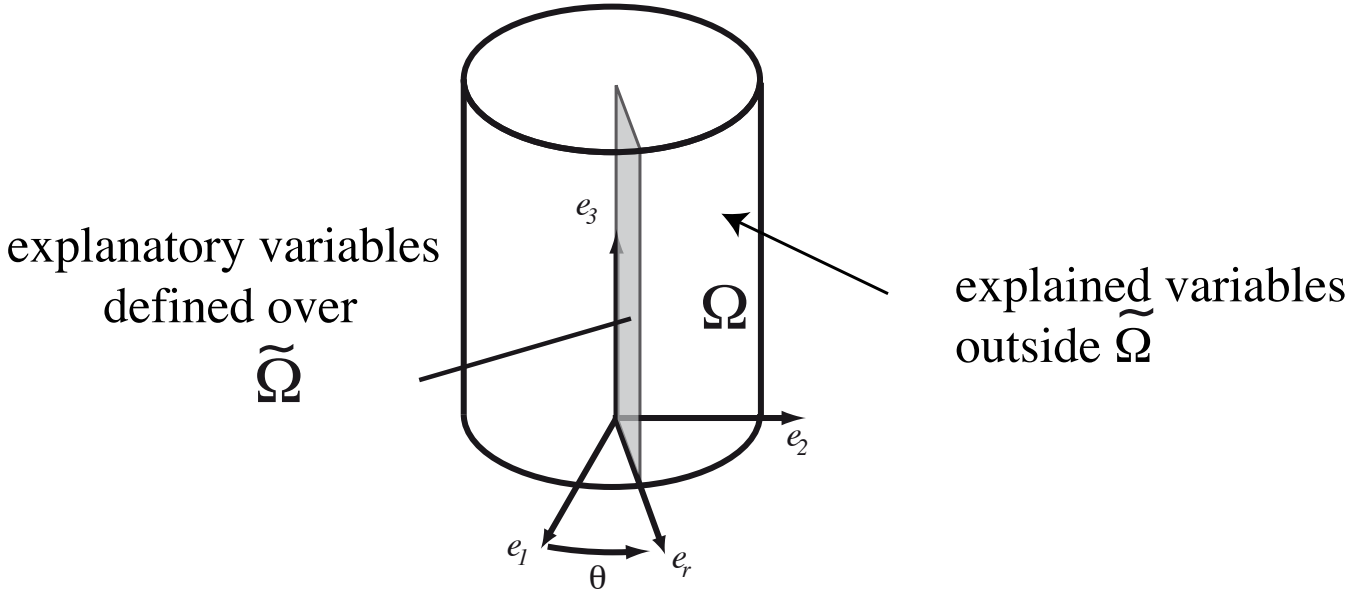
Plan

A postulate for the extension of the concept of symmetry.

If $u \in \mathcal{U}_{ROM} \subset \mathcal{U}$, there might be **explanatory variables** in mechanics of materials that should permit to **reduced the extent of the domain** over which equilibrium conditions are considered.

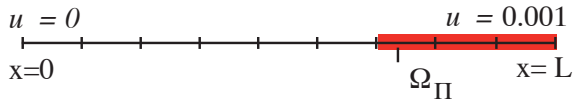


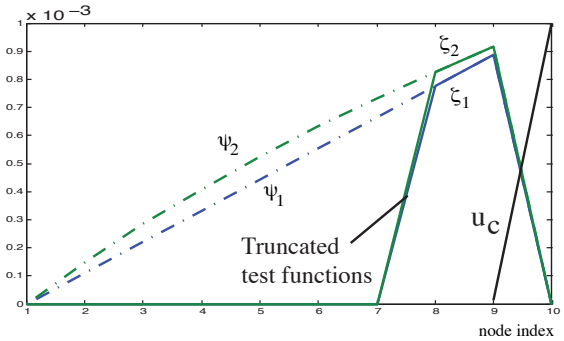
Canonical analysis and the PLS (Partial Least Square) method [Wold 1966] can help to find the best explanatory variables. These methods are convenient for **Data Mining**. But they don't take into account equilibrium equations or global conditions. We don't have data, but we have equations.

A simple theoretical example of Hyper-reduction

Let's consider the following elliptic PDE ($E > 0$), with Dirichlet conditions :

$$\frac{\partial}{\partial x}\left(E(X,p)\,\frac{\partial u}{\partial x}\right) = 0 \quad \forall x \in [0,L] \quad u(0,p) = 0 \quad u(L,p) = 0$$





$$\psi_1(x) = x \qquad \psi_2(x) = x\,(2\,L - x)$$

If $u(x,p) = \psi_1(x)\,a_1(p) + \psi_2(x)\,a_2(p) + u_c(x)$ then, a_1 and a_2 are such that :

$$\int_{\Omega_\Pi} \left(\zeta_1(x)\,a_1^* + \zeta_2(x)\,a_2^*\right)\,\frac{\partial}{\partial x}\left(E(X,p)\,\frac{\partial u}{\partial x}\right) dx = 0 \quad \forall a_1^*\,a_2^*$$

Plan

$$\boxed{\textit{Parameters} := \mathbf{p}}$$
$$\Downarrow$$
$$\boxed{X_{in}(\mathbf{x}, t)} \longrightarrow \boxed{\{\Sigma_{\mathbf{p}}\}_{\mathbf{p}}} \longrightarrow \boxed{X_{out}(\mathbf{x}, t)}$$

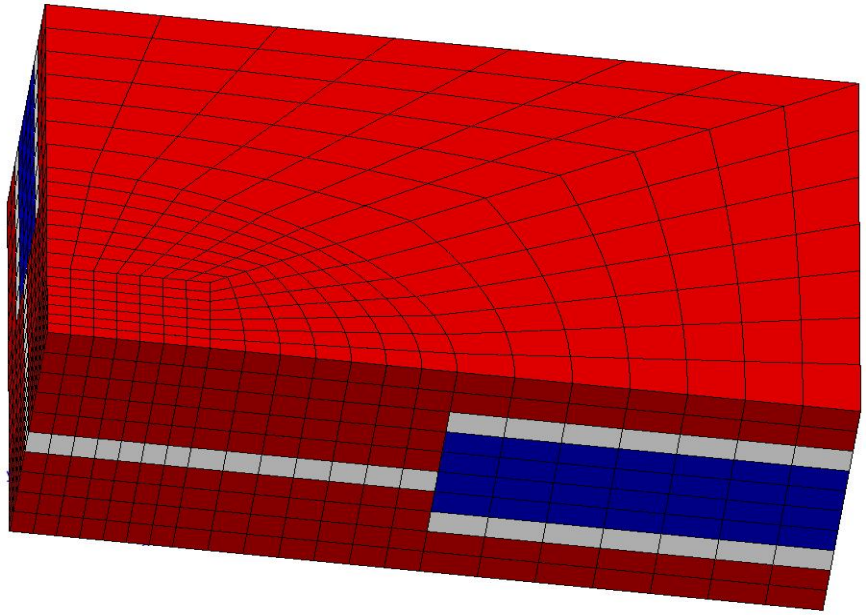
Objective :

The computation of the response in respect to parameters.

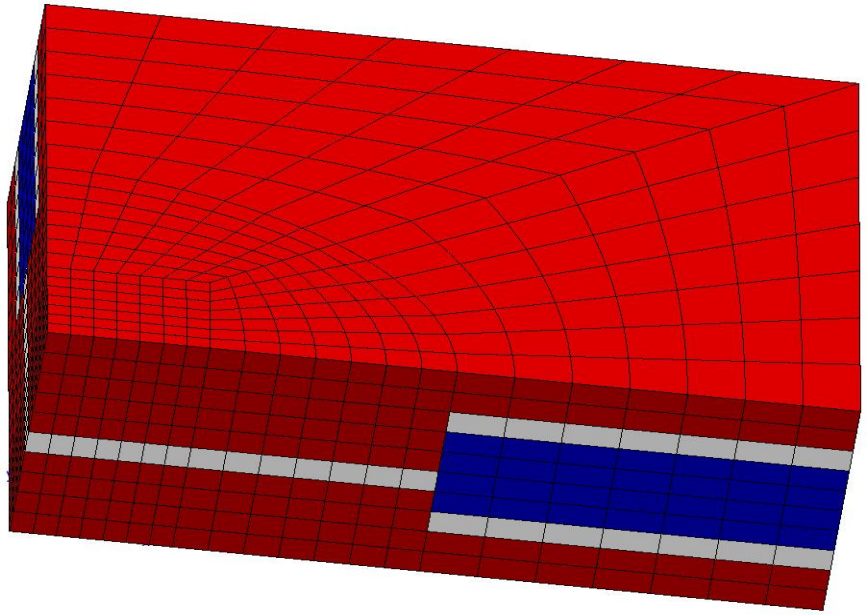
Tools :

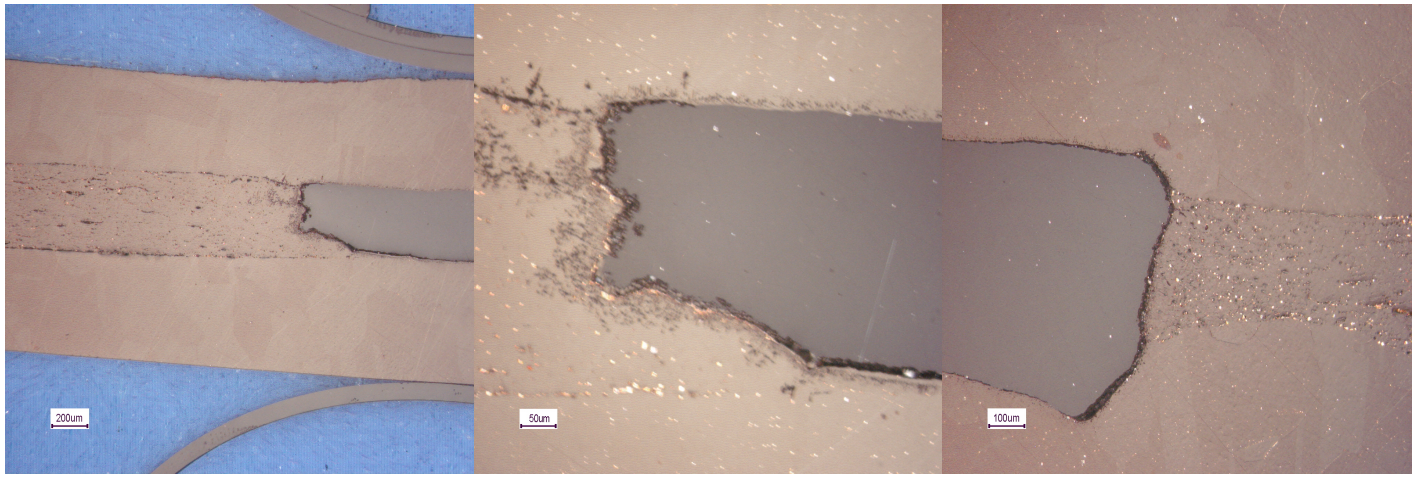
Given the limits of classical discretization methods for the multi-dimensional problems, the use of the reduced order models is necessary.

Material for mecatronic application

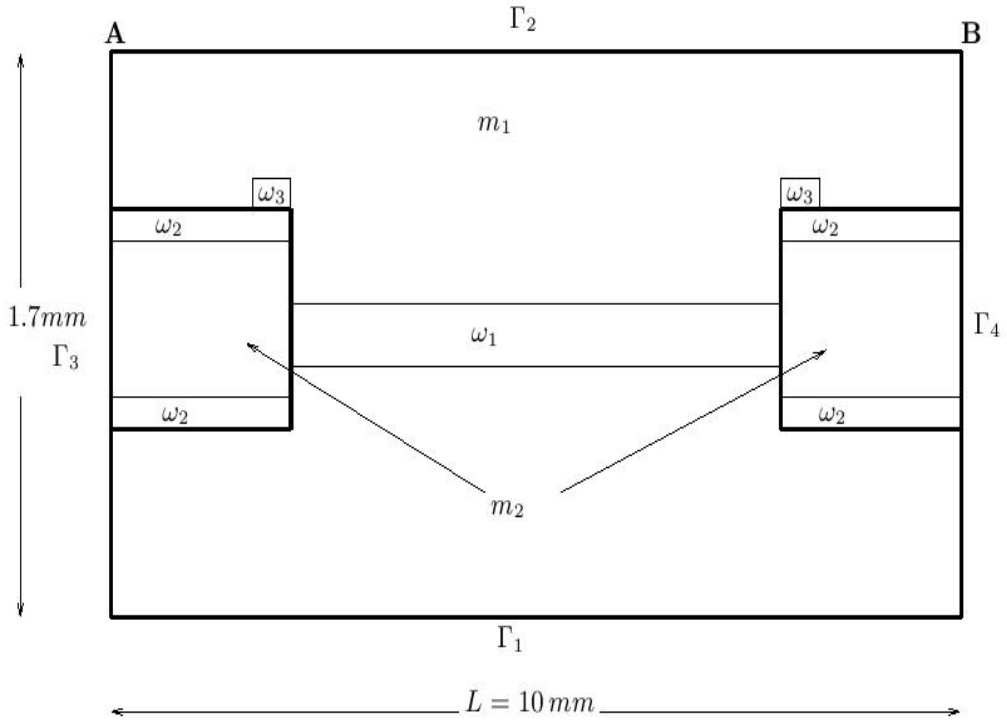


Material for mecatronic application





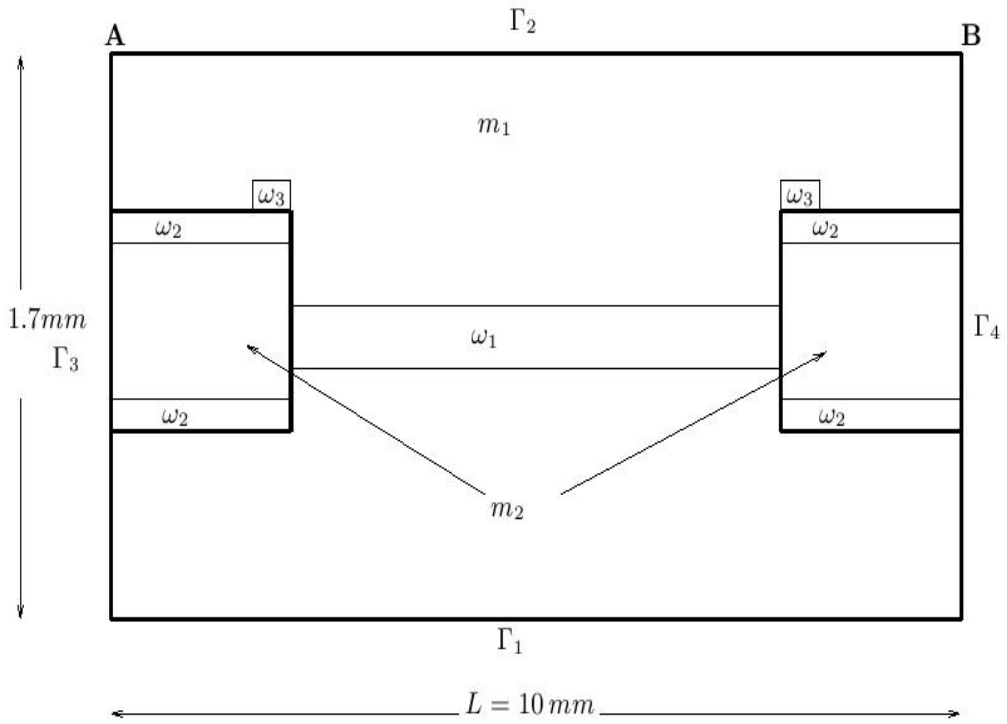
Thermal and mechanical perfomance indicators



Conductivity :

$$\mathcal{P}_T = e\,\frac{\dfrac{1}{|\Gamma_2|}\displaystyle\int_{\Gamma_2}\Phi(\mathbf{x})d\mathbf{x}}{\left(\dfrac{1}{|\Gamma_2|}\displaystyle\int_{\Gamma_2}T(\mathbf{x},t_{max})d\mathbf{x}-\dfrac{1}{|\Gamma_1|}\displaystyle\int_{\Gamma_1}T(\mathbf{x},t_{max})d\mathbf{x}\right)}$$

Thermal and mechanical perfomance indicators



Homogenized dilatation :

$$\mathcal{P}_E = \frac{(\mathbf{u}(\mathbf{A}, t_{max}, T(\mathbf{A}, t_{max})) - \mathbf{u}(\mathbf{B}, t_{max}, T(\mathbf{B}, t_{max}))_x}{L\left(\frac{1}{|\Gamma_2|}\int_{\Gamma_2} T(\mathbf{x}, t_{max})d\mathbf{x} - T_0\right)}$$

Behaviour law ($\mathcal{M}^E$)

$$\begin{cases} \mathrm{div}\sigma(\mathbf{u}(\mathbf{x},t,T),T) &= 0 \quad \text{in } \Omega\times(0,t_{max}] \\ \varepsilon(\mathbf{u}(\mathbf{x},t,T)) &= \dfrac{1+\nu}{E(\mathbf{x},T)}\sigma(\mathbf{u}(\mathbf{x},t,T),T) - \dfrac{\nu}{E(\mathbf{x},T)}tr(\sigma(\mathbf{u}(\mathbf{x},t,T),T))\mathbb{I} + \alpha(T(\mathbf{x},t)-T_D)\mathbb{I} \\ \varepsilon(\mathbf{u}(\mathbf{x},t,T)) &= \dfrac{1}{2}(\nabla\mathbf{u}(\mathbf{x},t,T)+\nabla\mathbf{u}(\mathbf{x},t,T)^T), \\ \sigma(\mathbf{u}).\mathbf{n} &= 0 \quad \text{on } \partial\Omega\times(0,t_{max}], \\ \mathbf{u} &= 0 \quad \text{on } \Gamma_2\times(0,t_{max}]. \end{cases}$$

Nonlinear thermal loading ($\mathcal{M}^k$)

$$\begin{cases} \rho(\mathbf{x})c(\mathbf{x},t)\dfrac{\partial T}{\partial t}(\mathbf{x},t) - \mathrm{div}\,(k(\mathbf{x},t,T)\nabla T(\mathbf{x},t)) &= 0 & \text{in } \Omega\times(0,t_{max}] \\ T(\mathbf{x},0) &= T_D & \text{in } \Omega \\ \nabla T.\mathbf{n} &= h(T(\mathbf{x},t)-T_D) & \text{on } \Gamma_1\times(0,t_{max}] \\ \nabla T.\mathbf{n} &= \Phi & \text{on } \Gamma_2\times(0,t_{max}] \\ \nabla T.\mathbf{n} &= 0 & \text{on } \Gamma_3\cup\Gamma_4\times(0,t_{max}]. \end{cases}$$

Correction with damage parameters

$$\begin{cases} k_{\omega_i} &= (1 - p^T_{\omega_i}) k(\mathbf{x}, t, T), \\ E_{\omega_i} &= (1 - p^E_{\omega_i}) E(\mathbf{x}, t, T). \end{cases}$$

With $\mathbf{p} = (p^T_{\omega_1}, p^T_{\omega_2}, p^T_{\omega_3}, p^E_{\omega_1}, p^E_{\omega_2}, p^E_{\omega_3}) \in [0,1]^6$, we obtain $\left\{ (\mathcal{M}^{E_{\mathbf{p}}}, \mathcal{M}^{k_{\mathbf{p}}}) \right\}_{\mathbf{p}}$

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Thermal performance function

$$\begin{array}{rccl} \mathcal{P}_T : & [0,1]^6 & \rightarrow & \mathbb{R}^+ \\ & \mathbf{p} & \rightarrow & e \dfrac{\dfrac{1}{|\Gamma_2|} \displaystyle\int_{\Gamma_2} \Phi(\mathbf{x}) d\mathbf{x}}{\left(\dfrac{1}{|\Gamma_2|} \displaystyle\int_{\Gamma_2} T(\mathbf{x}, \mathbf{p}, t_{max}) d\mathbf{x} - \dfrac{1}{|\Gamma_1|} \displaystyle\int_{\Gamma_1} T(\mathbf{x}, \mathbf{p}, t_{max}) d\mathbf{x} \right)}, \end{array}$$

Mechanical performance function

$$\begin{array}{rccl} \mathcal{P}_E : & [0,1]^6 & \rightarrow & \mathbb{R}^+ \\ & \mathbf{p} & \rightarrow & \dfrac{(\mathbf{u}(\mathbf{A}, \mathbf{p}, t_{max}, T(\mathbf{A}, \mathbf{p}, t_{max})) - \mathbf{u}(\mathbf{B}, \mathbf{p}, t_{max}, T(\mathbf{B}, \mathbf{p}, t_{max}))_x}{L \left(\dfrac{1}{|\Gamma_2|} \displaystyle\int_{\Gamma_2} T(\mathbf{x}, \mathbf{p}, t_{max}) d\mathbf{x} - T_0 \right)} \end{array}$$

Plan

Weak formulation of PDE :

$\hookrightarrow$ find $\mathbf{u}(\mathbf{x}, t)$ such that

$$\boxed{\int_\Omega \mathcal{L}(\mathbf{x}, t, \mathbf{u}(\mathbf{x}, t))\, \mathbf{v}(\mathbf{x}) d\mathbf{x} = 0, \quad \forall \mathbf{v} \in \mathcal{V}, \quad \forall t \in (0, t_{max}]}$$

$$\Downarrow$$

Finite element method (FEM) :

$\hookrightarrow$ find $\{\mathbf{a}_j(t)\}_{j=1}^{N_{dof}}$ such that $\mathbf{u}(\mathbf{x}, t) = \sum_{j=1}^{N_{dof}} \mathbf{a}_j(t) \mathbf{N}_j(\mathbf{x})$ and

$$\boxed{\int_\Omega \mathcal{L}(\mathbf{x}, t^n, \mathbf{u}(\mathbf{x}, t^n))\, \mathbf{v}_h(\mathbf{x}) d\mathbf{x} = 0, \quad \forall \mathbf{v}_h \in \mathcal{V}_h, \quad \forall n = 1, \cdots, N_t}$$

- $\star$ $\mathcal{V}_h = span\{\mathbf{N}_j\}_{j=1}^{N_{dof}}$ shape functions related to a given mesh of Ω,
- $\star$ $\cup_{n=1}^{N_t} \left[t^{n-1}, t^n\right] = (0, t_{max}]$.

Proper Orthogonal Decomposition (POD) :

$\hookrightarrow$ find $\{\varphi_j(t)\}_{j=1}^{N_{rom}}$ such that $\mathbf{u}(\mathbf{x},t) = \sum_{j=1}^{N_{rom}} \varphi_j(t)\psi_j(\mathbf{x})$

$$\boxed{\int_\Omega \mathcal{L}(\mathbf{x}, t^n, \mathbf{u}(\mathbf{x}, t^n))\, \mathbf{v}_{rom}(\mathbf{x})\, d\mathbf{x} = 0, \quad \forall \mathbf{v}_{rom} \in \mathcal{V}_{rom}, \quad \forall n = 1, \cdots, N_t}$$

- $\star$ $\mathcal{V}_{rom} = span\{\psi_j\}_{j=1}^{N_{rom}}$.
- $\star$ $\{\psi_j\}_{j=1}^{N_{rom}}$ the eigenvectors corresponding to the N_{rom} largest eigenvalues of the correlation matrix related to the snapshots (with $N_{rom} << N_{dof}$).

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☹ POD needs a priori knowing of the solutions and does not offer the possibility to adapt the reduced basis.

A priori hyper reduction method (APHR)

Prediction Step : For $\mathcal{V}_{rom}^{(m)} = span\{\psi_j^{(m)}(\mathbf{x})\}_{j=1}^{N_{rom}}$, find $\{\varphi_j(t^n)\}_{j=1}^{N_{rom}}$ such that

$$\mathbf{u}(\mathbf{x}, t^n) = \sum_{j=1}^{N_{rom}} \varphi_j(t^n)\psi_j^{(m)}(\mathbf{x})$$

$$\boxed{\int_{\Omega_\Pi \subset \Omega} \mathcal{L}(\mathbf{x}, t^n, \mathbf{u}(\mathbf{x}, t^n))\, \hat{\mathbf{v}}_{rom} d\mathbf{x} = 0, \quad \forall \mathbf{v}_{rom} \in \mathcal{V}_{truncated\ rom}^{(m)}}$$

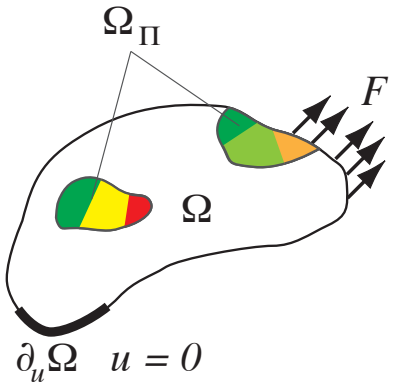
Correction Step (using eventually a FETI solver for // computing) :

$\hookrightarrow$ Find $\delta\mathbf{u}(\mathbf{x}) = \sum_{j=1}^{N_{dof}} \mathbf{a}_j \mathbf{N}_j(\mathbf{x})$ such that

$$\boxed{\int_{\Omega} \mathcal{L}(\mathbf{x}, t^n, \delta\mathbf{u} + \mathbf{u}(\mathbf{x}, t^n))\, \mathbf{v}_h d\mathbf{x} = 0, \quad \forall \mathbf{v}_h \in \mathcal{V}_h}$$

$\hookrightarrow$ Enrich the reduced basis $\Rightarrow \mathcal{V}_{rom}^{(m+1)}$

The construction of the RID Ω_Π



Ω_Π is built by aggregating balls whose the radius is $h/2$ or $2h$. Each vector of reduced bases provide a center of a ball.

- for each mode $\Upsilon_{\mathcal{A}k}$ related to an internal variable $\mathcal{A}$ the center of the ball is :

$$\mathbf{x} = \arg \max_{\in\Omega} |\Upsilon_{\mathcal{A}k}()| \tag{1}$$

- for each displacement mode ψ_k the center of the ball is :

$$\mathbf{x} = \arg \max_{\in\Omega} (\, \varepsilon(\psi_k)() : \varepsilon(\psi_k)() \,) \tag{2}$$

Plan

Multidimensional model :

$$\boxed{\mathcal{L}(\mathbf{x}, \mathbf{p}, t, \mathbf{u}(\mathbf{x}, \mathbf{p}, t)) = 0 \quad \text{on } \Omega \times \mathbf{P} \times]0, t_{max}].}$$

$$\Downarrow$$

Simultaneous resolution :

$$\boxed{\mathcal{L}(\mathbf{x}', t, \mathbf{u}(\mathbf{x}', t)) = 0 \quad \text{on } \Omega' \times]0, t_{max}]}.$$

$$\Downarrow$$

Weak formulation :

$$\boxed{\int_{\Omega'} \mathcal{L}(\mathbf{x}', t^n, \mathbf{u}(\mathbf{x}', t^n)) \mathbf{v}'(\mathbf{x}')\, d\mathbf{x}' = 0, \quad \forall \mathbf{v}' \in L^2(\mathbf{P}; \mathcal{V})}$$

Where $L^2(\mathbf{P}; \mathcal{V}) = \left\{ \mathbf{v} : \mathbf{P} \to \mathcal{V}, \int_{\mathbf{P}} \|\mathbf{v}(., \mathbf{p})\|^2_{\mathcal{V}} d\mathbf{p} < \infty \right\}$

Multidimensional problems should have more pseudo-symmetries.

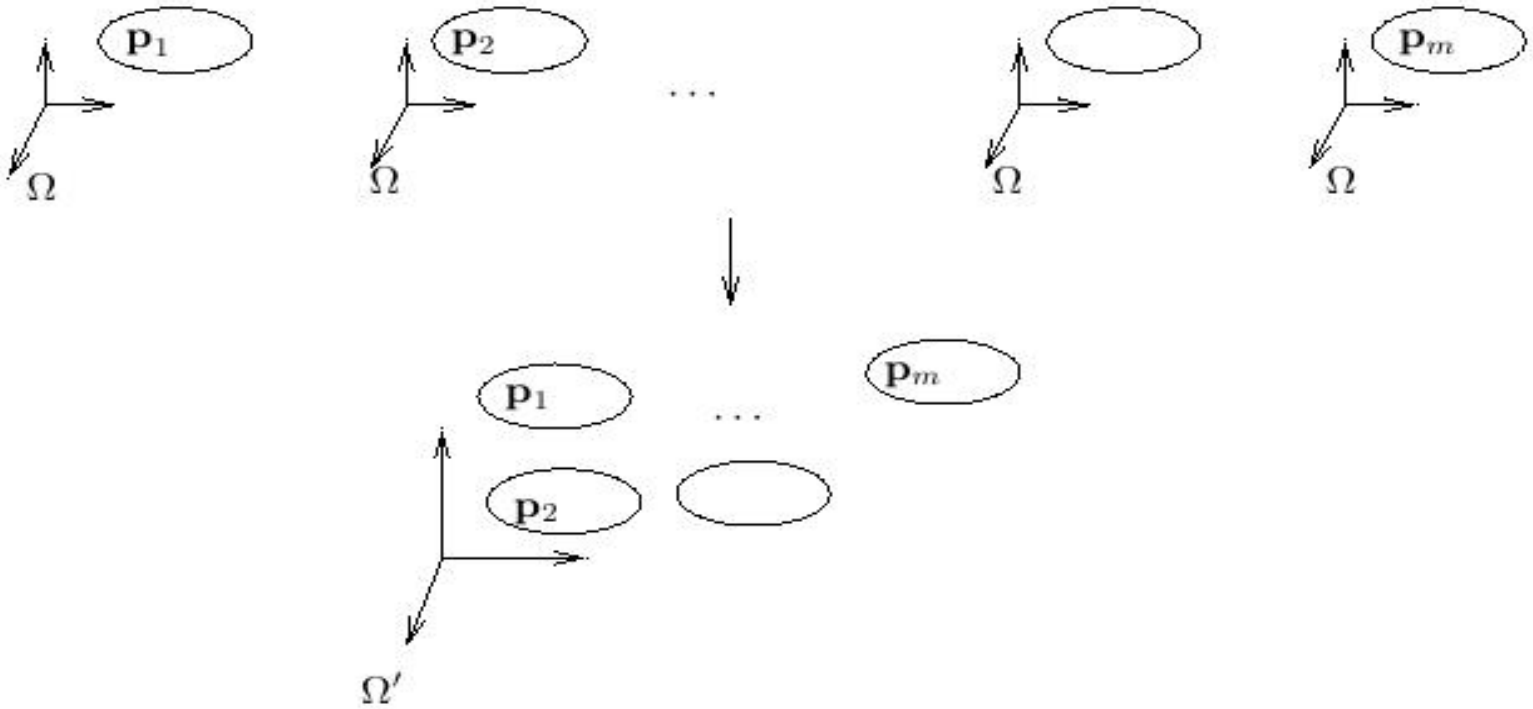


FIGURE: The virtual domain $\Omega' = \Omega \times \mathbf{P}$.

Mechanical problem

$$\int_{\Omega\times\mathbf{P}} \sigma(\mathbf{u}(\mathbf{x},\mathbf{p},t)) : \varepsilon(\mathbf{v}(\mathbf{x},\mathbf{p}))d\mathbf{x}\,d\mathbf{p} = \alpha \int_{\Omega\times\mathbf{P}} \left(T(\mathbf{x},\mathbf{p},t) - T_D\right) \operatorname{div}\mathbf{v}(\mathbf{x},\mathbf{p})d\mathbf{x}\,d\mathbf{p},$$

$$\forall t \in (0,t_{max}], \forall \mathbf{v} \in \mathcal{V} \;=\; \left\{\mathbf{v} \in L^2\left(\mathbf{P};\mathbf{H^1}(\Omega)\right), \mathbf{v}(.,t)|_{\Gamma_2\times\mathbf{P}} = 0\right\}.$$

Thermal problem

$$\int_{\Omega\times\mathbf{P}} \rho(\mathbf{x})c(\mathbf{x},t)\frac{\partial T}{\partial t}(\mathbf{x},\mathbf{p},t)w(\mathbf{x},\mathbf{p})d\mathbf{x}\,d\mathbf{p} + \int_{\Omega\times\mathbf{P}} k(\mathbf{x},t,T)\nabla T(\mathbf{x},\mathbf{p},t)\nabla w(\mathbf{x},\mathbf{p})d\mathbf{x}\,d\mathbf{p} +$$
$$+ \int_{\Gamma_1\times\mathbf{P}} h\left(T(\mathbf{x},\mathbf{p},t) - T_D\right) w(\mathbf{x},\mathbf{p})d\mathbf{x}\,d\mathbf{p} = \int_{\Gamma_2\times\mathbf{P}} \Phi w(\mathbf{x},\mathbf{p})d\mathbf{x}\,d\mathbf{p}$$

$$\forall t \in (0,t_{max}], \forall w \in L^2\left(\mathbf{P};H^1(\Omega)\right).$$

Is there an optimal strategy for parallel computing ?

Each case of damage simulation can be run on a processor of a parallel computer.

Does the hyper-reduction of the multidimensional problem is efficient when using parallel computing ?

Plan

Objective :

Variability of the thermal and the mechanical performances for $p^T_{\omega_1}, p^T_{\omega_2}, p^T_{\omega_3}, p^E_{\omega_1}, p^E_{\omega_2}, p^E_{\omega_3} \in \{0.8, 0.9\}$ (64 cases) such that

$$\mathcal{P}_T(T, \mathbf{p}) = e \frac{\frac{1}{|\Gamma_2|}\int_{\Gamma_2} \Phi(\mathbf{x})d\mathbf{x}}{\left(\frac{1}{|\Gamma_2|}\int_{\Gamma_2} T(\mathbf{x}, \mathbf{p}, t_{max})d\mathbf{x} - \frac{1}{|\Gamma_1|}\int_{\Gamma_1} T(\mathbf{x}, \mathbf{p}, t_{max})d\mathbf{x}\right)},$$

$$\mathcal{P}_E(\mathbf{u}, T, \mathbf{p}) = \frac{(\mathbf{u}(\mathbf{A}, \mathbf{p}, t_{max}, T(\mathbf{A}, \mathbf{p}, t_{max})) - \mathbf{u}(\mathbf{B}, \mathbf{p}, t_{max}, T(\mathbf{B}, \mathbf{p}, t_{max}))_x}{L\left(\frac{1}{|\Gamma_2|}\int_{\Gamma_2} T(\mathbf{x}, \mathbf{p}, t_{max})d\mathbf{x} - T_0\right)}$$

Domain and mesh

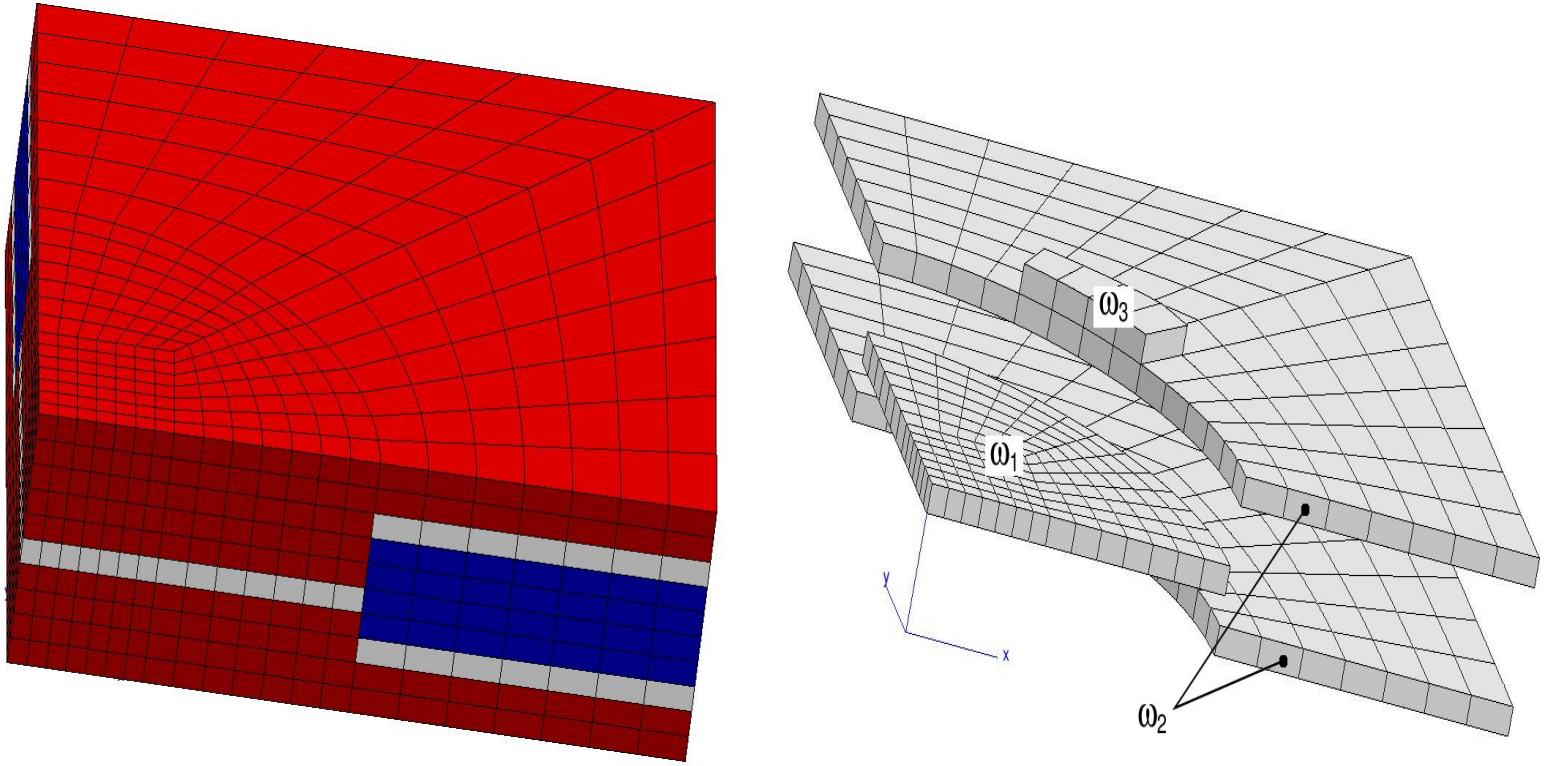
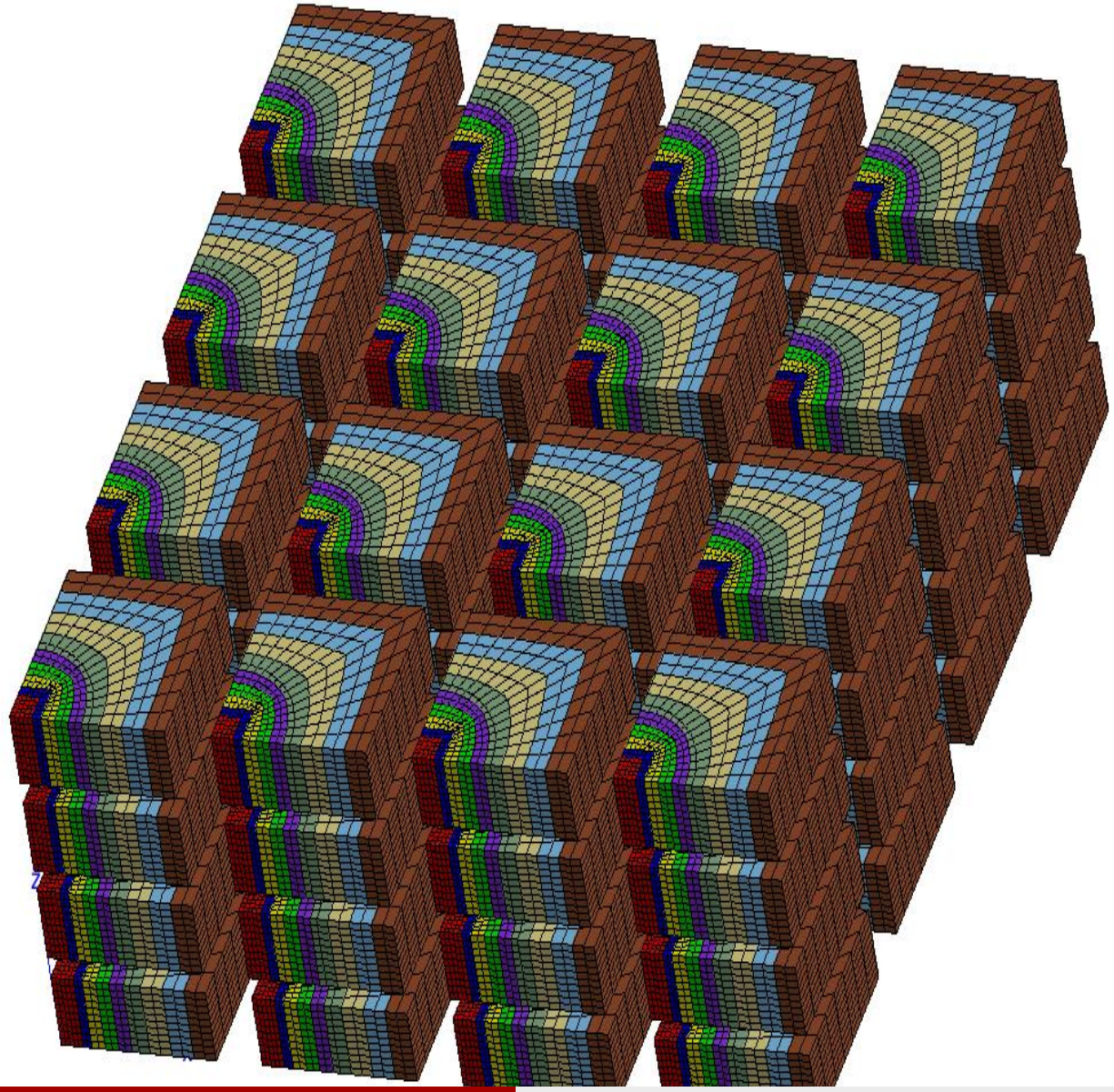
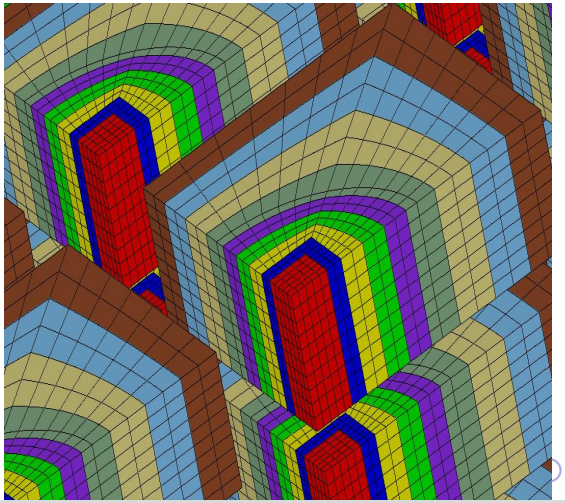


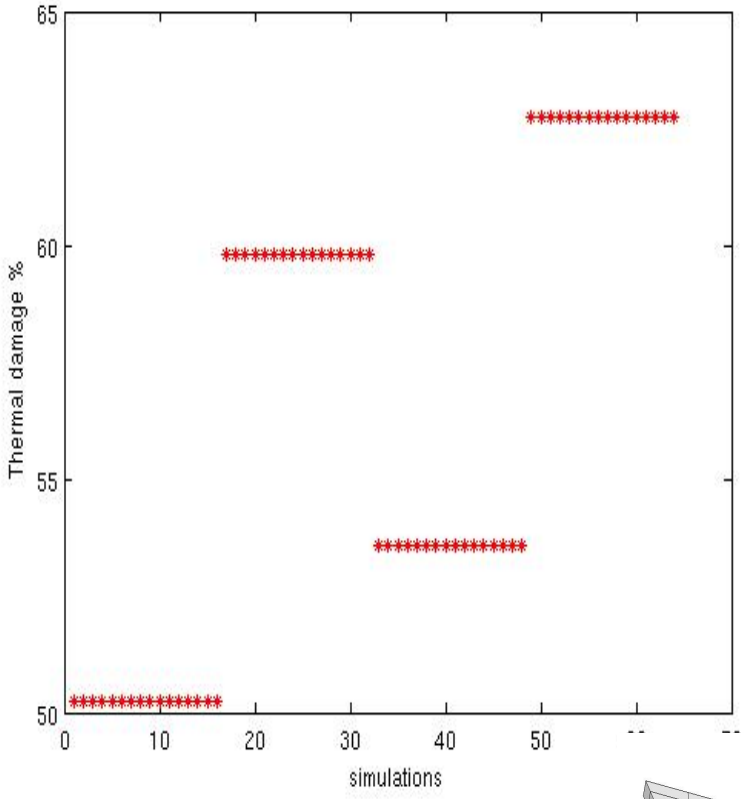
FIGURE: The composite Ω $(m_1 \cup m_2)$ and damage zones ω_1, ω_2 and ω_3.

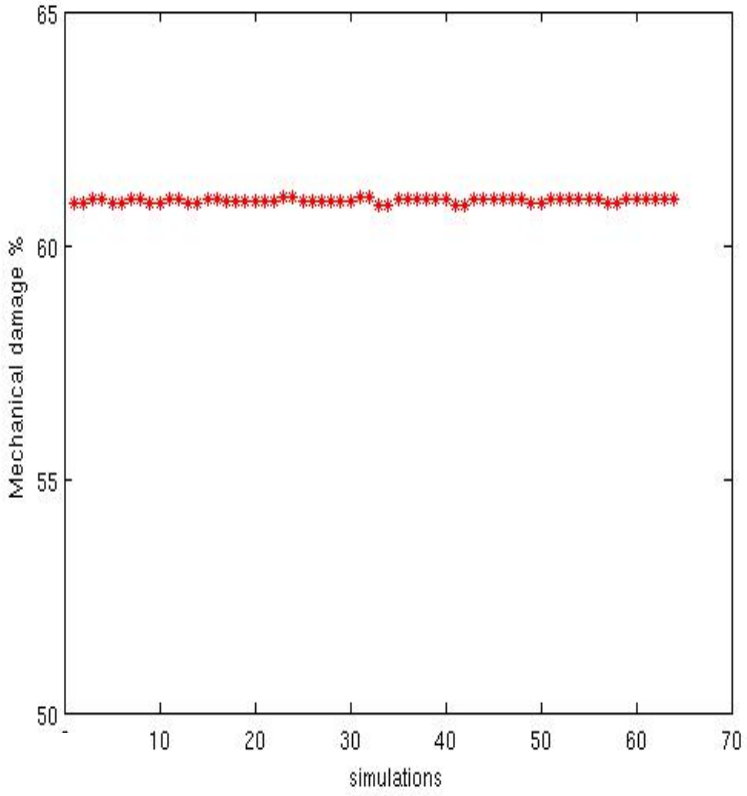
Simultaneous mesh $\Omega' = \Omega \times \mathbf{P}$ (64 simulations)

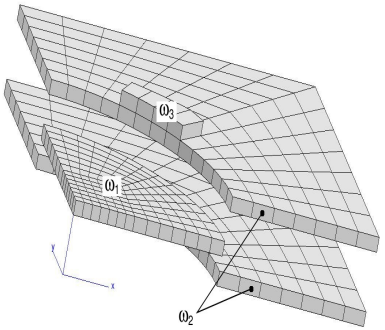




Damage rates







The relative errors

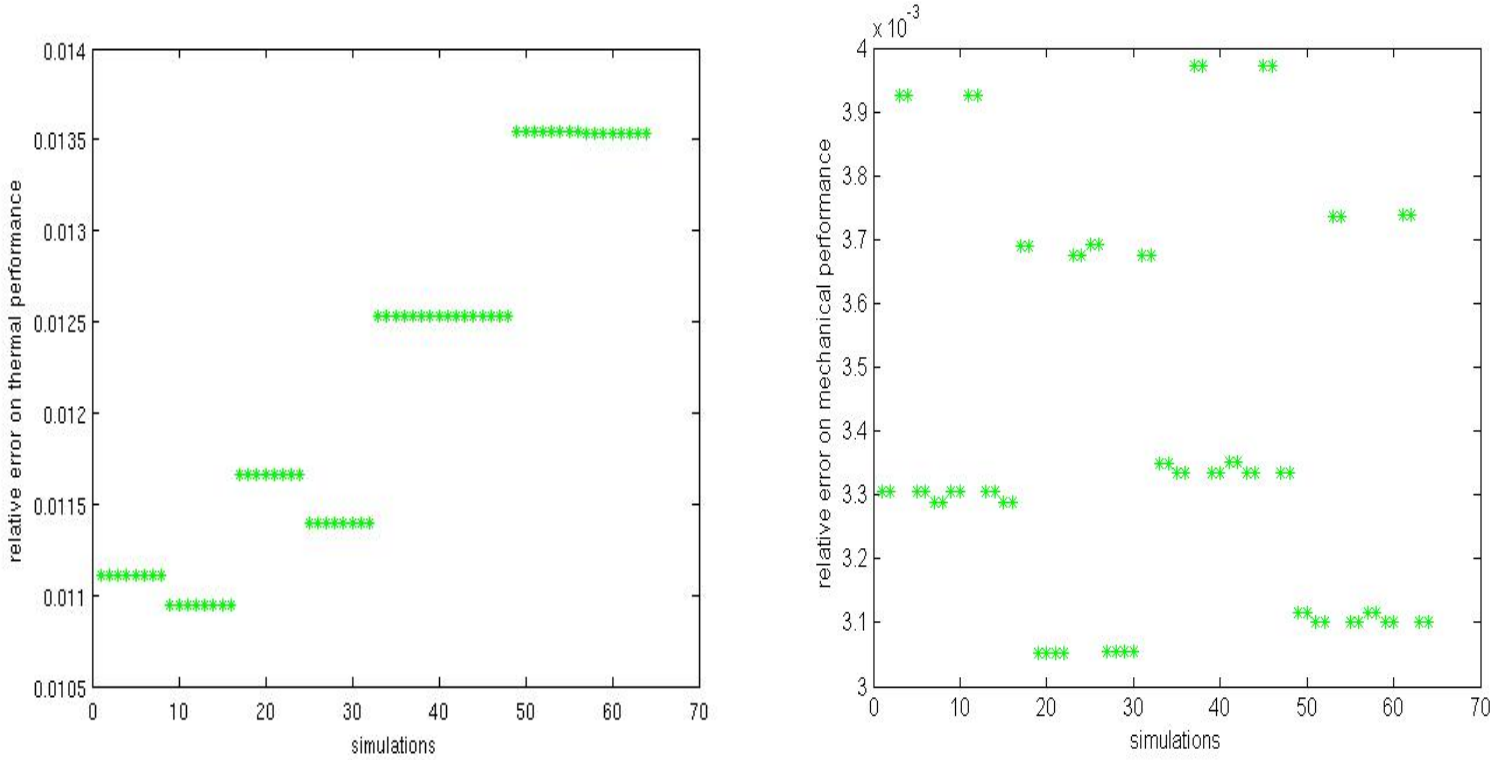


FIGURE: The relative error between APHR computation and FEM for respectively thermal and mechanical performances

Reduced integration domains (RID)

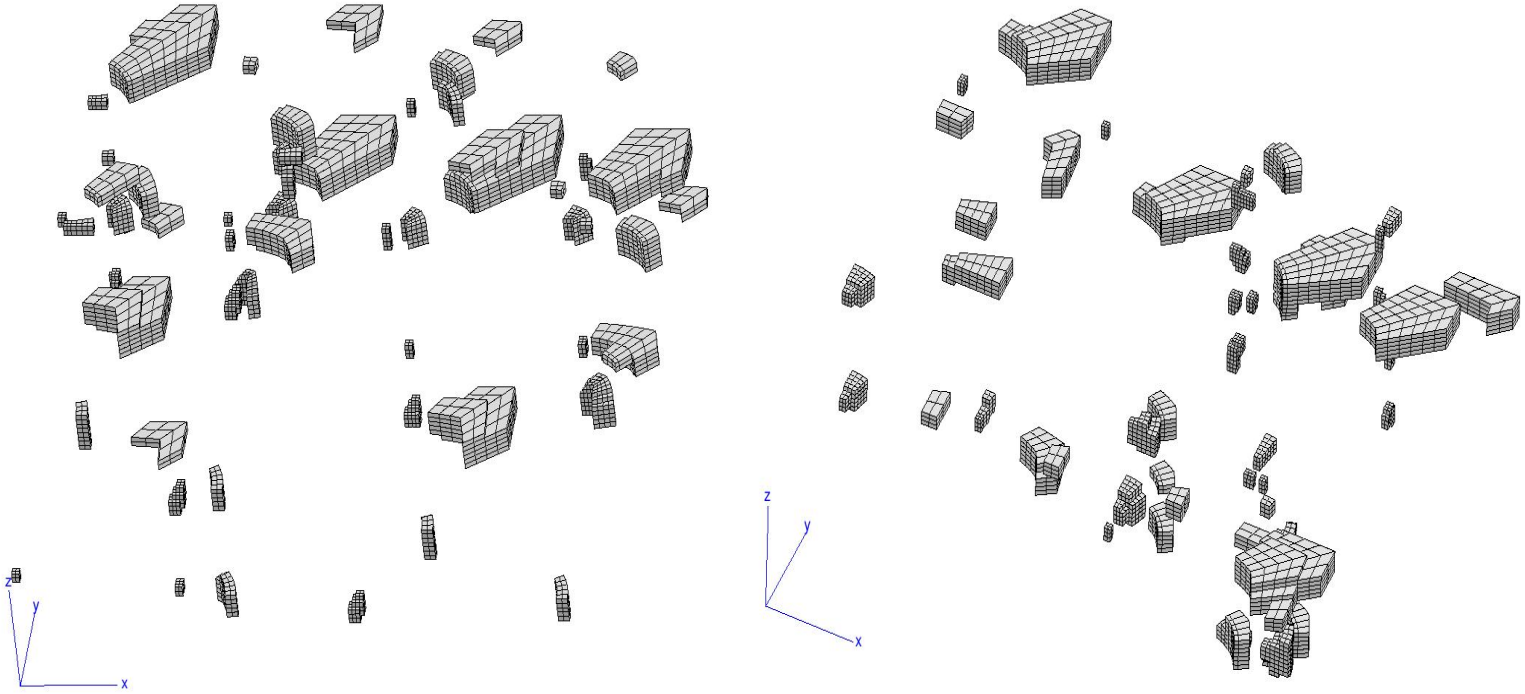


FIGURE: In the left hand side : the reduced integration domain $\Omega_{\Pi^t} \subset \Omega \times \mathbf{P}$ of the thermal problem and in the right hand side : the reduced integration domain $\Omega_{\Pi^m} \subset \Omega \times \mathbf{P}$ of the mechanical problem.

Scalability

Saving on CPU time

	(4 procs, 32 cases)	(8 procs, 64 cases)
Thermal problem	15.28 %	48.10 %
Mechanical problem	46.47 %	54.64 %

Saving on time of computation of the internal variables

	(4 procs, 32 cases)	(8 procs, 64 cases)
Thermal problem	72.86 %	84.32 %
Mechanical problem	88 %	91.36 %

Conclusion

If multidimensional problems have more symmetries or pseudo symmetries, the hyper-reduction is more efficient in these cases than in classical dimensions. An optimal parallel strategy should exist for the simulations based on hyper-reduced multidimensionnal model.