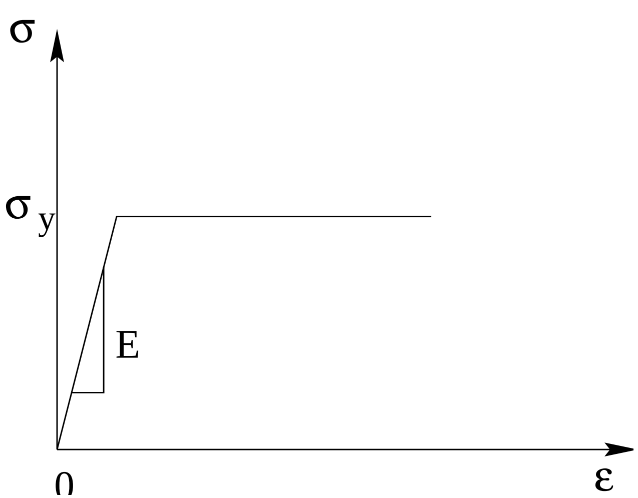
Rheology

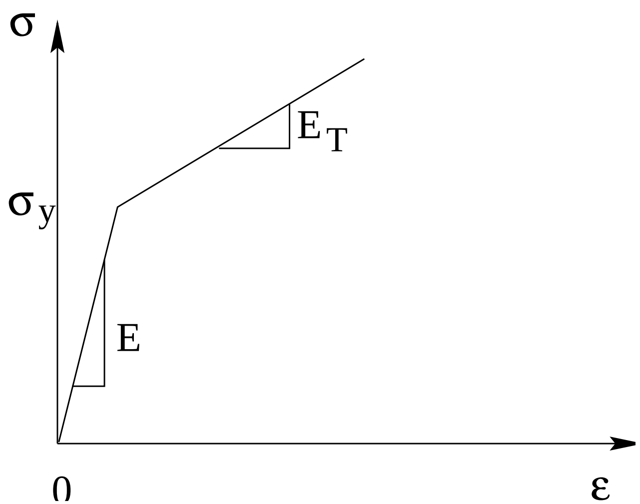
- **Onedimensional rheology**
- **Elastoplasticity, viscoelasticity, viscoplasticity**
- **Basic models**
- **Classical plasticity criteria**

–Review the foundation of inelastic behavior–

Plastic behavior for monotonic tensile loading



a. Elastic–perfectly plastic

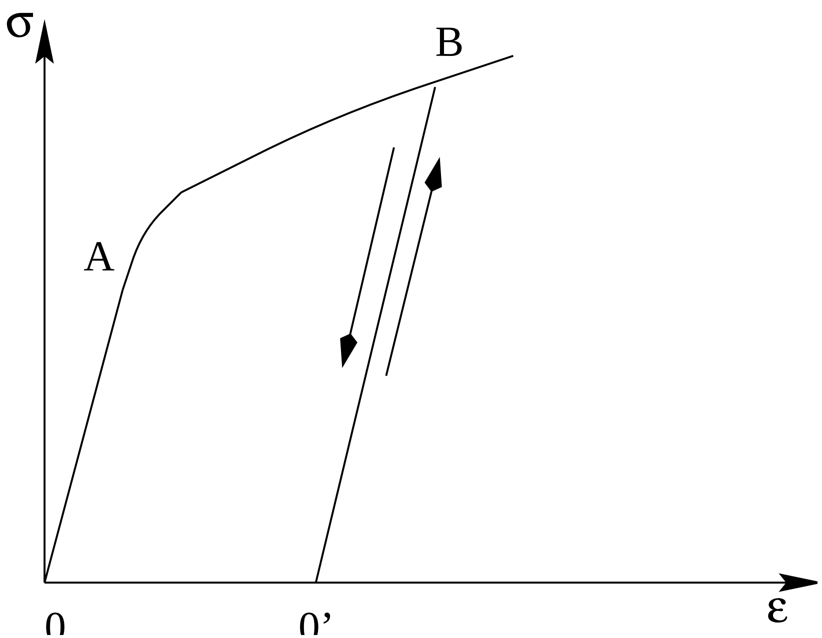


b. Elastic– linear plastic

Elastoplastic modulus, $E_T = d\sigma / d\varepsilon$.

The elastoplastic modulus is zero for perfectly plastic material, constant for a linear elastoplastic behavior; it depends on plastic strain in the general case

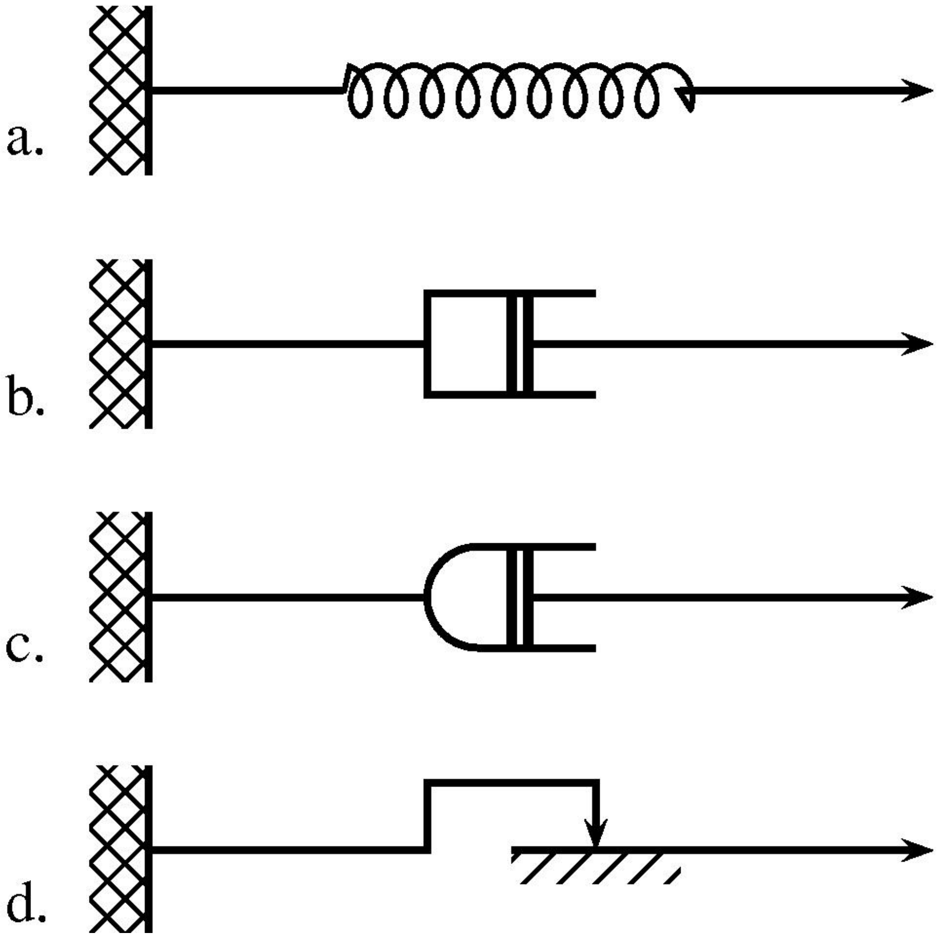
Various regimes in plasticity



Basic ingredients of the model :

- Strain partition assumption (*small perturbation analysis*) : $\varepsilon = \varepsilon^e + \varepsilon^p$
- The *elastic domain*, defined by a function f
- *Hardening*, defined through *hardening variables*, Y_I.

Building bricks for material modeling



$$\sigma = E\varepsilon$$

$$\sigma = \eta\,\dot{\varepsilon}$$

$$\sigma = \eta\,\dot{\varepsilon}^{1/N}$$

$$-\sigma_y \leq \sigma \leq \sigma_y$$

Various types of rheologies

- Time independent plasticity

$$\varepsilon = \varepsilon^e + \varepsilon^p \qquad d\varepsilon^p = f(...)d\sigma$$

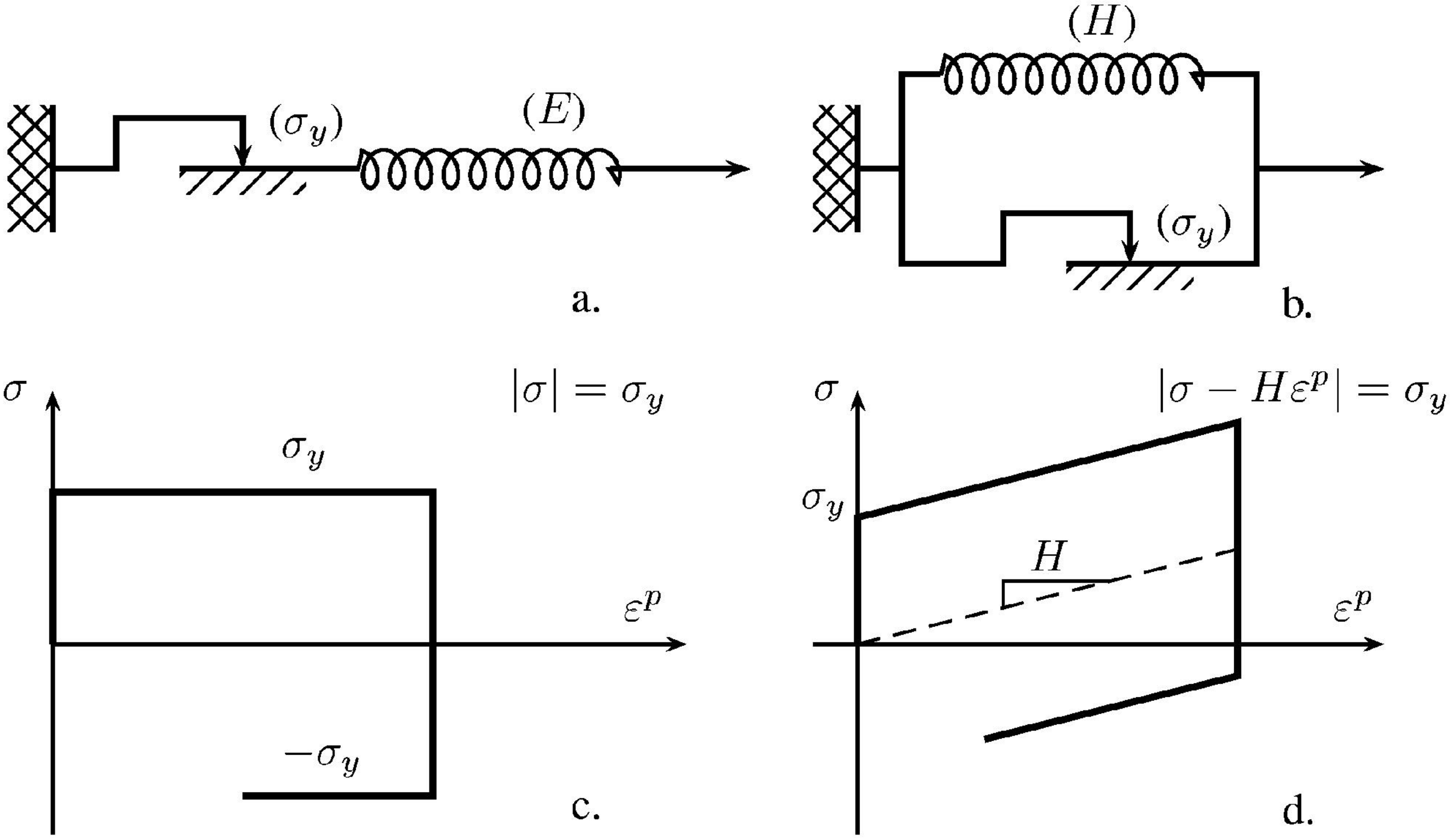
- Elastoviscoplasticity

$$\varepsilon = \varepsilon^e + \varepsilon^p \qquad d\varepsilon^p = f(...)dt$$

- Viscoelasticity

$$F(\sigma, \dot{\sigma}, \varepsilon, \dot{\varepsilon}) = 0$$

Time independent plasticity



Elastic–perfectly plastic material

$$f(\sigma) = |\sigma| - \sigma_y$$

– Elasticity domain if : $f < 0$ $\quad (\dot{\varepsilon} = \dot{\varepsilon}^e = \dot{\sigma}/E)$

– Elastic unloading if : $f = 0$ and $\dot{f} < 0$ $\quad (\dot{\varepsilon} = \dot{\varepsilon}^e = \dot{\sigma}/E)$

– Plastic flow : $f = 0$ and $\dot{f} = 0$ $\quad (\dot{\varepsilon} = \dot{\varepsilon}^p)$

Prager's rule

$$f(\sigma, X) = |\sigma - X| - \sigma_y \qquad \text{with } X = H\varepsilon^p$$

Plastic flow iff $f = 0$ and $\dot{f} = 0$.

$$\frac{\partial f}{\partial \sigma}\,\dot{\sigma} + \frac{\partial f}{\partial X}\,\dot{X} = 0$$

$$sign(\sigma - X)\,\dot{\sigma} - sign(\sigma - X)\,\dot{X} = 0$$

$$\dot{\sigma} = \dot{X}, \text{ and : } \quad \dot{\varepsilon}^p = \dot{\sigma}/H$$

$$\dot{\varepsilon}^p = \frac{E}{E+H}\,\dot{\varepsilon}$$

The state variable is plastic strain

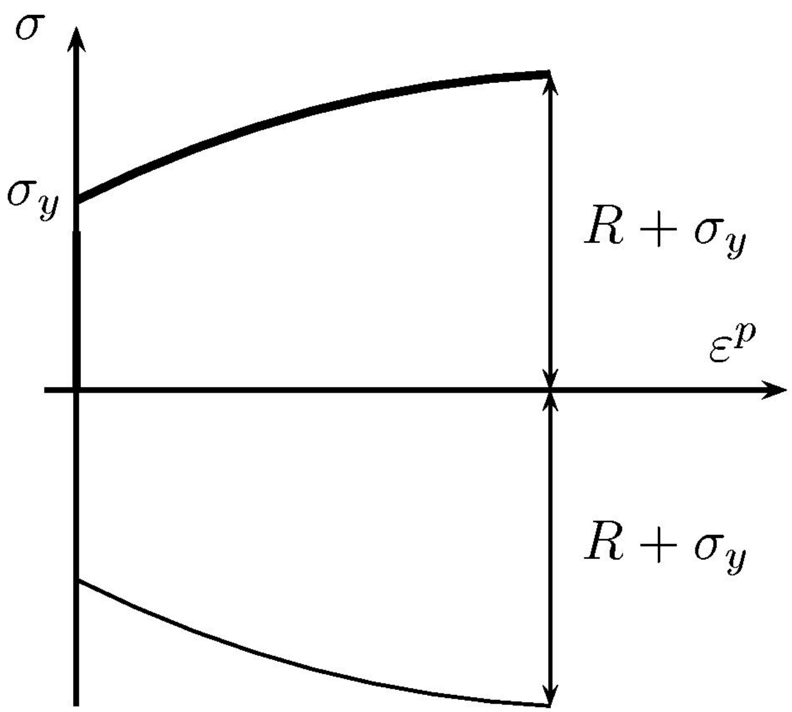
Generic expression for onedimensional time independent plasticity

- Elastic domain if : $f(\sigma, Y_i) < 0$ $(\dot{\varepsilon} = \dot{\sigma}/E)$
- Elastic unloading if : $f(\sigma, Y_i) = 0$ and $\dot{f}(\sigma, Y_i) < 0$ $(\dot{\varepsilon} = \dot{\sigma}/E)$
- Plastic flow if : $f(\sigma, Y_i) = 0$ and $\dot{f}(\sigma, Y_i) = 0$ $(\dot{\varepsilon} = \dot{\sigma}/E + \dot{\varepsilon}^p)$

Consistency condition :

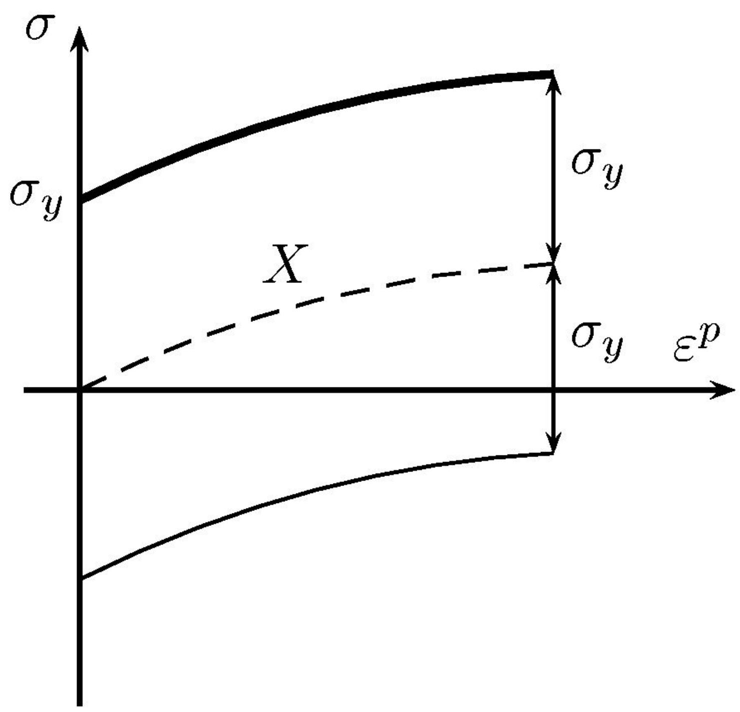
$$\dot{f}(\sigma, Y_i) = 0$$

Two hardening types



a. Isotropic

$$f(\sigma, R) = |\sigma| - R - \sigma_y$$



b. Kinematic

$$f(\sigma, X) = |\sigma - X| - \sigma_y$$

Isotropic hardening model

$dR/dp = H$ (p, accumulated plastic strain : $\dot{p} = |\dot{\varepsilon}^p|$)

$$f(\sigma, R) = |\sigma| - R - \sigma_y$$

Plastic flow iff $f = 0$ and $\dot{f} = 0$

$$\frac{\partial f}{\partial \sigma}\,\dot{\sigma} + \frac{\partial f}{\partial R}\,\dot{R} = 0$$

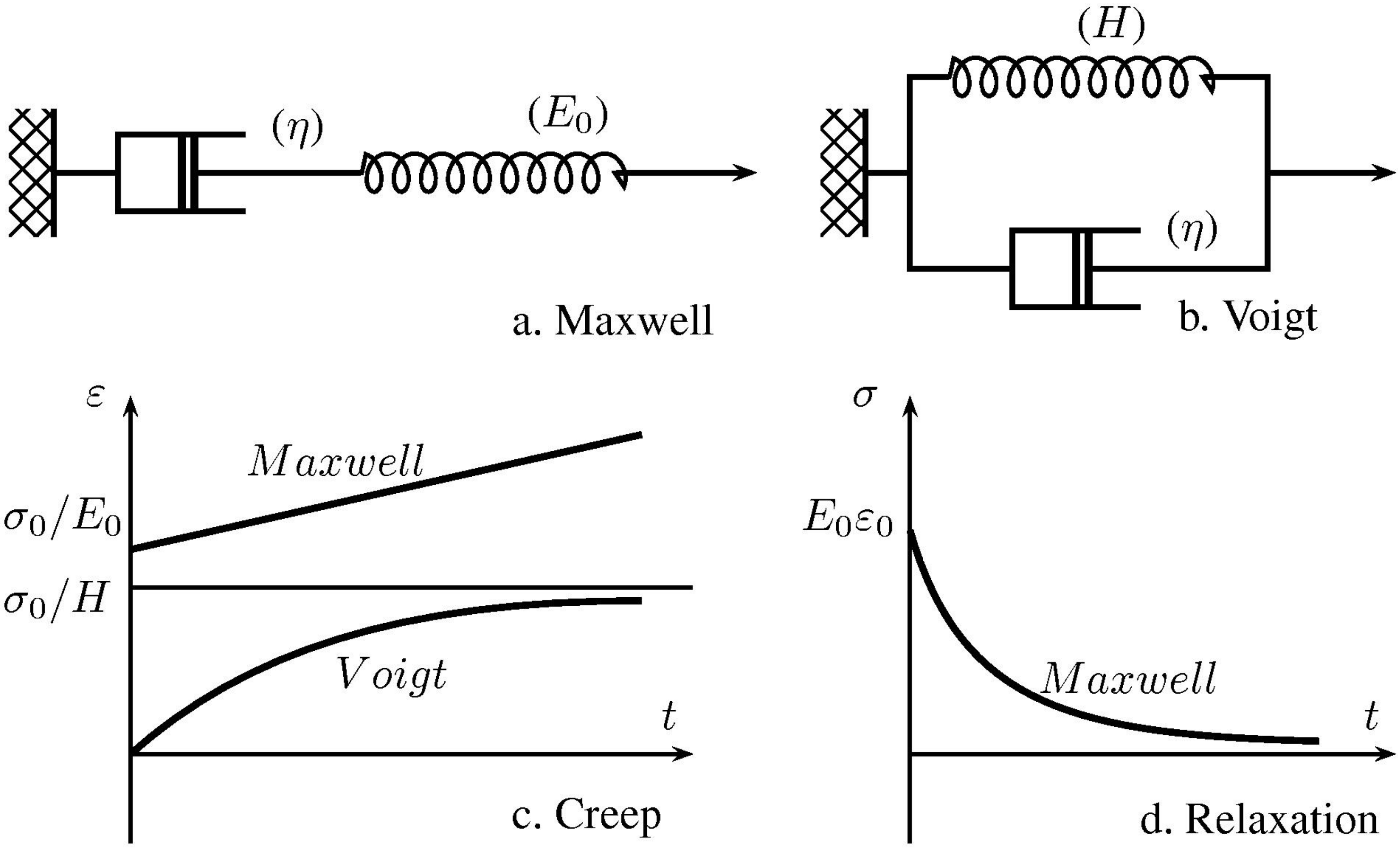
$$sign(\sigma)\,\dot{\sigma} - \dot{R} = 0$$

$$\dot{\sigma} = sign(\sigma)\,\dot{R}, \text{ and : } \quad \dot{p} = sign(\sigma)\,\dot{\sigma}/H$$

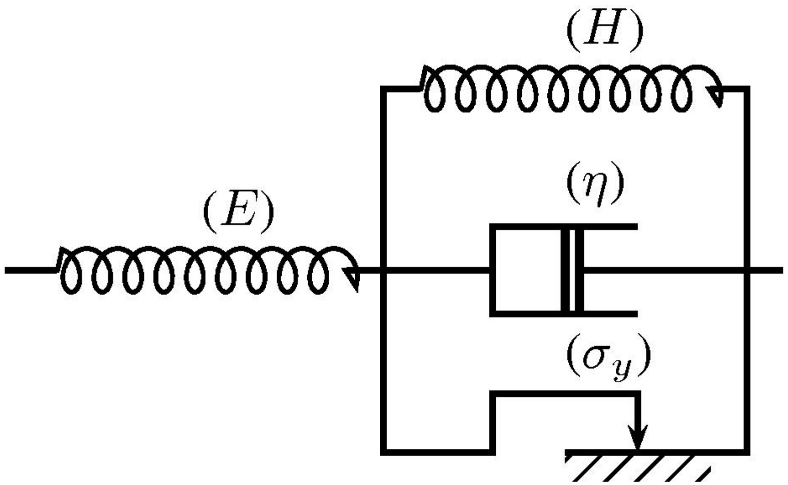
- Ramberg-Osgood : $\sigma \;=\; \sigma_y + K p^m$
- Exponential rule : $\sigma \;=\; \sigma_u + (\sigma_y - \sigma_u)\exp(-bp)$

The state variable is accumulated plastic strain

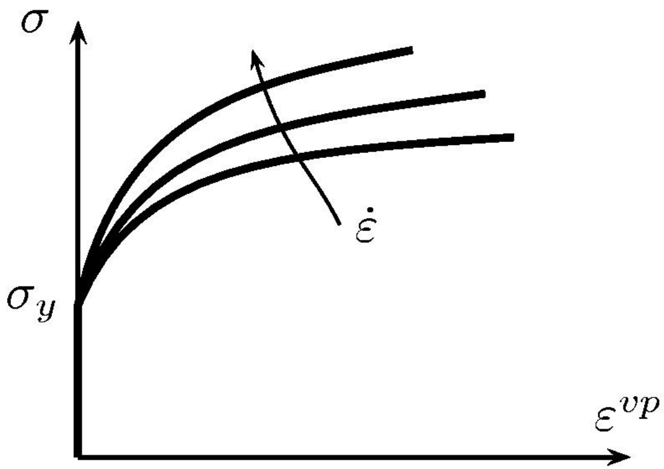
Viscoelasticity



Elastoviscoplasticity

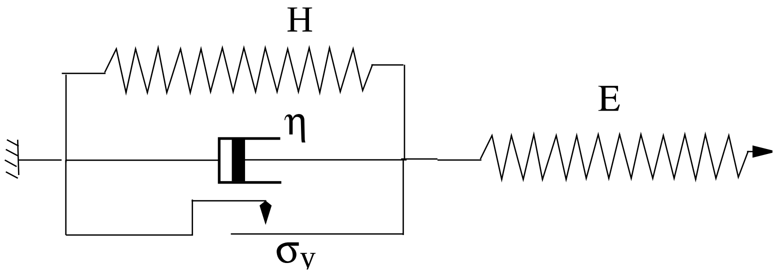


a. Generalized Bingham model

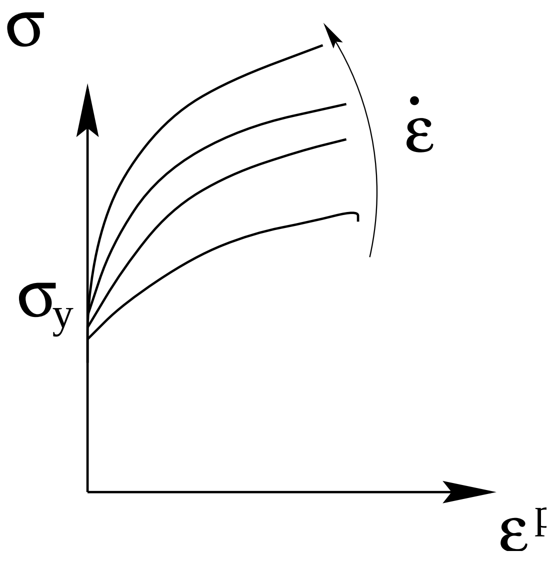


b. Behavior for tensile loading

Generalized Bingham's model



a. Construction of the model



b. Response under tensile loading

$$X = H\varepsilon^{vp} \qquad \sigma_v = \eta\dot{\varepsilon}^{vp} \qquad \sigma_p \leq \sigma_y$$

$$\sigma = X + \sigma_v + \sigma_p$$

Boundary of the elastic domain reached when $|\sigma_p| = \sigma_y$

Equations of the model

Three regimes:

$$\begin{array}{llll}
(a) & \dot{\varepsilon}^{vp} = 0 & |\sigma_p| = |\sigma - H\varepsilon^{vp}| & \leq \sigma_y \\
(b) & \dot{\varepsilon}^{vp} > 0 & \sigma_p \ = \sigma - H\varepsilon^{vp} - \eta\,\dot{\varepsilon}^{vp} & = \sigma_y \\
(c) & \dot{\varepsilon}^{vp} < 0 & \sigma_p \ = \sigma - H\varepsilon^{vp} - \eta\,\dot{\varepsilon}^{vp} & = -\sigma_y
\end{array}$$

(a) interior of the elastic domain ($|\sigma_p| < \sigma_y$)

(b) and (c) flow ($|\sigma_p| = \sigma_y$ and $|\dot{\sigma}_p| = 0$)

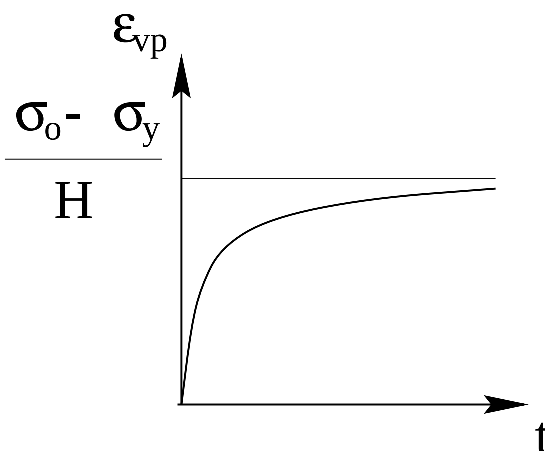
Let us set $< x >= max(x, 0)$:

$$\eta\,\dot{\varepsilon}^{vp} = \langle |\sigma - X| - \sigma_y \rangle \ sign(\sigma - X)$$

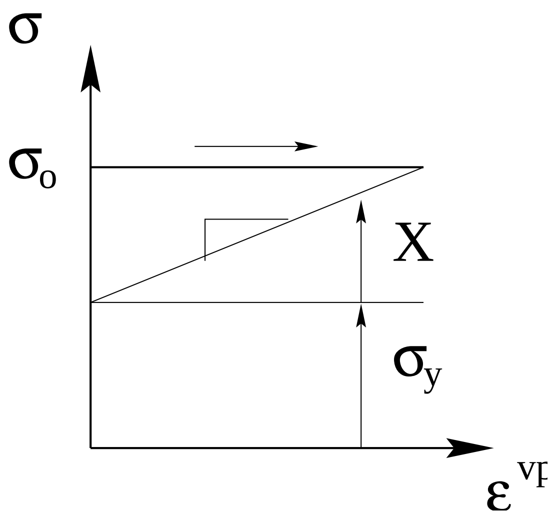
or :

$$\dot{\varepsilon}^{vp} = \frac{< f >}{\eta} sign(\sigma - X)\,, \qquad \text{with } f(\sigma, X) = |\sigma - X| - \sigma_y$$

Creep with Bingham's model



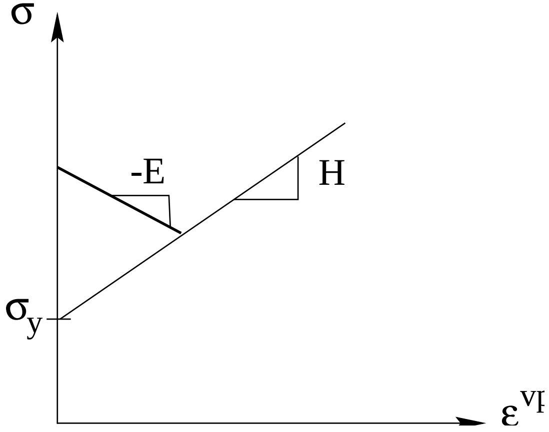
a. Viscoplastic strain *versus* time



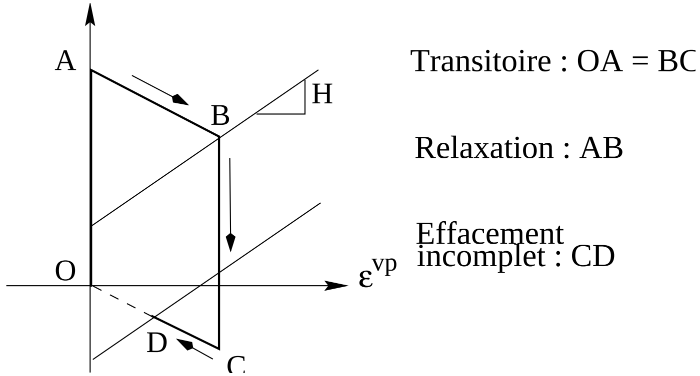
b. Evolution in the stress–viscoplastic strain plan

$$\varepsilon^{vp} = \frac{\sigma_o - \sigma_y}{H}\left(1 - \exp\left(-\frac{t}{\tau_f}\right)\right) \text{ with : } \tau_f = \eta / H$$

Relaxation with Bingham's model



a. Relaxation



b. Erasing

$$\sigma = \sigma_y \frac{E}{E+H}\left(1 - \exp\left(-\frac{t}{\tau_r}\right)\right) + \frac{E\varepsilon_o}{E+H}\left(H + E\exp\left(-\frac{t}{\tau_r}\right)\right)$$

$$\text{with : } \tau_r = \frac{\eta}{E+H}$$

A few classical models in viscoplasticity (1)

$$\dot{\varepsilon}^{vp} = \phi(f) \text{ , with } \qquad \phi(0) = 0 \text{ ,} \phi \text{ monotonic}$$

Model with isotropic and kinematic hardening under tensile loading:

$$\sigma = \sigma_y + X + R + \phi^{-1}(\dot{\varepsilon}^{vp}) = \sigma_y + X + R + \sigma_v$$

$\sigma_v = f(\dot{\varepsilon}^{vp})$ is the *Viscous stress*, overstress from yield surface

- Concept of additive and multiplicative hardening.
- The elastic domain can shrink to the origin ($\sigma = 0$).

$$\dot{\varepsilon}^{vp} = \left(\frac{\sigma}{K}\right)^n sign(\sigma) \qquad , \qquad \dot{\varepsilon}^{vp} = \left(\frac{\sigma}{K}\right)^n (\varepsilon^p)^m \, sign(\sigma)$$

A few classical models in viscoplasticity (2)

$$\dot{\varepsilon}^{vp} = \left\langle \frac{|\sigma| - \sigma_y}{K} \right\rangle^n sign(\sigma) \quad , \quad \dot{\varepsilon}^{vp} = \dot{\varepsilon}_0 \left\langle \frac{|\sigma|}{\sigma_y} - 1 \right\rangle^n sign(\sigma)$$

$$\dot{\varepsilon}^{vp} = \left(\frac{\sigma}{\sigma_{eq}} - 1 \right)^n$$

$$\dot{\varepsilon}^{vp} = A \,\mathrm{sh} \left(\frac{|\sigma|}{K} \right) sign(\sigma)$$

$$\dot{\varepsilon}^{vp} = \left\langle \frac{|\sigma - X| - R - \sigma_y}{K} \right\rangle^n sign(\sigma - X)$$

– kinematic hardening (X is the *internal stress*);

– isotropic hardening ($R + \sigma_y$ is the *friction stress*);

– hardening on the viscous stress (K is the *drag stress*).

Summary in plasticity and viscoplasticity

For both cases:

- elastic domain defined by the load function $f < 0$;
- isotropic and kinematic hardenings

For plastic materials:

- plastic flow defined by the consistency condition, $f = 0, \dot{f} = 0$;
- plastic flow is *time independent* :

$$d\varepsilon^p = g(\sigma, \ldots)d\sigma$$

For viscoplastic materials:

- viscoplastic flow is defined by the value of the overstress $f > 0$;
- possible hardening on the viscous stress;
- viscoplastic flow if *time dependent* :

$$d\varepsilon^{vp} = g(\sigma, \ldots)dt$$

State variables

- Isotropic hardening depend on p, the *accumulated plastic strain* defined as :

$$\dot{p} = |\dot{\varepsilon}^p|$$

- Linear kinematic hardening depend on ε^p, the *present plastic strain*

- Nonlinear kinematic hardening depend on α, defined as :

$$\dot{\alpha} = (1 - D\alpha sign(\dot{\varepsilon}^p))\,\dot{\varepsilon}^p$$

asymptotic value of α = 1 / D

Hardening variables

- Isotropic hardening :

$$R = Q(1 - \exp(-bp))$$

- Linear kinematic hardening :

$$X = C\varepsilon^p$$

- Nonlinear kinematic hardening ($X = C\alpha$) :

$$\dot{X} = (C - DX\,sign(\dot{\varepsilon}^p))\,\dot{\varepsilon}^p$$

for tensile loading :

$$X = (C/D)(1 - \exp(-D\varepsilon^p))$$

Flow

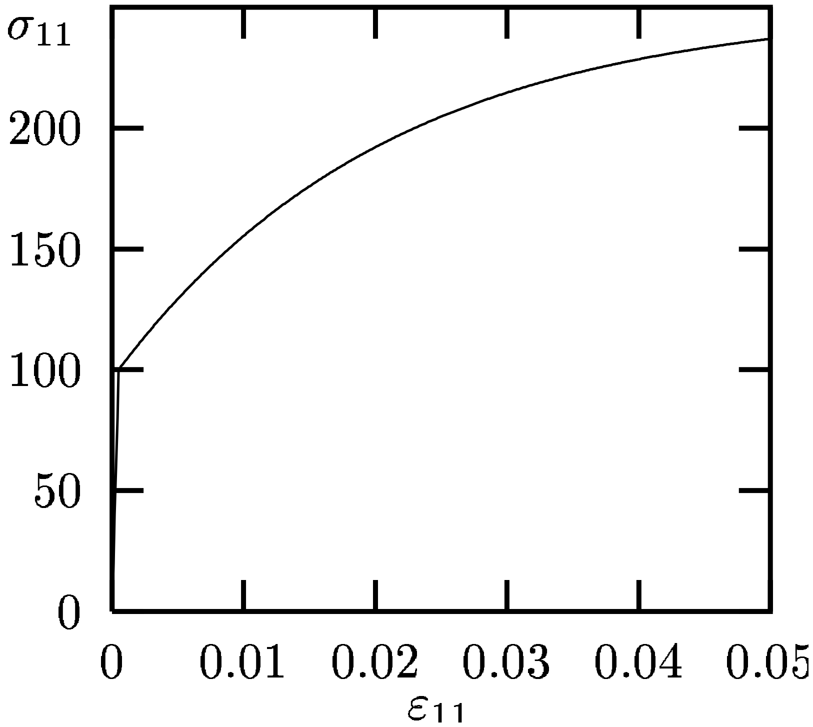
- Viscoplastic flow :

$$\dot{\varepsilon}^p = \left\langle \frac{|\sigma - X| - R - \sigma_y}{K} \right\rangle^n sign(\sigma - X)$$

- Tensile loading :

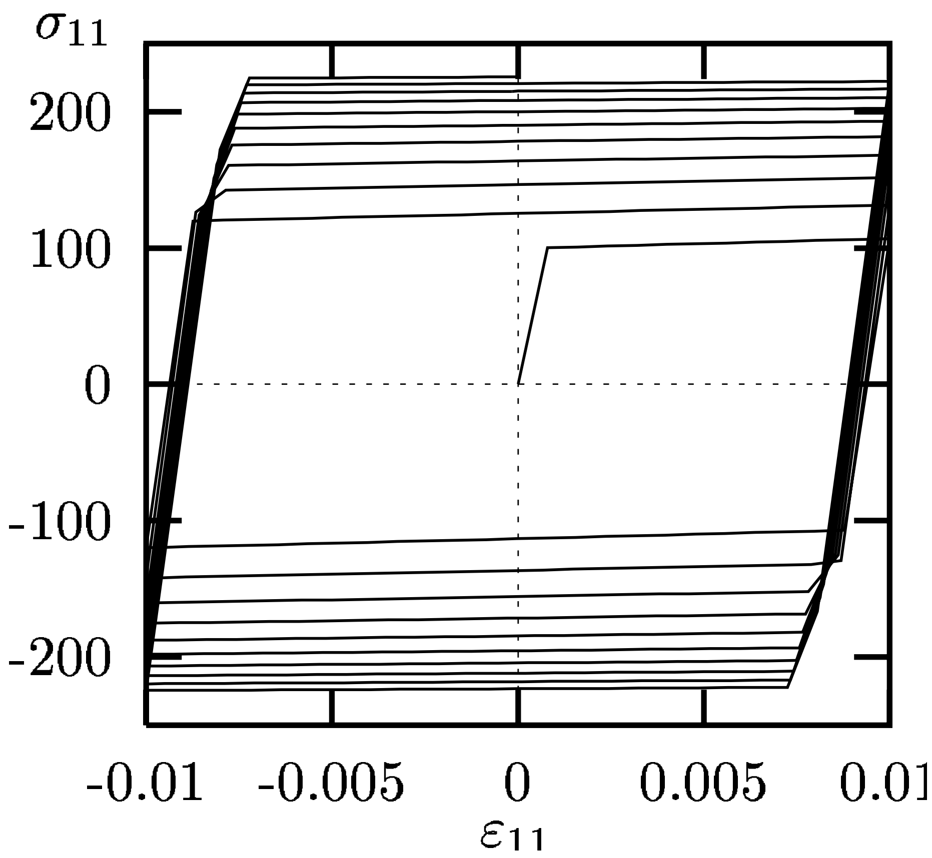
$$\sigma = \sigma_y + Q(1 - \exp(-b\varepsilon^p)) + \frac{C}{D}(1 - \exp(-D\varepsilon^p)) + K(\dot{\varepsilon}^p)^{1/n}$$

Tensile test



	Isotrope	Cin NL
σ_y	100	100
Q	150	0
b	50	0
C	0	7500
D	0	50

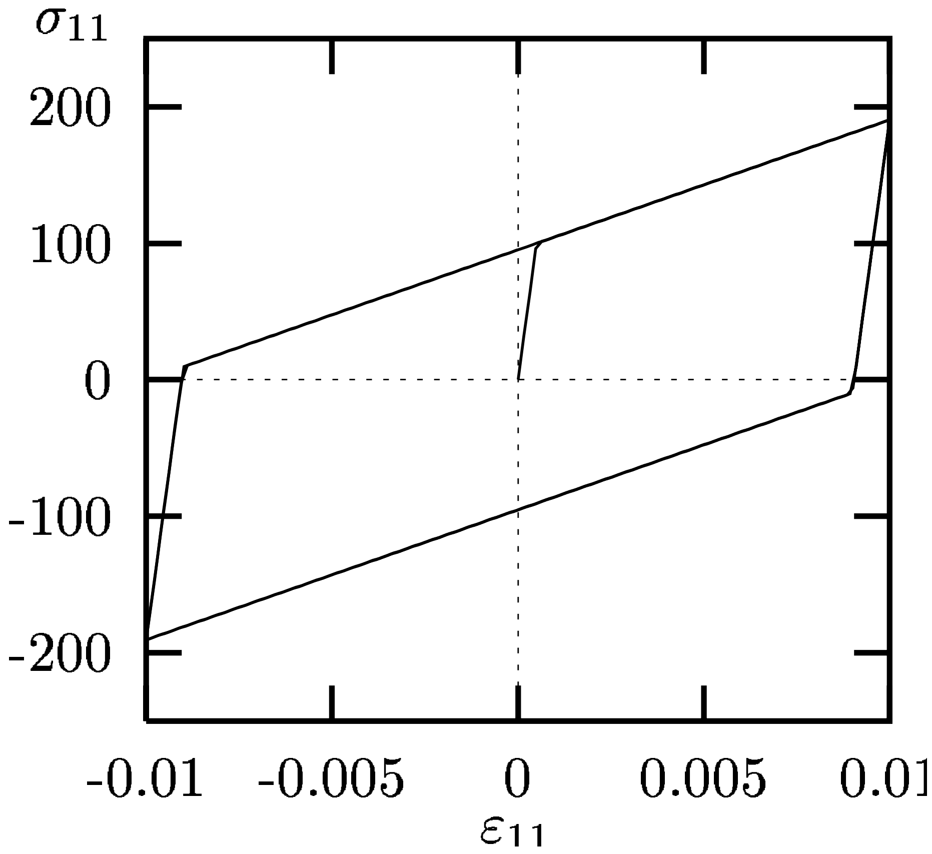
Cyclic Iso



σ_y	100
Q	150
b	5
C	0
D	0

a. Isotropic hardening

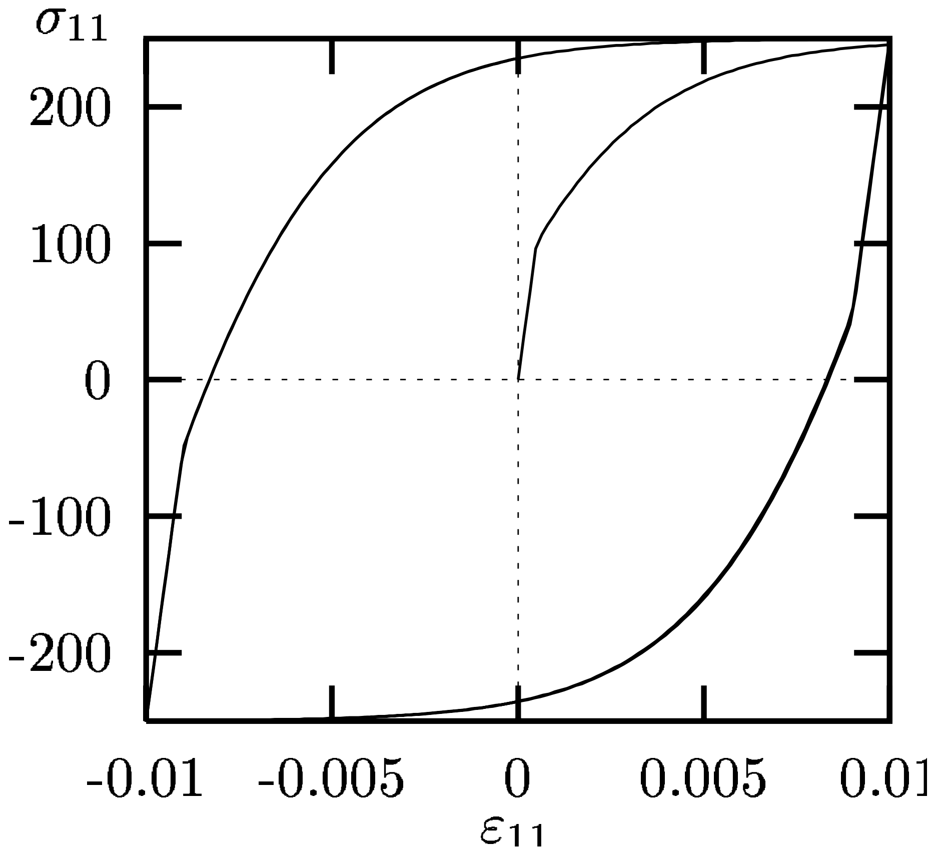
Cyclic Lin Kin



σ_y	100
Q	0
b	0
C	10000
D	0

b. Linear kinematic hardening

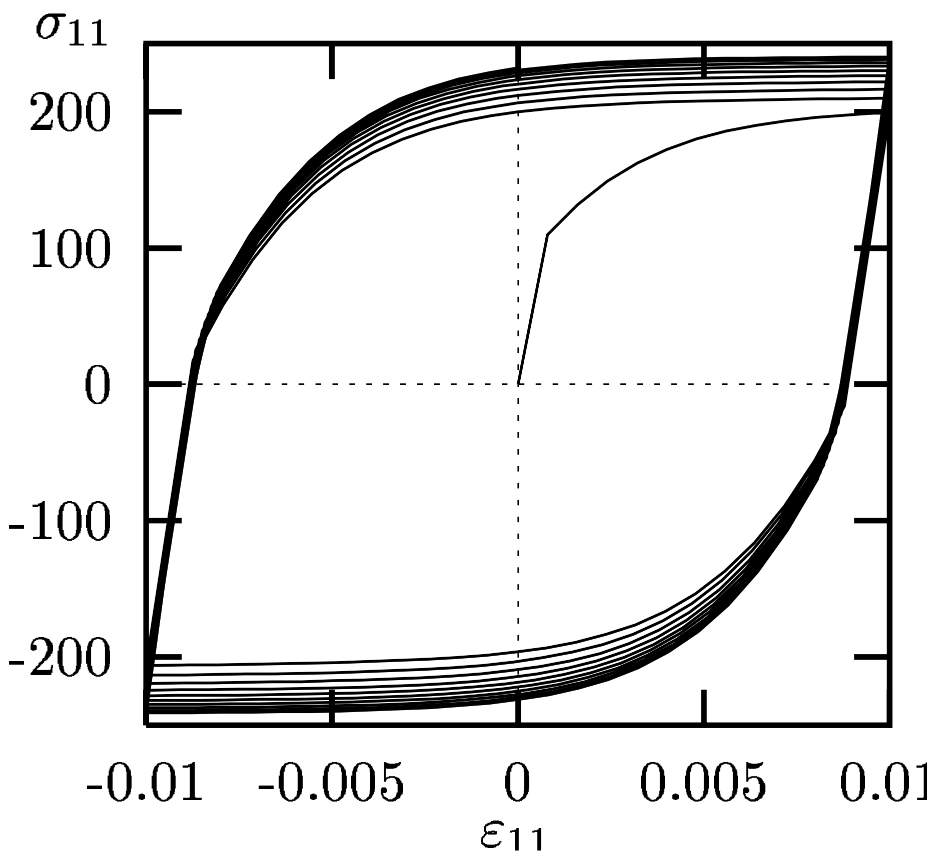
Cyclic Nonlin Kin



σ_y	100
Q	0
b	0
C	60000
D	400

c. Nonlinear kinematic hardening

Cyclic Iso + Nonlin Kin



σ_y	100
Q	50
b	5
C	40000
D	400

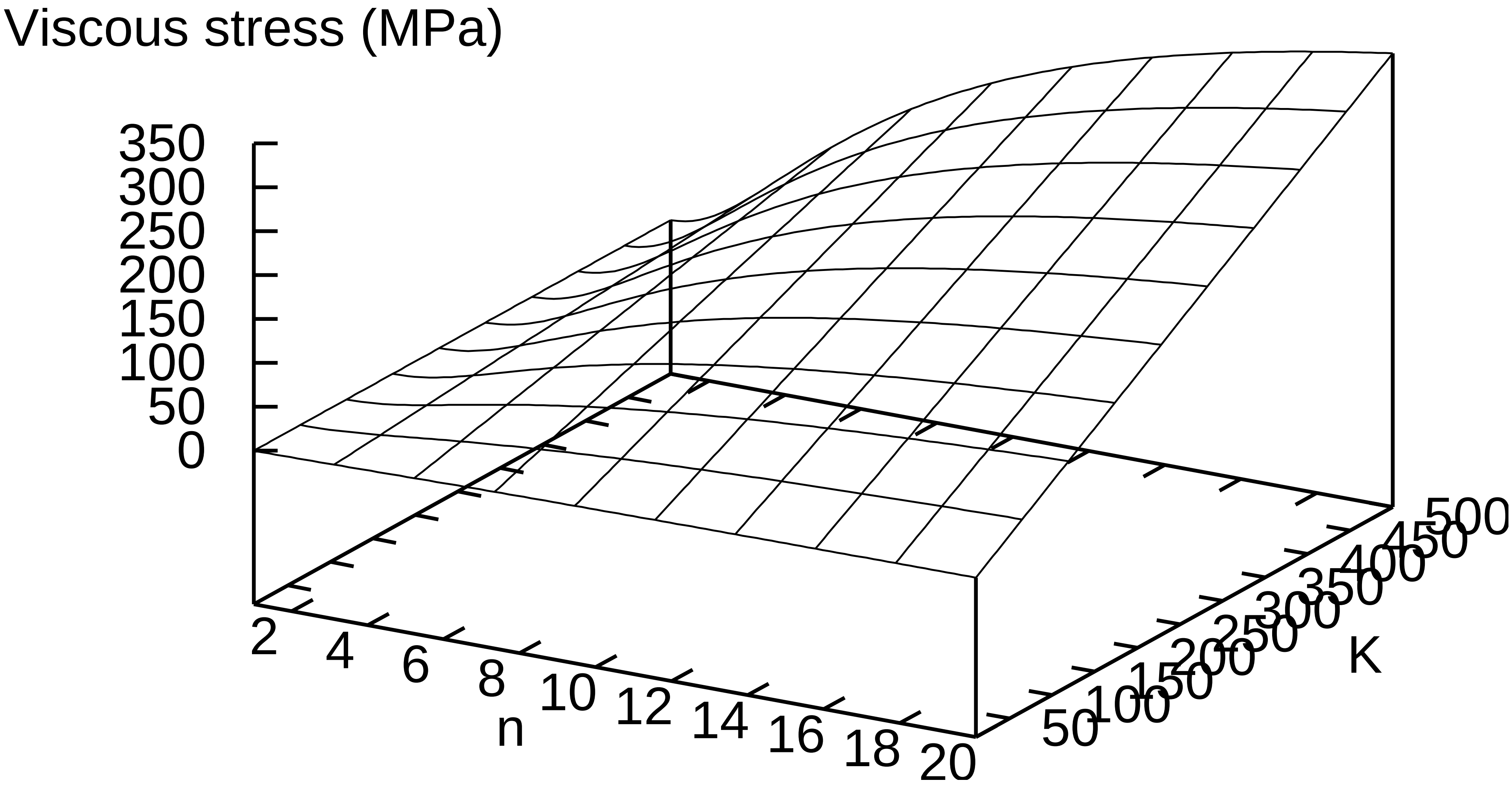
d. Isotropic + Nonlinear kinematic hardening

Rôle of each coefficient

R0	σ_y, initial yield stress
Q	cyclic hardening or softening
b	convergence rate to Q
C/ D	asymptotic value of X
D	convergence rate to C/D
K	viscous stress for $\dot{\varepsilon}^p = 1s^{-1}$
n	$\rightarrow 1$ for high temperature

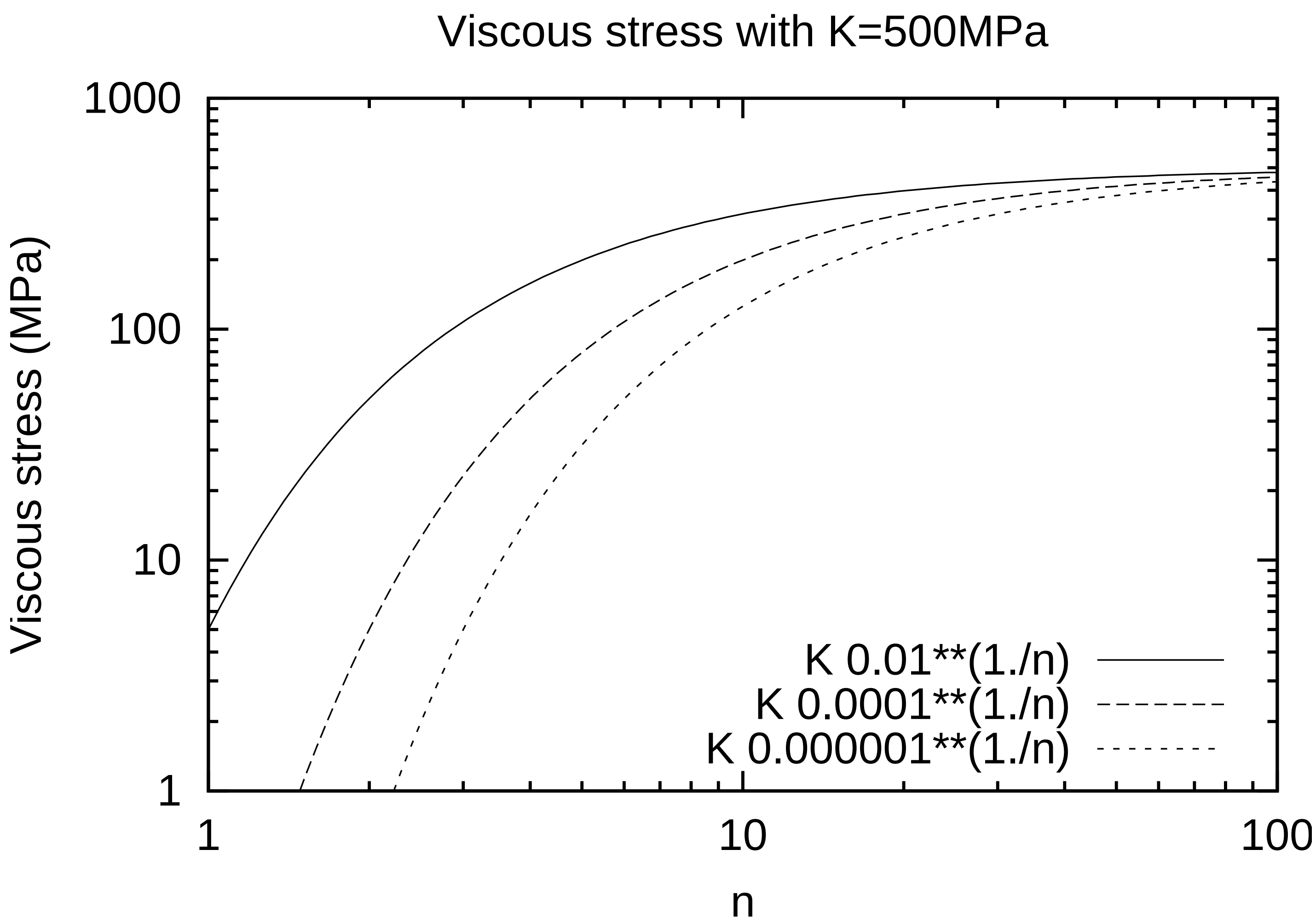
- for $\sigma_y = R = X = 0$, Norton model
- for $\sigma_y = R = 0$, no threshold (non linear viscoelasticity)
- for small K, no more viscous effect ($\rightarrow$ time independent plasticity)

Viscous stress



No more strain rate effect for $K \to 0$

Viscous stress



No more strain rate effect for high values of n

Phenomenological aspects

- Modeling of R_m (assuming $\dot{\varepsilon}^p \approx \dot{\varepsilon} = 0.001 s^{-1}$)

$$R_m = R0 + Q + (C/D) + K \times 0.001^{1/n}$$

- Modeling of $R_{0.2}$ (assuming $\dot{\varepsilon}^p \approx \dot{\varepsilon} = 0.001 s^{-1}$)

$$R_{0.2} = R0 + Q(1 - \exp(-0.002 \times b)) + (C/D)(1 - \exp(-0.002 \times D)) + K \times 0.001^{1/n}$$

- Modeling of the cyclic hardening curve (assuming $\dot{\varepsilon}^p \approx \dot{\varepsilon} = 0.001 s^{-1}$)

$$\Delta\sigma/2 = R0 + Q + (C/D)\tanh(D\Delta\varepsilon^p/2) + K \times 0.001^{1/n}$$

- Secondary creep rate

$$\dot{\varepsilon}^p = \left\langle \frac{\sigma - (C/D) - R - R0}{K} \right\rangle^n$$

- Asymptotic stress in relaxation

$$\sigma_\infty = R0 + Q + (C/D)$$

Chaboche's model

```
***behavior gen_evp
**elasticity isotropic young 160000.  poisson 0.3
**potential gen_evp ep
*criterion mises
*flow norton               K    300.     n    7.
*kinematic linear          C  10000.
*kinematic nonlinear       C 180000.     D  600.
*isotropic nonlinear       R0   300.     Q  100.  b   10.
***return
```

$$
\begin{aligned}
\sigma &= R0 + Q(1 - e^{-b\varepsilon^p}) && \textit{isotropic} \\
&+ H\varepsilon^p && \textit{kinematic} \\
&+ C/D\,(1 - e^{-D\varepsilon^p}) && \textit{kinematic} \\
&+ K\,(\dot{\varepsilon}^p)^{1/n} && \textit{viscous}
\end{aligned}
$$

$\longrightarrow$ 8 material parameters

Plasticity criteria

- Isotropic materials: - Invariants of the stress tensor:

$$I_1 = trace(\underset{\sim}{\boldsymbol{\sigma}}) \qquad = \sigma_{ii}$$

$$I_2 = (1/2)\, trace(\underset{\sim}{\boldsymbol{\sigma}})^2 = (1/2)\, \sigma_{ij}\sigma_{ji}$$

$$I_3 = (1/3)\, trace(\underset{\sim}{\boldsymbol{\sigma}})^3 = (1/3)\, \sigma_{ij}\sigma_{jk}\sigma_{ki}$$

- Invariants of the deviator:

$$\underset{\sim}{\boldsymbol{s}} = \underset{\sim}{\boldsymbol{\sigma}} - (I_1/3)\, \underset{\sim}{\boldsymbol{I}}$$

$$J_1 = trace(\underset{\sim}{\boldsymbol{\sigma}}) \qquad = 0$$

$$J_2 = (1/2)\, trace(\underset{\sim}{\boldsymbol{s}})^2 = (1/2)\, s_{ij}s_{ji}$$

$$J_3 = (1/3)\, trace(\underset{\sim}{\boldsymbol{s}})^3 = (1/3)\, s_{ij}s_{jk}s_{ki}$$

- Let us set:

$$J = ((3/2)s_{ij}s_{ji})^{0,5} = \left((1/2)\left((\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right)\right)^{0,5} = |\sigma|$$

Physical meaning of J

- sphere in the deviatoric stress space
- octahedral shear stress:

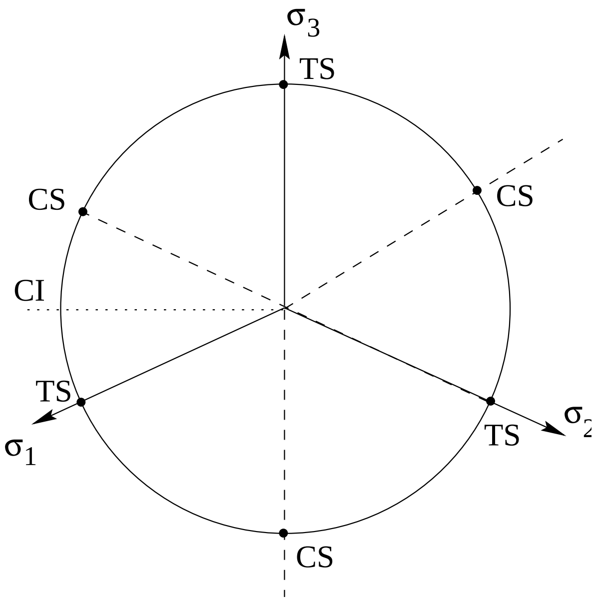
  for a facet with a normal (1,1,1), the stress vector has the following normal σ_{oct} and tangential τ_{oct} components:

$$\sigma_{oct} = (1/3)\, I_1 \quad ; \qquad \tau_{oct} = (\sqrt{2}/3)\, J$$

- elastic energy (associated to the deviatoric part of $\underset{\sim}{\boldsymbol{\sigma}}$ et $\underset{\sim}{\boldsymbol{\varepsilon}}$).

$$W_{ed} = \frac{1}{2}\, \underset{\sim}{\boldsymbol{s}} : \underset{\sim}{\boldsymbol{e}} = \frac{1}{6\mu}\, J^2$$

Contour of the von Mises criterion in the deviatoric plane



TS denote the points which can be mapped on pure tensile loading, CS the points which can be mapped into pure compression (for instance a biaxial loading, since a stress state for which the only non zero stress are $\sigma_1 = \sigma_2 = \sigma$ is equivalent to $\sigma_3 = -\sigma$), CI a shear state loading

$$f(\underset{\sim}{\boldsymbol{\sigma}}) = J - \sigma_y$$

Criteria insensitive to hydrostatic pressure

- von Mises

$$f(\underset{\sim}{\boldsymbol{\sigma}}) = J - \sigma_y$$

- Tresca

$$f(\underset{\sim}{\boldsymbol{\sigma}}) = Max_{i,j}|\sigma_i - \sigma_j| - \sigma_y$$

- Use of the third invariant

$$f(\underset{\sim}{\boldsymbol{\sigma}}) = f(J_2, J_3)$$

Comparison between Tresca and von Mises

- for tension–shear loadings

– von Mises : $f(\sigma,\tau) = \left(\sigma^2 + 3\tau^2\right)^{0,5} - \sigma_y$

– Tresca : $f(\sigma,\tau) = \left(\sigma^2 + 4\tau^2\right)^{0,5} - \sigma_y$

- in the principal stress plane (σ_1, σ_2)

– von Mises : $f(\sigma_1,\sigma_2) = \left(\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2\right)^{0,5} - \sigma_y$

– Tresca :

$f(\sigma_1,\sigma_2) = \sigma_2 - \sigma_y$ si $0 \leq \sigma_1 \leq \sigma_2$

$f(\sigma_1,\sigma_2) = \sigma_1 - \sigma_y$ si $0 \leq \sigma_2 \leq \sigma_1$

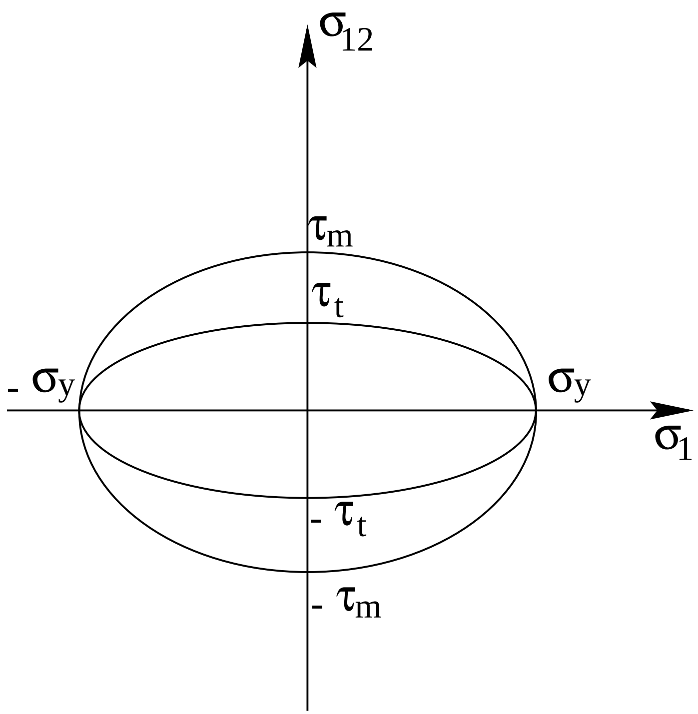
$f(\sigma_1,\sigma_2) = \sigma_1 - \sigma_2 - \sigma_y$ si $\sigma_2 \leq 0 \leq \sigma_1$

(symmetry with respect to the axis $\sigma_1 = \sigma_2$)

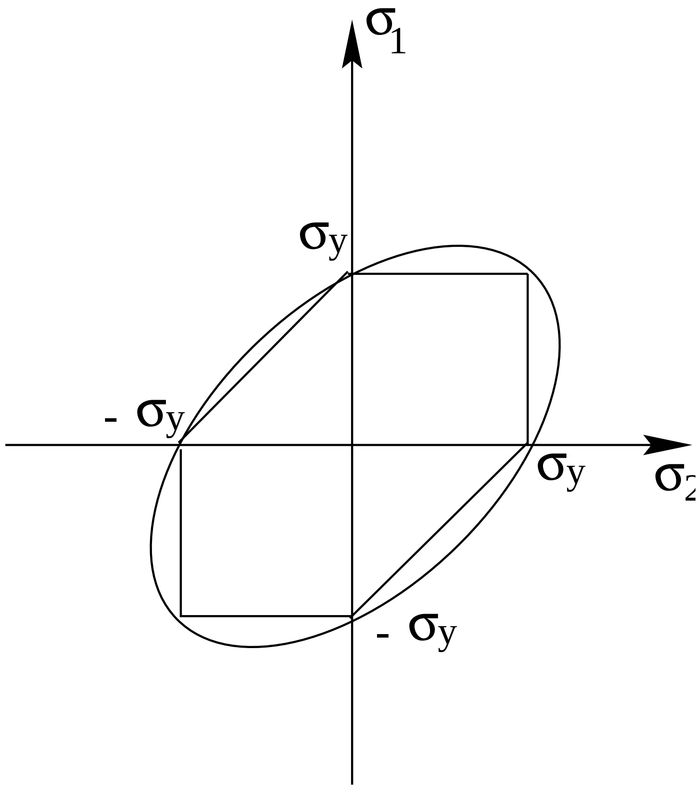
- in the deviatoric plane, von Mises = circle, Tresca = hexagone;

- in the space of the principles stresses, cylindres (1,1,1)

Comparisons of Tresca and von Mises



a. Tension–shear (von Mises : $\tau_m = \sigma_y/\sqrt{3}$, Tresca : $\tau_t = \sigma_y/2$)



b. Biaxial tension

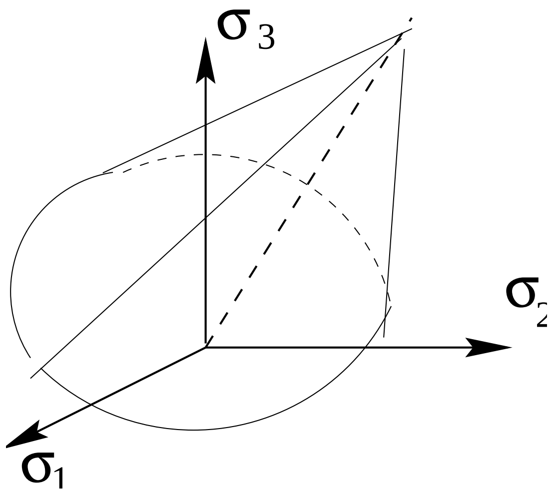
Criteria sensitive to the hydrostatic pressure

- Drucker-Prager criterion

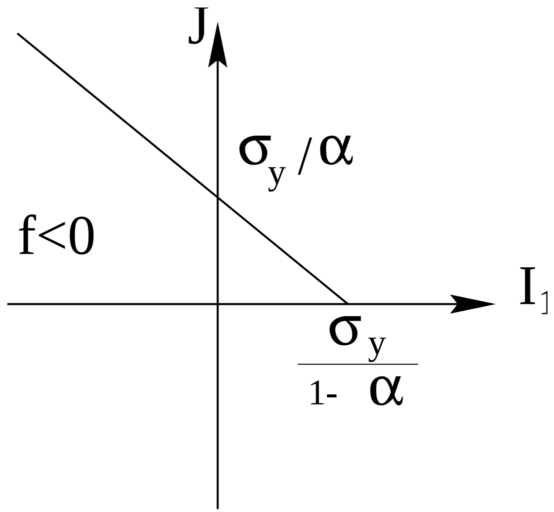
$$f(\underset{\sim}{\boldsymbol{\sigma}}) = J - \frac{\sigma_y - \alpha\, I_1}{1-\alpha}$$

Elastic yield in tension: σ_y , in compression: $-\sigma_y/(1-2\,\alpha)$

0 ¡ α ¡ 0.5



a. In the space of principle stresses



b. In the plan $I_1 - J$

Mohr-Coulomb criterion

$$f(\underset{\sim}{\boldsymbol{\sigma}}) = \sigma_1 - \sigma_3 + (\sigma_1 + \sigma_3)\sin\phi - 2C\cos\phi$$

$$\text{(with } \sigma_3 \leq \sigma_2 \leq \sigma_1\text{)}$$

$$|T_t| < -\tan(\phi)\, T_n + C$$

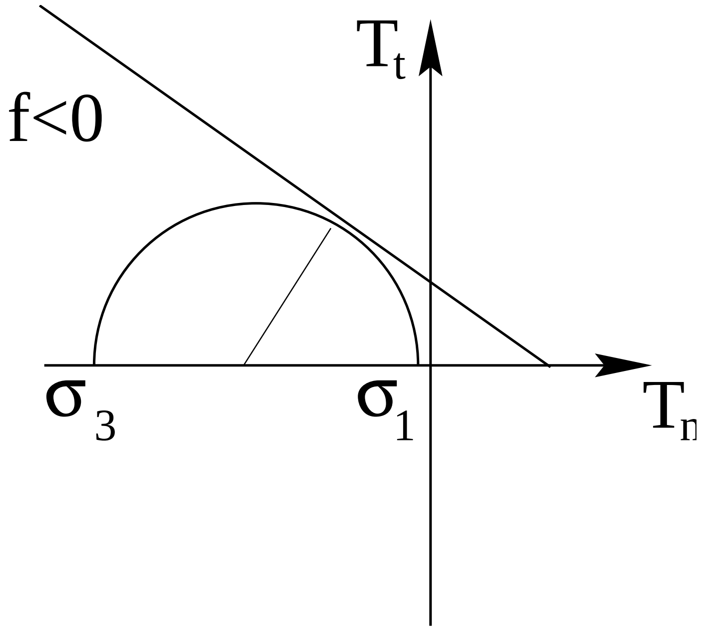
- C cohesion, ϕ internal friction of the material
- If C is zero and ϕ non zero, powder material.
- If ϕ is zero and C non nul, purely cohesive material.
- K_p; compressive yield limit, R_p :

$$f(\underset{\sim}{\boldsymbol{\sigma}}) = K_p\,\sigma_1 - \sigma_3 - R_p$$

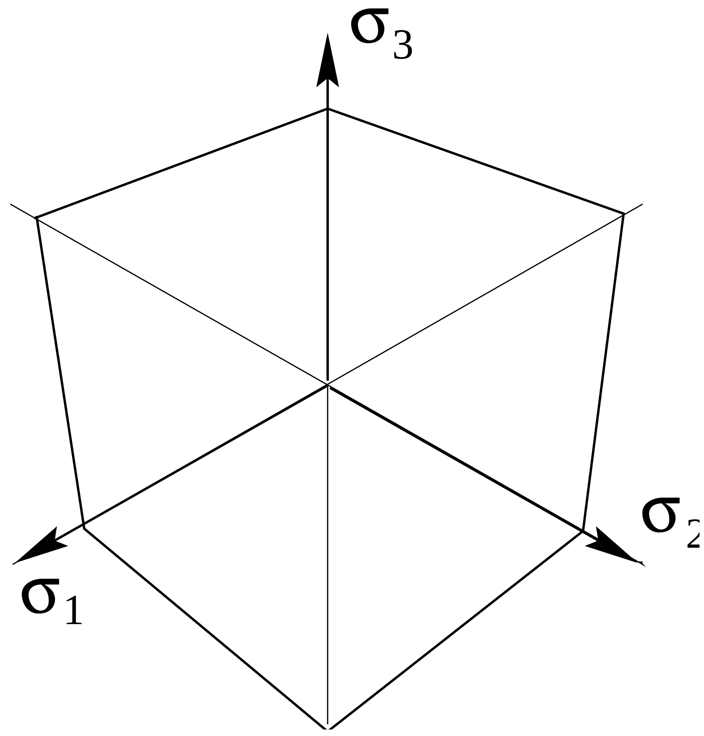
$$\text{with } K_p = \frac{1+\sin\phi}{1-\sin\phi} = \tan^2\left(\frac{\pi}{4} + \frac{\phi}{2}\right) \qquad R_p = \frac{2\,\cos\phi\, C}{1-\sin\phi}$$

Representation of Mohr-Coulomb's criterion

In the deviatoric plane, one get a regular hexagon
$(TS = 2\sqrt{6}(C\cos\phi - p\sin\phi)/(3+\sin\phi)$,
$CS = 2\sqrt{6}(-C\cos\phi + p\sin\phi)/(3-\sin\phi))$



a. In Mohr's plane



b. In the deviatoric plane

”Closed” criteria

The material must also present plastic flow in compression:

- *cap model*, allow to close by means of an ellipse Drucker–Prager’s criterion,
- *Cam–clay model* has a curve defined by two ellipses in the plane $(I_1 - J)$

Anisotropic criteria

$$f(\underset{\sim}{\boldsymbol{\sigma}}) = \left((3/2)\, H_{ijkl}\, s_{ij}\, s_{kl}\right)^{0,5} - \sigma_y \qquad \text{(ou } H_{ijkl}\, \sigma_{ij}\, \sigma_{kl})$$

Hill's criterion,

- in orthotropy axes :

$$f(\underset{\sim}{\boldsymbol{\sigma}}) = \left(F(\sigma_{11} - \sigma_{22})^2 + G(\sigma_{22} - \sigma_{33})^2 + H(\sigma_{33} - \sigma_{11})^2 + 2\, L\sigma_{12}^2 + 2\, M\sigma_{23}^2 + 2\, N\sigma_{13}^2\right)^{0,5} - \sigma_y$$

- transverse isotropy, 3 independent coefficients,
- cubic symmetry, one coefficient only

Rheology

- **Time independent plasticity**

$$\varepsilon = \varepsilon^e + \varepsilon^p \qquad d\varepsilon^p = f(...)d\sigma$$

- **Elastoviscoplasticity**

$$\varepsilon = \varepsilon^e + \varepsilon^p \qquad d\varepsilon^p = f(...)dt$$

- **Viscoelasticity**

$$F(\sigma, \dot{\sigma}, \varepsilon, \dot{\varepsilon}) = 0$$

–Review the foundation of inelastic behavior–